Artin groups and associated spaces
Part I

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Coxeter graph and Coxeter matrix

Definition. Let S be a finite set. A *Coxeter matrix* over S is a square symmetric matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of S with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ for all $s \in S$ and $m_{s,t} = m_{t,s} \geq 2$ for all $s, t \in S$, $s \neq t$. We represent M with a labeled graph Γ called *Coxeter graph*, defined as follows.

- The set of vertices of Γ is S.
- Two vertices $s, t \in S$ are connected by an edge if $m_{s,t} \geq 3$, and this edge is labeled with $m_{s,t}$ if $m_{s,t} \geq 4$.

Example. The following is a Coxeter matrix:

$$M = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Its Coxeter graph is:

Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. For $s, t \in S$ and $m \in \mathbb{N}$, $m \geq 2$, we denote by $\mathrm{Prod}(s, t, m)$ the word $sts \cdots$ of length m. So, $\mathrm{Prod}(s, t, 2) = st$, $\mathrm{Prod}(s, t, 3) = sts$, and $\mathrm{Prod}(s, t, 4) = stst$. The *Artin group* of Γ is the group $A[\Gamma]$ defined by the presentation:

$$A[\Gamma] = \langle S \mid \mathrm{Prod}(s, t, m_{s,t}) = \mathrm{Prod}(t, s, m_{s,t}) \text{ for } s, t \in S\,, \\ s \neq t\,,\ m_{s,t} \neq \infty \rangle\,.$$

The *Coxeter group* of Γ, denoted $W[\Gamma]$, is the quotient of $A[\Gamma]$ by the relations $s^2 = 1$, $s \in S$.

Example. Let Γ be:

Then:

$$A[\Gamma] = \langle s, t, r \mid stst = tsts\,,\ sr = rs\,,\ trt = rtr \rangle\,,$$

Coxeter group

$$W[\Gamma] = \langle s, t, r \mid s^2 = t^2 = r^2 = 1\,,\ stst = tsts\,,\ sr = rs\,,\ trt = rtr \rangle\,.$$

Exercise

Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Show that $W[\Gamma]$ has the following presentation:

$$W[\Gamma] = \langle S \mid s^2 = 1 \text{ for } s \in S\,,\ (st)^{m_{s,t}} = 1 \text{ for } s, t \in S\,,\ s \neq t\,,\ m_{s,t} \neq \infty \rangle\,.$$

Example. Let Γ be:

Then:

$$W[\Gamma] = \langle s, t, r \mid s^2 = t^2 = r^2 = 1\,,\ (st)^4 = (sr)^2 = (tr)^3 = 1 \rangle\,.$$

Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Take an abstract set $\Pi = \{\alpha_s \mid s \in S\}$ in one-to-one correspondence with S and set $V = \oplus_{s \in S} \mathbb{R}\, \alpha_s$ the real vector space having Π as a basis. The elements of Π are called *simple roots*. Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be the symmetric bilinear form defined by:

$$\langle \alpha_s, \alpha_t \rangle = \begin{cases} -\cos(\pi / m_{s,t}) & \text{if } m_{s,t} \neq \infty\,, \\ -1 & \text{if } m_{s,t} = \infty\,. \end{cases}$$

For each $s \in S$ we define the linear map $\rho_s : V \to V$ by:

$$\rho_s(v) = v - 2\, \langle v, \alpha_s \rangle\, \alpha_s\,.$$

Lemma

Let $s \in S$. Then $\rho_s : V \to V$ is a linear reflection.

Proof. First, note that:

$$\langle \alpha_s, \alpha_s \rangle = -\cos(\pi/1) = 1\,.$$

Let $v \in V$. Then:

$$\rho_s^2(v) = \rho_s(v - 2\langle v, \alpha_s \rangle \alpha_s) = \rho_s(v) - 2\langle v, \alpha_s \rangle \rho(\alpha_s) =$$
$$v - 2\langle v, \alpha_s \rangle \alpha_s - 2\langle v, \alpha_s \rangle \alpha_s + 4\langle v, \alpha_s \rangle \langle \alpha_s, \alpha_s \rangle \alpha_s = v\,.$$

Thus, $\rho_s^2 = \mathrm{id}$. It remains to show that $\mathrm{Ker}(\rho_s - \mathrm{id})$ is a hyperplane. Let $\varphi : V \to \mathbb{R}$ be the linear form defined by:

$$\varphi(v) = -2\langle v, \alpha_s \rangle\,.$$

We have $\varphi \neq 0$ since $\varphi(\alpha_s) = -2\langle \alpha_s, \alpha_s \rangle = -2 \neq 0$. Moreover, for each $v \in V$,

$$(\rho_s - \mathrm{id})(v) = v - 2\langle v, \alpha_s \rangle \alpha_s - v = \varphi(v)\,\alpha_s\,,$$

hence $\mathrm{Ker}(\rho_s - \mathrm{id}) = \mathrm{Ker}(\varphi)$ is a hyperplane. □

Exercise

Show that the map $S \to \mathrm{GL}(V)$, $s \mapsto \rho_s$, induces a linear representation $\rho : W[\Gamma] \to \mathrm{GL}(V)$.

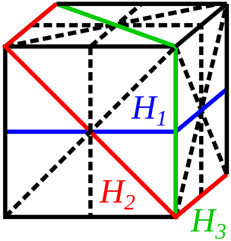
Definition. The linear representation $\rho : W[\Gamma] \to \mathrm{GL}(V)$ of the above exercise is called the *canonical representation* of $W[\Gamma]$.

Theorem (Tits–Bourbaki [1968])

Let Γ be a Coxeter graph. Then the canonical representation $\rho : W[\Gamma] \to \mathrm{GL}(V)$ is faithful.

Example. Let $V = \mathbb{R}^3$ endowed with the standard scalar product, $\langle \cdot, \cdot \rangle$. Let $\{e_1, e_2, e_3\}$ be the canonical basis of V.

Let $C = [-1, 1]^3 \subset V$ be the standard cube:



We denote by W the group of isometries of V that preserve C. Then W is a finite group generated by three orthogonal reflections: the orthogonal reflection r_1 with respect to the plane $H_1 = \{x_3 = 0\}$, the orthogonal reflection r_2 with respect to the plane $H_2 = \{x_2 + x_3 = 0\}$, and the orthogonal reflection r_3 with respect to the plane $H_3 = \{x_1 - x_2 = 0\}$.

Set:

$$\alpha_1 = e_3\,,\ \alpha_2 = \frac{1}{\sqrt{2}}(-e_1 - e_3)\,,\ \alpha_3 = \frac{1}{\sqrt{2}}(e_1 - e_2)\,.$$

Note that α_i is orthogonal to H_i for all $i \in \{1, 2, 3\}$. Also, $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of V and the matrix of $\langle \cdot, \cdot \rangle$ in this basis is:

$$\begin{pmatrix} 1 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 1 & \frac{-1}{2} \\ 0 & \frac{-1}{2} & 1 \end{pmatrix}\,.$$

Note that:

$$\frac{-1}{\sqrt{2}} = -\cos(\pi/4)\,,\ \frac{-1}{2} = -\cos(\pi/3)\,,\ 0 = -\cos(\pi/2)\,.$$

Actually we have:

Exercise

The above group W has the following presentation:

$$W = \langle r_1, r_2, r_3 \mid r_1^2 = r_2^2 = r_3^2 = 1\,,\ (r_1 r_2)^4 = (r_1 r_3)^2 = (r_2 r_3)^3 = 1 \rangle\,.$$

In other words, W is the Coxeter group of the Coxeter graph:

Recall that a *convex cone* in V is a subset $I \subset V$ such that:

- for each $u \in I$ and each $\lambda > 0$, we have $\lambda u \in I$;
- for each $u, v \in I$ and each $t \in [0,1]$, we have $tu + (1-t)v \in I$.

On the other hand, we set $R = \{wsw^{-1} \mid s \in S\,,\ w \in W[\Gamma]\}$. Let $r \in R$. Then the linear transformation $\rho(r)$ is a reflection whose fixed hyperplane is denoted by H_r.

Canonical representation

Theorem (Tits–Bourbaki [1968])

Let Γ be a Coxeter graph. There exists a non-empty open cone $I \subset V$ stable under the action of $W[\Gamma]$ such that:

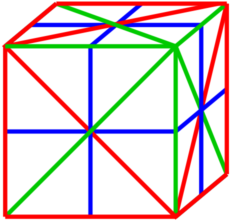
- $W[\Gamma]$ acts properly discontinuously on I;
- $W[\Gamma]$ acts freely and properly discontinuously on $I \setminus (\cup_{r \in R} H_r)$.

Definition. The cone I of the above theorem is called the *Tits cone*.

Example. Suppose again that Γ is :

Then $I = V = \mathbb{R}^3$ and the reflecting hyperplanes are:

$$\{x_1 = 0\}\,,\ \{x_2 = 0\}\,,\ \{x_3 = 0\}\,,\ \{x_1 - x_2 = 0\}\,,\ \{x_1 - x_3 = 0\}\,,$$
$$\{x_2 - x_3 = 0\}\,,\ \{x_1 + x_2 = 0\}\,,\ \{x_1 + x_3 = 0\}\,,\ \{x_2 + x_3 = 0\}\,.$$



Definition. Let Γ be a Coxeter graph. Set:

$$M[\Gamma] = (I \times I) \setminus \left(\bigcup_{r \in R} (H_r \times H_r) \right) .$$

Note that $M[\Gamma]$ is a $2\,|S|$ dimensional manifold and $W[\Gamma]$ acts freely and properly discontinuously on $M[\Gamma]$. Then we set:

$$N[\Gamma] = M[\Gamma]/W[\Gamma].$$

Theorem (Van der Lek [1983])

Let Γ be a Coxeter graph. Then $\pi_1(N[\Gamma]) = A[\Gamma]$.

Definition. Let G be a group and let X be a connected CW-complex (topological space). We say that X is a *classifying space* for G if $\pi_1(X) = G$ and the universal covering of X is contractible.

Theorem

Let G be a (discrete) group.

(1) Let X and Y be two classifying spaces for G. Then X and Y are homotopically equivalent.

(2) Let X be a classifying space for G. Then $H_*(G, \mathbb{Z}) = H_*(X, \mathbb{Z})$.

Conjecture ($K(\pi, 1)$ conjecture)

Let Γ be a Coxeter graph. Then $N(\Gamma)$ is a classifying space for $A[\Gamma]$.

Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. For $X \subset S$ we set $M_X = (m_{s,t})_{s,t \in X}$ and we denote by Γ_X the Coxeter graph of M_X. Note that Γ_X is the full subgraph of Γ spanned by X. We know by Bourbaki [1968] that the inclusion map $X \hookrightarrow S$ induces an injective homomorphism $W[\Gamma_X] \hookrightarrow W[\Gamma]$. So, we can assume that $W[\Gamma_X]$ is the subgroup of $W[\Gamma]$ generated by X. On the other hand, we denote by $\lg : W[\Gamma] \to \mathbb{N}$ the word length with respect to S. We say that $w \in W[\Gamma]$ is *X-minimal* if it is of minimal length in the coset $wW[\Gamma_X]$.

Proposition (Bourbaki [1968])

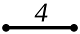
Let Γ be a Coxeter graph, let X be a subset of S, and let $w \in W[\Gamma]$. Then there exists a unique X-minimal element lying in $wW[\Gamma_X]$.

Example. Let Γ be:

Recall:

$$W[\Gamma] = \langle s, t, r \mid s^2 = t^2 = r^2 = 1\,,\ (st)^4 = (sr)^2 = (tr)^3 = 1 \rangle\,.$$

Let $X = \{s, t\}$. Then Γ_X is:



and:

$$\langle X \rangle = W[\Gamma_X] = \langle s, t \mid s^2 = t^2 = 1\,,\ (st)^4 = 1 \rangle\,.$$

We have:

$$|W[\Gamma]| = 48\,,\ |W[\Gamma_X]| = 8\,,$$

hence $W[\Gamma_X]$ has $48/8 = 6$ left cosets. The X-minimal elements of $W[\Gamma]$ are:

$$1\,,\ r\,,\ tr\,,\ str\,,\ tstr\,,\ rtstr\,.$$

Definition. We say that a Coxeter graph Γ is of *spherical type* if $W[\Gamma]$ is finite. If Γ is a Coxeter graph, then we denote by $\mathcal{S}_{\text{sph}}$ the set of subsets $X \subset S$ such that Γ_X is of spherical type.

Lemma

Let Γ be a Coxeter graph. Let $\preceq$ be the relation on $W[\Gamma] \times \mathcal{S}_{\text{sph}}$ defined by:

$(u, X) \preceq (v, Y) \iff [X \subset Y,\ v^{-1}u \in W[\Gamma_Y]$ and $v^{-1}u$ is X-minimal$]$

Then $\preceq$ is an order relation.

Thank you for your attention!