On Fusion control in FC-type Artin-Tits Groups

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ADVERTISING

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(hybrid mode)

(in memory of Patrick Dehornoy)

https://conf.lmno.cnrs.fr/Braids2020

I Artin-Tits groups

consider a (finite) simple labelled graph Γ where the labelled are in $\{3,4,5,\dots\} \cup \{\infty\}$

Associated Artin-Tits group:

$$A_\Gamma = \langle S \mid st = ts \text{ for } s,t \in S \text{ and no edge between } s \text{ and } t$$

$$\underbrace{sts\cdots}_{m} = \underbrace{tst\cdots}_{m} \quad \overset{s}{\circ}\overset{m}{\text{---}}\overset{t}{\circ} \quad m \neq \infty \rangle$$

Ex ① $\overset{\circ}{s_1}\text{---}\overset{\circ}{s_2}\text{---}\circ \cdots \overset{\circ}{s_{n-1}}\text{---}\overset{\circ}{s_n}$ $\quad A_\Gamma = B_{n+1}$

② no edge $\rightarrow$ free abelian groups

③ full graph with all labels $= \infty \rightarrow$ Free groups

$$A_\Gamma = \left\langle S \;\middle|\; \begin{array}{ll} st = ts & \text{if } \overset{s}{\circ} \quad \overset{t}{\circ} \\ sts\ldots = tst\ldots & \text{if } \underset{s}{\circ}\overset{m}{\text{—}}\overset{t}{\circ} \quad m \neq \infty \end{array} \right\rangle$$

$$A_\Gamma^+ = \langle \qquad \shortparallel \qquad \rangle^+$$

(Artin-Tits monoid)

$$W_\Gamma = \langle S \mid \underset{s^2=1 \quad s\in S}{\shortparallel} \rangle$$

(Coxeter group)

Théorème (PARIS) $A_\Gamma^+ \hookrightarrow A_\Gamma$

→ Few Results are known in the general case

→ Results obtained for sub families

— Spherical type A-T groups (W_Γ finite)

— RAAG

— FC-type A-T groups

— (extra / sufficiently) Large A.T. groups

— 2-dimensional A.T. groups.

II Parabolic subgroups

Definition

Let $A_\Gamma = \langle S \mid - \rangle$ be an A.T gp

Let $T \subseteq S$ ~~and~~ $\langle T \rangle$ is called a standard parabolic subgroup of A_Γ

($\Rightarrow$ Charney: special subgroup)

Proposition (Van der Lek)

Let $T \subseteq S$ and Γ_1 be the full subgraph of A_Γ generated by T.

1) Then
$$\begin{array}{ccc} A_{\Gamma_1} & \hookrightarrow & A_\Gamma \\ t & \longmapsto & t \end{array} \quad (\text{So } \langle T \rangle \simeq A_{\Gamma_1})$$

2) $T_1, T_2 \subseteq S$
$$A_{T_1} \cap A_{T_2} = A_{T_1 \cap T_2}$$

3) $A_T \cap A^+ = A_T^+$

These subgroups are very useful to study A-T groups.

III Fusion control

Definition

Let G be a group and H, K be subgroups so that $H \subseteq K$.
K controls fusion in H (with respect to G) if any two element of H conjugated in G are conjugated in K.
(H "controls fusion" when it controls fusion in itself.)

→ Terminology is due to BRAUER.
→ Fusion mainly studied in the case of finite group G.

Ex: → in an abelian group every subgroup controls fusion

→ in $F_n = F(X)$
$F(Y)$ controls fusion ($Y \subseteq X$)
↳ kind of rigidity

What is known for other A-T groups?

<u>Th 1</u> $A_{\Gamma} = \mathcal{B}_{n+1}$ $\left(\Gamma = \overset{s_1}{\circ}\!-\!\overset{s_2}{\circ} \cdots \circ\!-\!\overset{s_n}{\circ}\right)$

(Gonzales-Menesses) 2014

$\Gamma_1 = \overset{s_1}{\circ}\!-\!\circ \cdots \circ\!-\!\overset{s_k}{\circ}$ $(k < n)$

A_{Γ_1} controls fusion.

<u>Th 2</u> (Calvez, Cisneros de la Cruz, Cumplido) 2020

→ in a spherical type A-T. group standard parabolic subgroups does not always control fusion (Explicit list of Exceptions)

IV FC type A-T. groups

<u>RQ</u>: $\mathbb{Z}^n$ and $\mathbb{F}_n$ are FC type A.T groups

<u>Definition</u>

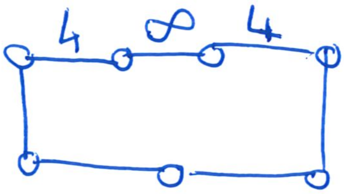
An A.T- group $A_{\Gamma} = \langle S \rangle$ is of FC type if the following condition holds:

$\forall\, \Gamma_1 \subseteq \Gamma$, $\circ\overset{\infty}{-}\circ \notin \Gamma_1 \Rightarrow A_{\Gamma_1}$ is of spherical type

→ can be verified on the graph

Ex 1) $\mathbb{Z}^n$ and $\mathbb{F}_m$

2) spherical type A.T groups are of FC type
(← FC without $\circ\!\overset{\infty}{-}\!\circ$)

3)  A_Γ is of FC type.

Q: Can we caracterise A_Γ so that all (standard) parabolic subgroups control fusion?

<u>Proposition</u> (G) Let $A_\Gamma = \langle S \rangle$ be of FC type

Let $X \subseteq S$ and $g, h \in A_{\Gamma_X}$ and $t \in S$

$t^{-1} g t = h \Rightarrow t \in X$ or $g = h$ and $m_{s,x} = 2$ for all $x \in \mathrm{Supp}(g)$

$\left[\mathrm{Supp}(g) = \bigcap_{g \in A_Y} Y \;;\; g \in A_{Y_1}, g \in A_{Y_2} \Rightarrow g \in A_{Y_1} \cap A_{Y_2} = A_{Y_1 \cap Y_2} \right]$

→ in particular: if two distinct elements of A_X are conjugated by some element t of S, then t has to belong to X.

(Rigidity as in $\mathbb{F}_n$)

→ Answer to a question of Arye Juhász

→ First step in the direction of an answer to previous question

V Amalgamated product:

How to prove such a result?

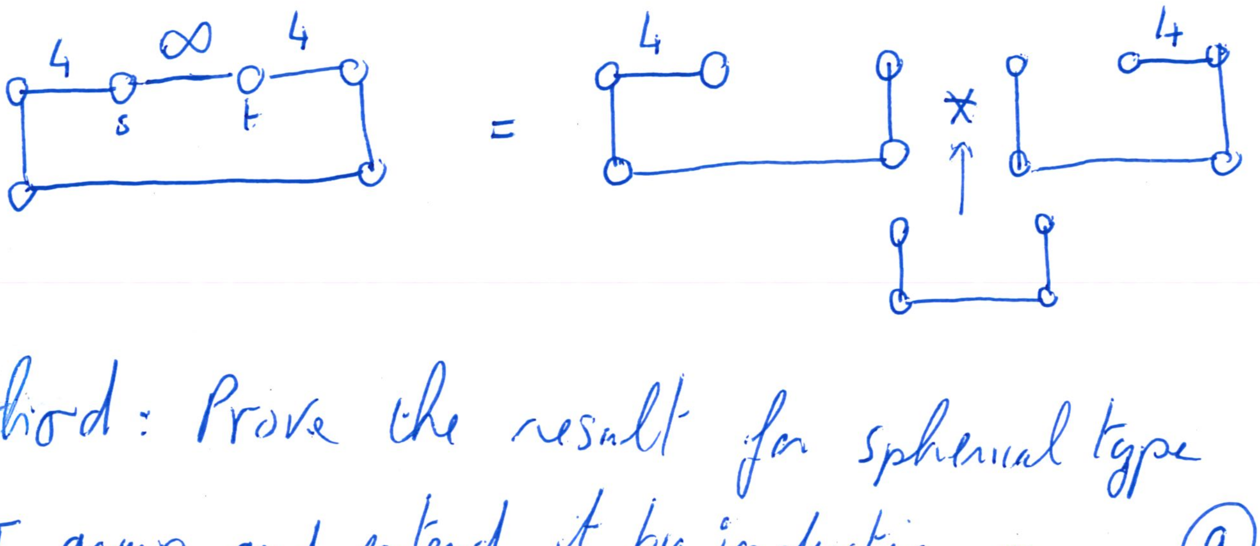
→ use an important tool for FC type A-T groups
Amalgamation and transversals.

Proposition (Charney?)
Let A_S be of FC type. Consider $s \overset{\infty}{—} t$

$S_1 = S \setminus \{s\}$
$S_2 = S \setminus \{t\}$
$S_{1,2} = S \setminus \{s,t\}$

Then $A_S = A_{S_1} \underset{A_{S_{1,2}}}{*} A_{S_2}$

Ex



method: Prove the result for spherical type A-T group and extend it by induction

VI The spherical type case

Fact: spherical type A-T group are Garside groups

<u>Theorem</u> (Charney)

Assume A_S is of spherical type

1) $\forall\, g \in A_S \quad \exists!\, (g_1, g_2) \in A_S^+ \times A_S^+ \;/\; \begin{cases} g = g_1^{-1} g_2 \\ g_1 \wedge_{\leq} g_2 = 1 \end{cases}$

(Charney normal form

2) If $g \in A_S \quad g = g_1^{-1} g_2$ (CNF)

and $g = h_1^{-1} h_2$ with $h_1, h_2 \in A_S^+$

Then $\exists\, h \in A^+ \;/\; \begin{cases} h_1 = h g_1 \\ h_2 = h g_2 \end{cases}$

3) $g \in A_S \quad g = g_1^{-1} g_2$ (CNF). Let $X \subseteq S$

If $g \in A_X$, then $g_1 \in A_X^+$ and $g_2 \in A_X^+$.

Idea of the proof of the proposition for spherical type A.T. groups

$t^{-1} g t = h \qquad t \in X;\ g, h \in A_X$.

($\overset{?}{\rightsquigarrow}$ $t \in X$ or $g = h$ and $m_{s,t} = 2$

$\forall s \in \mathrm{Supp}(g)$

$$t^{-1} g t = h \longrightarrow t^{-1} g_1^{-1} g_2 t = h_1^{-1} h_2$$

$$g = g_1^{-1} g_2 \quad h = h_1^{-1} h_2 \quad \text{CNF}$$

$$\Rightarrow \begin{cases} g_1, g_2, h_1, h_2 \in A_X^+ \\ \exists k \in A_S^+ \ / \ \begin{cases} k h_2 = g_2 t \\ k h_1 = g_1 t \end{cases} \end{cases}$$

$\rightsquigarrow$ use braid relations to conclude if $t \notin X$

VII Go to the FC type

$\rightarrow$ How can we prove the induction step?

(partial) answer: use the transversals!

Definition

Let G be a group and H be a subgroup

a transversal of H in G is a subset T so that

- $1 \in T$
- $\forall g_1, g_2 \in G \quad g_1 H = g_2 H \Rightarrow g_1 = g_2$

Proposition 1

Let $G = G_1 \underset{H}{*} G_2$

T_1, T_2 be transversals of H in G_1 and G_2 respectively

Then

$\forall g \in G \ \exists! (g_1, \dots g_n, h) \in (T_1 \cup T_2)^n \times H$

such that (a) $g = g_1 \cdots g_n h$

(b) $\forall i \ g_i \neq 1$ and g_i, g_{i+1} are not in the the same transversal

Proposition 2 (Altobelli)

If A_S is an FC-type Artin-Tits group with $s \overset{\infty}{\quad} t$

$A_S = A_{S \setminus \{s\}} \underset{A_{S \setminus \{s,t\}}}{*} A_{S \setminus \{t\}}$; then for every $X \subseteq S$ there exists a nice transversal $T(X, S)$ of A_X in A_S

The definition is technical and the transversals are built by induction using Proposition 1

Proposition (G, 2020)

Let A_S be a FC type A-T group

Let $X, Y \subseteq S$

Then $T(X,S)$ meets A_Y crosswise:

$$T(X,S) \cap A_Y \text{ is a transversal of } A_{X \cap Y} \text{ in } A_Y$$

VIII The transversals in the spherical type case

NOTATION

Assume A_S is of spherical type

If $g \in A_S$ we denote by $D(g)^{-1} N(g)$ its Charney Normal form.

Définition A_S of spherical type and $S_0 \subseteq S$.

We say that $g \in A_S$ is S_0-minimal if

(a) $\forall h \in A_{S_0}$ $|D(gh)| \geq |D(g)|$

(b) $\forall h \in A_{S_0}$ $|D(gh)| = |D(g)| \Rightarrow |N(gh)| \geq |N(g)|$

Proposition (Altobelli / Dehornoy - G)

Let $T(S_0, S) = \{ g \in A_S \mid g \text{ is } S_0\text{-minimal} \}$

Then $T(S_0,S)$ is a transversal of A_{S_0} in A_S.

Example 
$$\begin{cases} sts = tst \\ tut = utu \\ su = us \end{cases}$$

$X = \{s,t\}$

$u^{-1}tust$ is not X-minimal

$u^{-1}tust\,s = u^{-1}tu\,tst = u^{-1}u\,tust = tust = tu(st)$

tu is S_0-minimal

IX The induction step

$\rightarrow$ No clean argument for the induction step ☹

$\rightarrow$ Prove a stronger result that allows an induction step ☺

We want : $t \in S$, $g, h \in A_X$

$t \notin X$ and $t^{-1} g t = h \Rightarrow g = h$ and $m_{s,t} = 2$ for $s \in Supp(g)$.

$$t^{-1}gt = h \iff t^{-1}g = h t^{-1}.$$

$$\Rightarrow t^{-1}g A_{\{t\}} = h A_{\{t\}}$$

$\Leftarrow$ If $|g| = |h|$

$\rightsquigarrow$ consider $t^{-1}g A_{S_0}$ and $h A_{S_0}$ (with $S_0 \subseteq S$)

Proposition (6)

Let A_S be of spherical type

Let $X, S_0 \subseteq S$ $t \in S$ $g, h \in T(S_0, S)$

Assume $g \in A_X$ $h \in A_X$

If $t^{-1} g A_{S_0} = h A_{S_0}$ then

$t \in X$ or $t \in S_0$, $g = h$ and $m_{s,t} = 2$ for $s \in Supp(g)$

→ For $S_0 = \{t\}$ we obtain the expected proposition

→ Result extends to FC type by induction

Thanks !