Tame dimension vectors for wild quivers

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$Q = (Q_0, Q_1, s, t)$ quiver w/o oriented cycles,
$K = \overline{K}$ alg. closed field,
$\mathbf{d} \in \mathbb{N}_0^{Q_0}$ dimension vector

Theorem (Kac, 1980)
$\mathbf{d}$ dim. vector of an indec. repn. for $Q \Leftrightarrow \mathbf{d}$ positive root of the KM-Lie algebra for Q.

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d dim. vector of an indec. repn. for $Q \Leftrightarrow$ **d** positive root of the KM-Lie algebra for Q.

positive roots $= \Delta_{\mathrm{re}}^{+} \dot{\cup} \Delta_{\mathrm{im}}^{+}$
$\Delta_{\mathrm{re}}^{+} = W \cdot \Pi \cap \mathbb{N}_0^{Q_0}$, $\Pi = \{\text{simple roots } \mathbf{e_i} \mid i \in Q_0\}$
$\Delta_{\mathrm{im}}^{+} = W \cdot F_Q$, $F_Q = \{\mathbf{d} \in \mathbb{N}_0^{Q_0} \mid (\mathbf{d}, \mathbf{e_i}) \leq 0 \; \forall i \in Q_0\}$
(fundamental domain)
W Weyl group for Q
$(-,-)$ Euler form for Q

Furthermore:

Theorem (Kac, 1982)

- $\mathbf{d} \in \Delta^+_{\mathrm{re}} \Rightarrow \exists!$ (up to isomorphism) indec. repn. with dim. vector $\mathbf{d}$
- $\mathbf{d} \in \Delta^+_{\mathrm{i}m} \Rightarrow \exists$ infinitely many indec. non isomorphic repns. with dim. vector $\mathbf{d}$, and families of indec. repns. are parametrised by at most $1 - q(\mathbf{d})$ parameters (and $1 - q(\mathbf{d})$ parameters do occur!).

Tame dimension vectors

Question: Which are the dim. vectors (for wild quivers)

- with a *one parameter family* of *indec.* repns. and
- for which *each* family of (not nec. indec.) repns. depends on *at most one* parameter?

(=: **tame dim. vectors**)

Need a description of the

- *minimal* dim. vectors admitting m-parameter families of indec. repns. with $m \geq 2$

(=: **hypercritical dim. vectors**)

Hypercritical dimension vectors (for trees)

- up to a reduction step: finitely many
- complete list

Definition

$\mathbf{d}$ is *combinatorially hypercritical* if

(i) $q(\mathbf{d}) < 0$, and

(ii) $q(\mathbf{d}') \geq 0$ for all $\mathbf{d}' < \mathbf{d}$,

where q Tits form for Q.

Theorem
$\mathbf{d} \in \mathbb{N}_0^{Q_0}$ dim. vector for Q, Q a tree.
$\mathbf{d}$ hypercritical $\Leftrightarrow$ $\mathbf{d}$ comb. hypercritical

On the proof

1. "Classify" all comb. hypercritical dim. vectors!
2. Show that all of them are roots!
3. Show: $\mathbf{d}$ hypercritical $\Leftrightarrow$ $\mathbf{d}$ comb. hypercritical (using the list from step 1.)!

Lemma 1
The Tits form of a dimension vector with fixed dimensions at the branching vertices becomes minimal if and only if the dimension jumps on the arms and on central lines are distributed as evenly as possible.

Example

$$q\left(\begin{array}{ccccccccccccc}
 & & & & 2 & & & & & & & & 1\\
 & & & & & \diagdown & & & & & & \diagup & \\
 & & & & & & 4 & - & 3 & - & 2 & & \\
 & & & & & \diagup & & & & & & \diagdown & \\
 & & & & 3 & & & & & & & & 1\\
 & & & \diagup & & & & & & & & & \\
 & & 2 & & & & & & & & & & \\
 & \diagup & & & & & & & & & & & \\
1 & & & & & & & & & & & &
\end{array}\right) = -1,$$

but

$$
q\left(\begin{array}{ccccccccccccc}
 & & & & 2 & & & & & & & & 1 \\
 & & & & & \diagdown & & & & & & \diagup & \\
 & & & & & & 4 & - & {\color{red}2} & - & 2 & & \\
 & & & & & \diagup & & & & & & \diagdown & \\
 & & & & 3 & & & & & & & & 1 \\
 & & & \diagup & & & & & & & & & \\
 & & 2 & & & & & & & & & & \\
 & \diagup & & & & & & & & & & & \\
1 & & & & & & & & & & & &
\end{array}\right) = 0
$$

and

$$
q\left(\begin{array}{ccccccccccccc}
 & & & & {\color{red}1} & & & & & & & & 1 \\
 & & & & & \diagdown & & & & & & \diagup & \\
 & & & & & & 4 & - & 3 & - & 2 & & \\
 & & & & & \diagup & & & & & & \diagdown & \\
 & & & & 3 & & & & & & & & 1 \\
 & & & \diagup & & & & & & & & & \\
 & & 2 & & & & & & & & & & \\
 & \diagup & & & & & & & & & & & \\
1 & & & & & & & & & & & &
\end{array}\right) = 0
$$

Trick

Rewrite parts of the dim. vectors in terms of dim. jumps and use the formula for the Tits forms of stars on these parts!

Classification

1. All dim. vectors in the list are comb. hypercritical: quite easy (using Lemma 1).
2. There are no others: not so easy.

Trick

Take reduced dimension vectors, i. e. delete all double entries in the dimension vector.

Lemma 2

$\mathbf{d}_{\mathrm{red}}$ not comb. hypercritical $\Rightarrow$ $\mathbf{d}$ not comb. hypercritical.

Proof

1. $q(\mathbf{d}_{\mathrm{red}}) = q(\mathbf{d})$
2. If $\exists\ \mathbf{d}' < \mathbf{d}_{\mathrm{red}}$ with $q(\mathbf{d}') < 0$, we can construct $\mathbf{d}'' < \mathbf{d}$ with $q(\mathbf{d}'') = q(\mathbf{d}')$ by repeating the corresp. dim. entries.

$\square$

One branching vertex

Theorem is true (star case).

Two branching vertices

Get the following restrictions (using also Lemma 1):

- arm lengths: $(2,3)$ and $(2,2)$ and
  - minimal dim. entry in first arm $= 2$ or
  - minimal dim. entry in second arm $= 1$ or
  - minimal dim. entries in third and fourth arm $= 1$.
- arm lengths: $(2,m)$ and $(2,2)$ and minimal dimension entry in first arm $= 1$

Further restrictions lead to eight cases for which the Tits form can be calculated explicitly and is shown to be non negative.

More than two branching vertices

Proposition
There are *no* comb. hypercritical dimension vectors for quivers with more than two branching vertices.

Lemma 3
$\mathbf{d}$ dim. vector with a 1-entry on a central line, $q(\mathbf{d}) = 0$ and $q(\mathbf{d}') \geq 0$ for all $\mathbf{d}' < \mathbf{d}$, then $\mathbf{d} \geq \mathbf{c}$, where $\mathbf{c}$ is a critical dim. vector for a tame quiver.

Proof (Induction on number b of branching vertices)
$b = 1$:
star case: $\mathbf{d}$ tame $\Rightarrow$ $\mathbf{d}$ s-tame $\Rightarrow$ $\mathbf{d} \geq \mathbf{c}$ w. r. t. subspace order $\Rightarrow$ $\mathbf{d} \geq \mathbf{c}$
$b > 1$:
Write $\mathbf{d} = \mathbf{d_1} - 1 - \mathbf{d_2} \Rightarrow q(\mathbf{d_1} - 1) \geq 0$ and $q(\mathbf{d_2} - 1) \geq 0 \Rightarrow$ $q(\mathbf{d_1} - 1) = 0$ or $q(\mathbf{d_2} - 1) = 0$
So one of $\mathbf{d_1}$ or $\mathbf{d_2}$ has the same properties as $\mathbf{d}$, but $\leq b - 1$ branching vertices. $\Rightarrow \mathbf{d_1} \geq \mathbf{c}$ or $\mathbf{d_2} \geq \mathbf{c}$ for $\mathbf{c}$ a critical dim. vector for a tame quiver.

$\square$

Now show, using Lemma 3, that for Q a quiver with more than two branching vertices and a dimension vector $\mathbf{d}$ for Q with $q(\mathbf{d}) < 0$,

$$\mathbf{d}_{\mathrm{red}} \geq \begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 1 & - & 2 & - & 1 \\ & / & & \backslash & \\ 1 & & & & 1 \end{array}$$

or $\mathbf{d}_{\mathrm{red}} \geq \mathbf{c_1} - 1 - \mathbf{c_2}$ where $\mathbf{c_1} - 1$ and $\mathbf{c_2} - 1$ are critical dimension vectors of tame quivers!