# Geometric Approaches to Artin Groups

Certain classes of Artin groups (spherical-type, Euclidean-type) have Garside structures.

These structures provide combinatorial methods to prove algebraic properties of the group.

Most Artin groups do not have such structures and we can't answer many basic questions:

- Does $A$ have solvable word problem?
- Does $A$ contain torsion elements?
- Does $A$ have non-trivial center?
- Does $A$ have a finite classifying space?
- Does the $K(\pi,1)$-conjecture hold for $A$?

Most of the progress to date on understanding more general Artin groups uses geometric techniques.

There are lots of exciting new ideas for how to do this!

This talk: survey some old and some new applications of geometry to Artin groups.

**Geometric Group Theory:** Study a group $G$ by finding actions of $G$ on some "nice" metric space $X$.

$$G \curvearrowright X, \quad g \in G, \quad g: X \to X \text{ isometry}$$

Need

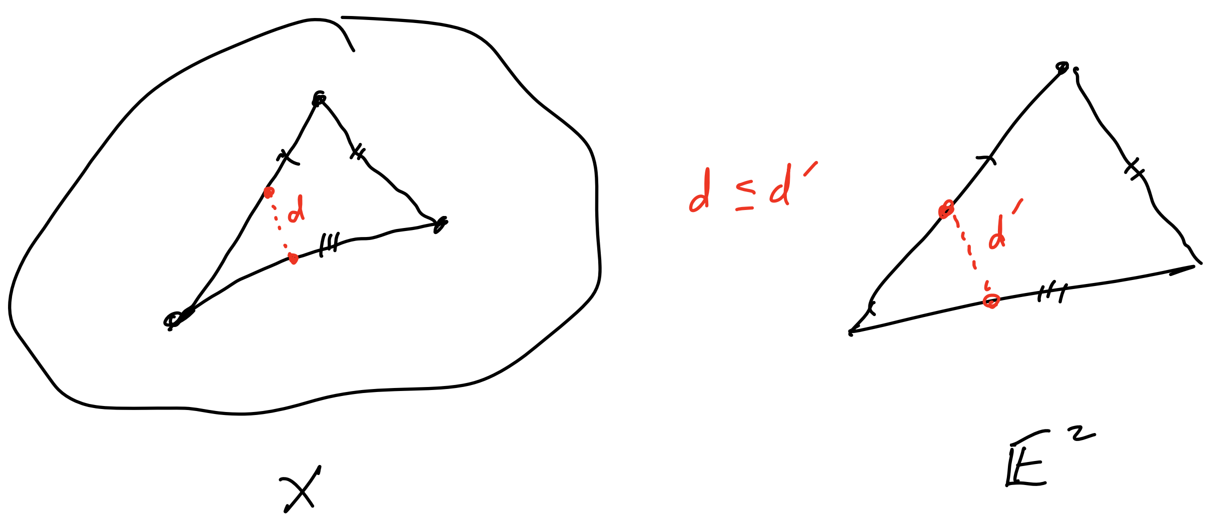
- some conditions on the action

  best possible $\begin{cases} \text{cocompact } (X/G \text{ is compact}) \\ \text{properly discontinuous} \end{cases}$

- $X$ satisfies some kind of "non-positive curvature" condition

A non-positive curvature condition:

A geodesic metric space $X$ is CAT(0) if triangles in $X$ are at least as thin as triangles in the Euclidean plane



Good news: groups acting on CAT(0) spaces have many amazing properties (see Bridson-Haefliger's 600 page book!!)

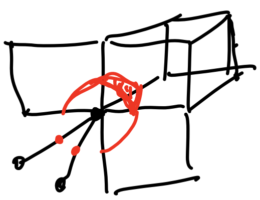
Bad news: it's generally very difficult to determine if a metric on $X$ is CAT(0)

Good news: there are some combinatorial analogues of the CAT(0) condition that are easy to check and have strong implications for groups that act on them.

## Combinatorial versions of non-positive curvature

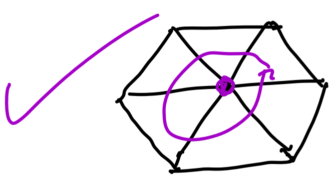
- CCC (CAT(0) cube complexes)

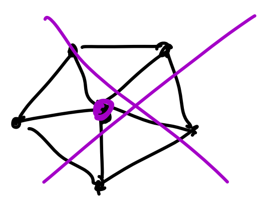
  $X$ = cube complex such that the $\underbrace{\text{link of every vertex}}$ is a $\underbrace{\text{flag complex}}_{\text{no empty simplices}}$



- Systolic complexes (Januszkiewicz - Świątkowski)

  $X$ = simplicial complex such that any cycle of length $< 6$ in the link of a simplex contains two sides of a triangle.





- Helly graphs (or Helly cell-complex)

  $X$ = graph in which every collection of pairwise intersecting balls has a non-empty common intersection.

  (Chalopin - Chepoi - Genevois - Hirai - Osajda, "Helly groups", 2020)

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**Which Artin groups act on such spaces and what can we learn from these actions?**

Notation:

$\Gamma$ = finite graph with vertex set $S = \{s_1, \ldots, s_n\}$ and edges $\overset{}{\underset{s_i \quad s_j}{\bullet\!\!-\!\!\!-\!\!\bullet}}$ labelled by $m_{ij} \in \{2, 3, 4, \ldots\}$

(This is different from the Coxeter graph which omits edges labelled 2 and includes edges labelled $\infty$)

$A_\Gamma$ = associated Artin group

$= \langle S \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}, \ \forall \text{ edges } \overset{m_{ij}}{\underset{s_i \quad s_j}{\bullet\!\!-\!\!\!-\!\!\bullet}} \rangle$

$W_\Gamma$ = associated Coxeter group

$T \subseteq S$, $A_T$ = subgp generated by $T$

= Artin group associated to the subgraph spanned by $T$

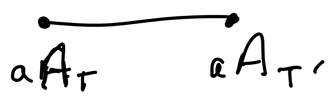
$A_T$ is called a special subgroup.

Conjugates of $A_T$, $a A_T a^{-1}$ are called parabolic subgroups.

# Geometric Constructions for Artin Groups

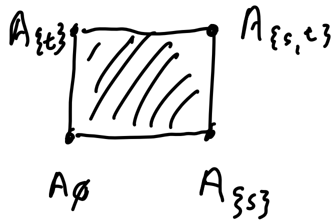
- Deligne complex (Deligne, Ch-Davis)

$\mathcal{D}_\Gamma$ = cubical complex with vertex set $\{aA_T \mid a \in A_\Gamma,\ A_T \text{ spherical}\}$ and edges




$T' = T \cup \{s_i\}$

eg: a 2-cube:




Thm (Ch-Davis, 1995)

① $\mathcal{D}_\Gamma$ is homotopy equivalent to the universal cover of the hyperplane complement for $W_\Gamma$ (so $K(\pi,1)$-conj holds for $A_\Gamma \Leftrightarrow \mathcal{D}_\Gamma \simeq *$)

② This cubical structure is CAT(0)
$\iff$ $A_\Gamma$ is FC-type
$\iff$ if $T \subseteq S$ spans a clique, then $A_T$ is spherical

Fact: Every CAT(0) space is contractible, so

Cor: $A_\Gamma$ FC-type $\Rightarrow$ $K(\pi,1)$-conj holds

Also use cubical structure to get nice solution to the word problem, and answer a variety of other questions.
(Altobelli, Godelle, Paris, .....)

- Clique cube complex (Paris - Godelle, 2012)

  $\mathcal{C}_\Gamma$ = cubical complex with vertex set $\{aA_T \mid a \in A_\Gamma,\ T \text{ spans a clique in } \Gamma\}$

  and edges $aA_T \, \text{———} \, aA_{T'}$ $\quad T' = T \cup \{s_i\}$

**Thm**: $\mathcal{C}_\Gamma$ is CAT(0) for all $\Gamma$.

Unfortunately,

1) We don't know how $\mathcal{C}_\Gamma$ is related to the hyperplane complement for $A_\Gamma$.

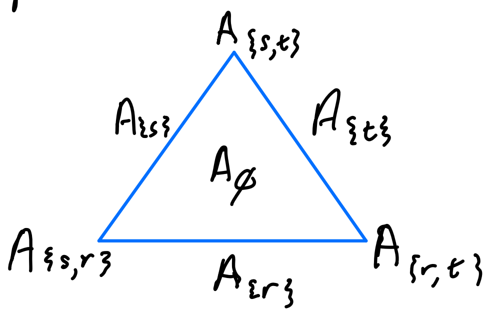
2) If $\Gamma$ itself is a clique (no $m_{ij} = \infty$), then $A_\Gamma \curvearrowright \mathcal{C}_\Gamma$ has a fixed point and the action is not useful.

Paris-Godelle: Using the action $A_\Gamma \curvearrowright \mathcal{C}_\Gamma$, can reduce many questions about Artin groups to the case where $\Gamma$ is a clique.

Ch-Morris-Wright: several other applications of $\mathcal{C}_\Gamma$. (More about this later)

- **Artin complex** (Cumplido – Martin – Vaskou, 2020)

  $X_\Gamma$ = simplicial complex of dim $|S|-1$, whose simplicies are cosets $aA_T$, $T \subsetneq S$,

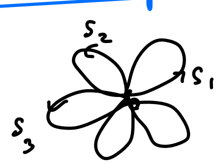



<u>Thm</u> (C-M-V) If $A_\Gamma$ is large type ($m_{ij} \geq 3$ $\forall i,j$) then $X_\Gamma$ is systolic.

They use this to deduce a number of new properties of parabolic subgps, such as

- if $g^n$ lies in $aA_Ta^{-1}$, then so does $g$.
- parabolic subgps are conjugacy stable

- **Salvetti complex** (Salvetti, Ch-Davis)

  $\mathcal{S}_\Gamma$ = 

 $\cup$ coxeter cell for each spherical $A_T$

<u>Thm</u> (Huang – Osajda) If $A_\Gamma$ is spherical or FC-type, then the universal cover $\tilde{\mathcal{S}}_\Gamma$ satisfies a cellular-Helly condition and $A_\Gamma$ is a Helly group

- Thickening of a lattice (Haettel, 2021)

$\left.\begin{array}{l}\text{Cayley graph of Garside gp} \\ \text{Deligne complex of Euclidean} \\ \quad \text{type } \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n\end{array}\right\} \overset{\text{thicken}}{\rightsquigarrow}$ Helly graph

Using the Helly property, this implies a variety of algebraic and topological properties of these groups

⋮

There are also a variety of new complexes that are conjectured or know to be hyperbolic (a form of negative curvature).

- Coned-off Deligne complex (Martin – Przytycki)
- Monoid Deligne complex (Ch – Boyd – Morris-Wright)
- Additional length graph (Calvez – Wiest)
- Parabolic subgroup graph (Cumplido - Gebharat – Gozalez-Meneses – Wiest)

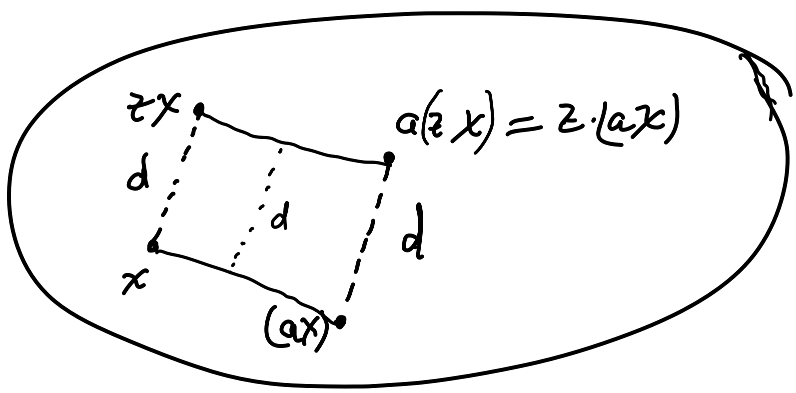
How do we turn geometric properties of $X$ into algebraic properties of $G$?

Let's look at an example.

<u>Thm</u> (Ch-Morris-Wright) If $\Gamma$ is not the star of a single vertex, then center$(A_\Gamma) = \{1\}$.

<u>Outline of proof:</u> Use the action $A_\Gamma \curvearrowright C_\Gamma$. Suppose $z \in$ Center$(A_\Gamma)$. Then we claim $z$ moves every point in $C_\Gamma$ by the same amount!

$C_\Gamma =$




$a \in A_\Gamma$

$\rightsquigarrow$ all points in orbit$(x) = A_\Gamma \cdot x$ are moved distance $d$

$\rightsquigarrow$ all points in convex hull of $A_\Gamma \cdot x$ are moved distance $d$

$\rightsquigarrow$ all points in $C_\Gamma$ are moved distance $d$. by the action on $z$.

This leaves two possibilities:

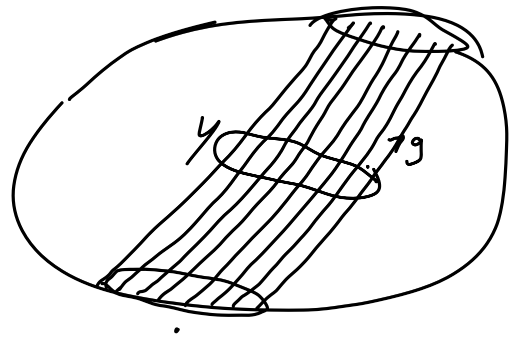
① $z$ fixes all of $\mathcal{C}_\Gamma$, $d(x, zx) = 0$.

$\Rightarrow$ $z$ fixes the vertex $A_\emptyset$, that is

$$zA_\emptyset = A_\emptyset = \{1\} \Rightarrow z = 1$$

② $d(x, zx) \neq 0$. Use a basic fact about CAT(0) spaces: if $g$ is an isometry of a CAT(0) space which does not have a fixed point, then the set of points moved a minimum distance by $g$ is of the form

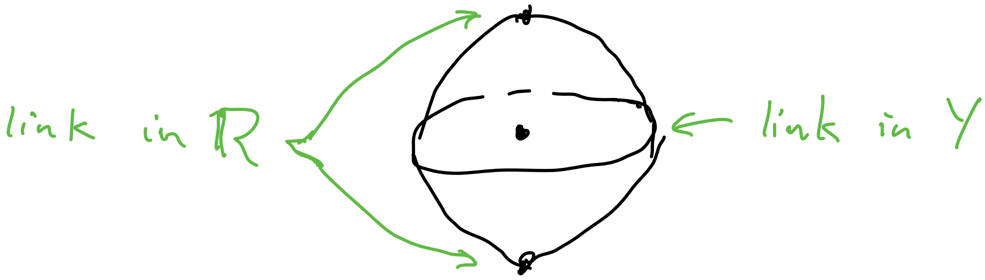
$$\min(g) = Y \times \mathbb{R} \leftarrow \text{axis for } g$$

$X =$ 


In our case $\min(z) = \mathcal{C}_\Gamma$, so

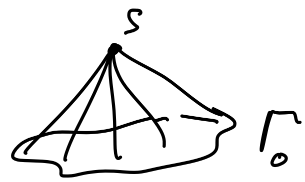
$$\mathcal{C}_\Gamma = Y \times \mathbb{R}$$

This implies that the link of any vertex in $\mathcal{C}_\Gamma$ must be a suspension




Checking the link of the vertex $A_\emptyset$, we discover that this is only possible if

$$\Gamma = \Gamma_0 * \{s\}.$$




$\square$

The End

Eg: <u>Conj</u>: $\forall \Gamma$, $A_\Gamma$ is torsion-free.

Suppose $g \in A_\Gamma$ is torsion. $\mathcal{C}_\Gamma$ CAT(0)
$\Rightarrow$ any finite order isometry $g : \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$
has a fixed point $\Rightarrow$ $g$ fixes a vertex $aA_T$

$$g \in \mathrm{Stab}(aA_T) = aA_T a^{-1} \cong A_T$$

But $aA_T$ vertex in $\mathcal{C}_\Gamma \Rightarrow T$ spans a clique

So if the conjecture holds for all cliques, then it holds for all $\Gamma$.