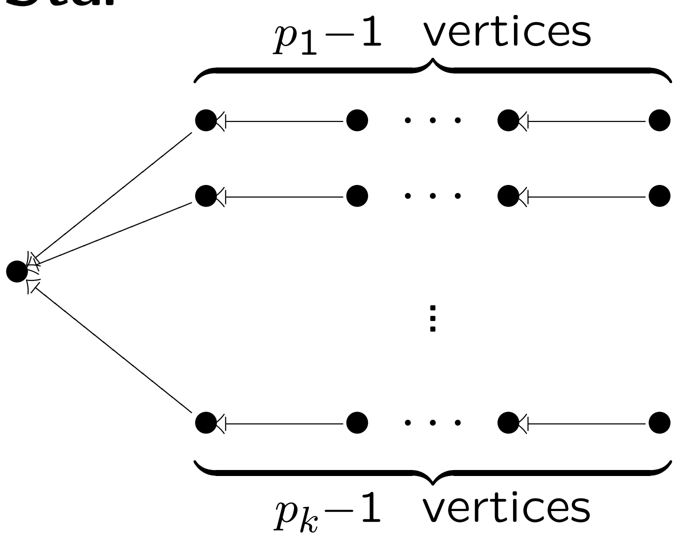
The s-tame dimension vectors for stars

§1: Basic definitions

Star



$Q = (Q_0, Q_1, s, t)$,
Q_0 vertices,
Q_1 arrows,
$s, t : Q_1 \to Q_0$
$s(\alpha) \xrightarrow{\alpha} t(\alpha)$

Representations (over $K = \overline{K}$)
$\mathbf{V} = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$,
V_i fin. dim. vector space $/K$,
$V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ K-linear map

Subspace representations
$\forall \alpha \in Q_1 : V_\alpha$ injective

Dimension vector for a representation (V_i, V_α)
$\mathbf{d} = (d_i)_{i \in Q_0}$, where $d_i = \dim V_i$ for all $i \in Q_0$

From now on, all dim. vectors are dim. vectors of **subspace representations**, i.e. they are increasing along their arms.

$\mathbf{V} \neq 0$ **indecomposable**
$\mathbf{V} = \mathbf{V_1} \oplus \mathbf{V_2} \Rightarrow \mathbf{V_1} = 0$ or $\mathbf{V_2} = 0$

$\mathbf{d}$ **s-tame**

(1) $\exists$ 1-param. family of indec. subspace repns. for $\mathbf{d}$, and
(2) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_1} \Rightarrow \nexists\ m$-param. family of indec. subspace repns. $\forall\ \mathbf{d_i}$, $i = 1, 2$, with $m \geq 2$.

§2: Classification of the s-tame dimension vectors

Need **Tits form**

$$q = q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

$$(x_i)_{i \in Q_0} \mapsto \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$$

Theorem

$\mathbf{d}$ is s-tame if and only if

(1)' $q(\mathbf{d}) = 0$, and

(2)' $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2} \Rightarrow q(\mathbf{d_i}) \geq 0 \; \forall i = 1, 2.$

Construct "minimal not s-tame" dim. vectors:

$\mathbf{d}$ is **s-hypercritical** if

(3) $\exists\ \ell$-param. family of indec. subspace repns. with $\ell \geq 2$ for $\mathbf{d}$, and

(4) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$, $\mathbf{d_i} \neq 0\ \forall i = 1, 2 \Rightarrow \nexists\ m$-param. family of subspace repns. for $\mathbf{d_i}$, $i = 1, 2$, with $m \geq 2$.

Proposition

$\mathbf{d}$ is s-hypercritical if and only if

(3)' $q(\mathbf{d}) < 0$, and

(4)' $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$, $\mathbf{d_i} \neq 0\ \forall i = 1, 2 \Rightarrow q(\mathbf{d_i}) \geq 0\ \forall i = 1, 2$.

Classify dim. vectors of subspace repns with conditions (1)' & (2)' and (3)' & (4)' in order to prove the Theorem.

§3: Finding the dimension vectors

Rewrite the dim. vectors:

$$\mathbf{d} \mapsto (\mathbf{a_1}, \ldots, \mathbf{a_k})$$

Take dimension jumps
along the arms.

$\forall i = 1, \ldots, k$:
$\sum_{j=1}^{p_i} a_{ij} =: n =$ dim. at the "central vertex"

Have $a_{ij} \geq 0$, since $\mathbf{d}$ is a dim. vector of subspace repns.

tuple of compositions of n

Examples

$n = 4$:

$$\begin{array}{ccccccc} & & 1 & \to & 2 & \searrow & \\ & & 1 & \to & 3 & \to & 4 \\ 1 & \to & 2 & \to & 3 & \nearrow & \end{array} \mapsto ((1,1,2),(1,2,1),(1,1,1,1))$$

$n = 9$:

$$\begin{array}{ccccccccc} & & & & 5 & \to & 7 & \searrow & \\ & & 3 & \to & 4 & \to & 5 & \to & 9 \\ 1 & \to & 2 & \to & 4 & \to & 6 & \nearrow & \end{array} \mapsto ((5,2,2),(3,1,1,4),(1,1,2,2,3))$$

Tits form for tuples of compositions

$$q(\mathbf{a_1}, \ldots, \mathbf{a_k}) = \frac{1}{2}\left(\sum_{i=1}^{k}\sum_{j=1}^{p_i} a_{ij}^2 + (2-k)n^2\right)$$

Properties

- independent of the order of the dimension jumps along the arms
- minimal value for fixed n and fixed arm lengths if and only if the dimension jumps are distributed as evenly as possible

Problem

Find conditions such that Tits form becomes non negative!

Have a list with conditions on dimension jumps and arm lengths:

($*$) q is positive

($\star$) q is non negative

($\circ$) q is neither positive nor non negative, and the dim. vectors $\mathbf{d}$ with smallest "central dimension" have $q(\mathbf{d}) < 0$.

Can show the following (where $\mathbf{d}' < \mathbf{d}$) (except for one case which has to be treated independently):

$\mathbf{d}$:	($*$)	($\star$)	dim. vectors with smallest central dim. in ($\circ$)
	$\downarrow$	$\downarrow$	$\downarrow$
$\mathbf{d}'$:	($*$)	($*$), ($\star$)	($*$), ($\star$)

§4: <u>Roots</u>

Define **reflections** r_i, $i \in Q_0$, on $\mathbb{Z}^{Q_0}$ as follows:

$$(r_i(\mathbf{x}))_j = \begin{cases} \sum\limits_{\substack{i \to k \\ \text{or } k \to i}} x_k - x_i, & \text{if } j = i \\ x_j, & \text{if } j \neq i \end{cases}$$

$W := W_Q := \langle r_i \mid i \in Q_0 \rangle$ – **Weyl group**

$\mathbf{e}_i$ **simple root** at vertex i, i.e.

$$(\mathbf{e}_i)_j = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$$

$$\Pi := \{\mathbf{e}_i \mid i \in Q_0\}$$

(symmetric) Euler form

$$(-,-) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

$$(\mathbf{x}, \mathbf{y}) = 2 \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} (x_{s(\alpha)} y_{t(\alpha)} + y_{s(\alpha)} x_{t(\alpha)})$$

Fundamental region

$$F_Q := \{\mathbf{d} \in \mathbb{N}_0^{Q_0} \backslash \{0\} \mid (\mathbf{d}, \mathbf{e}_i) \leq 0 \ \forall i \in Q_0\}$$

Kac (1980)
(positive) real roots $= \Delta_{re}^{+} = W\Pi \cap \mathbb{N}_0^{Q_0}$
(positive) imaginary roots $= \Delta_{im}^{+} = WF_Q$

$\Delta^{+} = \Delta_{re}^{+} \dot{\cup} \Delta_{im}^{+}$

Theorem (Kac) (1980/82)
$K = \overline{K}$

(1) $\exists$ indec. repn. of a dim. vector $\mathbf{d}$ $\Leftrightarrow$ $\mathbf{d} \in \Delta^{+}$.

(2) $\mathbf{d} \in \Delta_{re}^{+} \Rightarrow \exists!$ indec. repn. of dim. vector $\mathbf{d}$, and then $q(\mathbf{d}) = 1$.

(3) $\mathbf{d} \in \Delta_{im}^{+} \Rightarrow \exists$ a family of indec. repns. of dim. vector $\mathbf{d}$, and $\mu(\mathbf{d}) = 1 - q(\mathbf{d})$, where $\mu(\mathbf{d})$ = max. no. of parameters on which a family of indec. repns. with dim. vector $\mathbf{d}$ depends.

Lemma

$\mathbf{d}$ dim. vector with $\left\{ \begin{array}{c} (1)\text{' \& } (2)\text{'} \\ (3)\text{' \& } (4)\text{'} \end{array} \right\}$

$\Rightarrow \mathbf{d}$ is a root.

Lemma

Indec. repns. of stars with subspace orientation and "central dimension" $\neq 0$ are always subspace repns.

Proof of the Theorem

Let $\mathbf{d}$ be a dim. vector with properties (1)' and (2)'.

$\Rightarrow \mathbf{d}$ is a root (by Lemma).

$\Rightarrow \exists$ 1-param. family of indec. repns. for $\mathbf{d}$ (by property (1)' and Kac's Thm.)

$\Rightarrow$ property (1)

$\mathbf{d}' \leq \mathbf{d}$
$\Rightarrow \mathbf{d}'$ has also property (2)'
$\Rightarrow q(\mathbf{d}') = 0$ ($\Rightarrow \mu(\mathbf{d}') = 1$ or no indec. repn.)
or $q(\mathbf{d}') \geq 1$ (no families of indec. repns.)
$\Rightarrow$ property (2)

Let now $\mathbf{d}$ be a dim. vector with properties (1) and (2), and let $\mathbf{d}' \leq \mathbf{d}$.
$\Rightarrow q(\mathbf{d}') \geq 0$
(Otherwise, $\mathbf{d}' \geq \mathbf{d}''$ where $\mathbf{d}''$ has properties (3)' and (4)', so $\mathbf{d}''$ is a root (by Lemma), and hence there is an m-param. family of indec. repns. with $m \geq 2$.)
$\Rightarrow$ (2)'

(1) $\Rightarrow q(\mathbf{d}) \leq 0$,
but also have $q(\mathbf{d}) \geq 0$.
$\Rightarrow$ condition (1)'

□

§5: Construction of families of representations

- Restrict to smaller quivers with "known" repns., e.g. quivers of finite or tame type.
- Take canonical decomposition of restricted dimension vector (take characterisation by A. Schofield (1992)).
- Find representation(s) for the smaller quiver according to the canonical decomposition of the restricted dimension vector.
- Embed "remaining" vector spaces in an appropriate way.

§6: Remarks, References

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