Finding Reflection Factorizations

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Overview

The general plan for this mini-course is to:

- focus on finding minimal length reflection factorizations for spherical and and Euclidean isometries (Day 1).
- and then extend this (where possible) to isometries of arbitrary metric vector spaces (Day 2).

Connections to the other main speakers

This mini-course is closely related to the work of the other mini-course speakers.

- Luis Paris: Garside structures, general Artin group theorems, and key open problems (with Eddy Godelle)
- Jean Michel: Interval monoids and groups (which I learned about via John Crisp)
- François Digne: quasi-Garside structures on $\widetilde{A}_n$ and $\widetilde{C}_n$ (which my coauthors and I tried to imitate)
- Matthew Dyer: Hurwitz transitivity (with Barbara, Christian and Patrick), which simplifies things, and finally
- Ivan Marin and Gunter Malle: complex braid groups (which highlight the possibilities and the limits of this approach)

Coxeter groups and Artin groups

First, let's talk about why we're interested in such factorizations. We've all seen the standard presentations. Here's my notation.

Definition (Standard Presentations)

- Let Γ be a Coxeter/Artin diagram with vertex set S.
- Let $W = \text{Cox}(\Gamma)$ be the Coxeter group (generated by S).
- Let $A = \text{Art}(\Gamma)$ be the Artin group (also generated by S).
- Let $p \colon \text{Art}(\Gamma) \twoheadrightarrow \text{Cox}(\Gamma)$ be the natural projection map, sending S to S, whose kernel is the pure Artin group.

Coxeter Elements

Definition (Coxeter elements)

Fix a linear ordering of S. The product of the elements in S, in this order, is the Coxeter element w. And since we have a copy of $S \in \textsc{Art}(\Gamma)$, there is a natural copy of $w \in \textsc{Art}(\Gamma)$ as well.

Remark (Conjugacy)

Different orders create different elements. If Γ is a tree (when not drawing commutation relations) then all Coxeter elements are conjugate. In general, the different Coxeter elements have distinct properties and personalities.

Reflections

Definition (Reflections)

The elements of S are simple reflections and their conjugates in $W = \text{Cox}(\Gamma)$ are called reflections. Let $T = T^W$ be the set of all reflections in $\text{Cox}(\Gamma)$.

Geometrically this makes sense. You can also mimic this definition inside $A = \text{Art}(\Gamma)$. These are also sometimes called reflections, but the name no longer makes geometric sense.

Remark (Coxeter vs. Artin)

Be careful: the conjugates of S in $\text{Art}(\Gamma)$ form a much larger set T^A. We have $p(T^A) = T^W$ but $T^A \neq p^{-1}(T^W)$ in general.

Dual Artin Groups

Definition (Dual Artin groups)

The dual Artin group $\textsc{Art}^*(\Gamma, w)$ is the group defined by the interval $[1, w]$ constructed inside the Cayley graph of the group W with respect to its expanded generating set T. The generating set of $\textsc{Art}^*(\Gamma, w)$ is a subset T_0 of T, the set of reflections that actually occur in some minimum length factorization of w over T.

Remark

In all known worked examples, the group $\textsc{Art}^*(\Gamma, w)$ is isomorphic to the Artin group $\textsc{Art}(\Gamma)$.

Question: Why **Coxeter** factorizations?

Why are we looking at factorizations in $W = \textsc{Cox}(\Gamma)$? All Artin groups have internal dual presentations.

Proposition (Internal dual presentations)

Let $[1, w]_A$ be the interval defined in the Cayley graph of the Artin group $A = \textsc{Art}(\Gamma)$ with respect to the set T^A of reflection generators. For every Γ and every Coxeter element w, the interval group this defines is naturally isomorphic to the Artin group $\textsc{Art}(\Gamma)$.

Remark

- If we consider factorizations in $\textsc{Art}(\Gamma)$, we get the right group, but we can't compute the interval.
- If we consider factorizations in $\textsc{Cox}(\Gamma)$, we can compute the interval*, but we may not get the right group.

Answer: This is where the light is

Reflection Factorizations

How do we find all the minimal length reflection factorizations of of a Coxeter element in a Coxeter group?

Proposition (Getting started)

The reflection length of a Coxeter element w is the size of the generating set S: $\ell_T(w) = |S|$. In particular, the product defining w is a minimum length reflection factorization of w.

Proposition (BDSW)

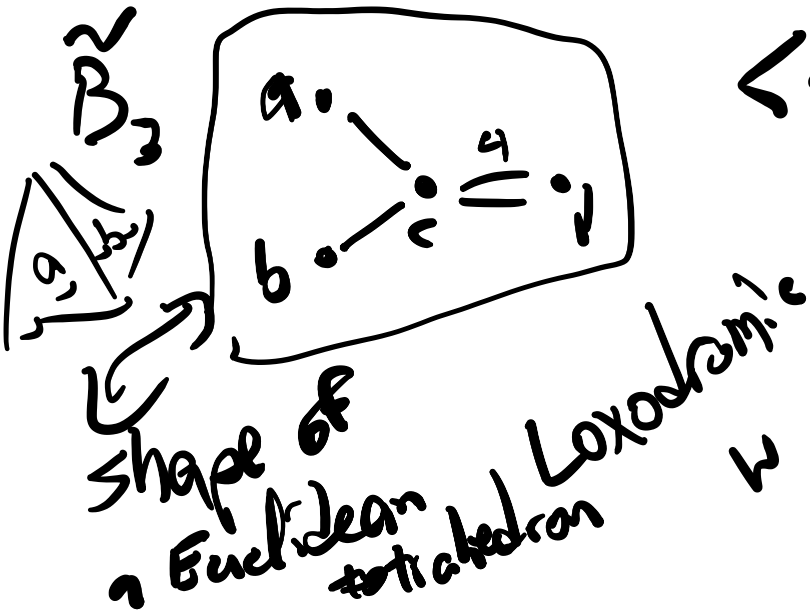
For every Coxeter group, the Hurwitz action is transitive on minimal length factorizations of a Coxeter element.

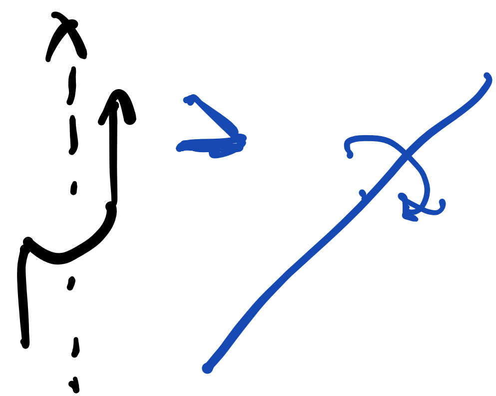
When $\text{Cox}(\Gamma)$ is finite/spherical, we can compute, but in general, this will run forever and we need a different idea.

Motivating question: Loxodromic isometries

The Coxeter element in $\mathrm{Cox}(\widetilde{B_3})$ is a loxodromic (screw-type) motion in $\mathbb{R}^3$. Which rotations are "below" it? The answer changes depending on which reflections you are allowed to use. Let's talk through some basic cases.

$$\left\langle a, b, c, d \;\middle|\; \begin{array}{l} ab = ba \;\ldots \\ aca = cac \\ cdcd = dcdc \end{array} \right\rangle$$





Using **all** reflections

Remark

When Noel Brady and I tried to understand this situation, we keep running up against the question “can we use this reflection?”. We decided to use them all and postpone that question to the end.

General Strategy

Step 1: geometrically understand how to factor using all reflections.

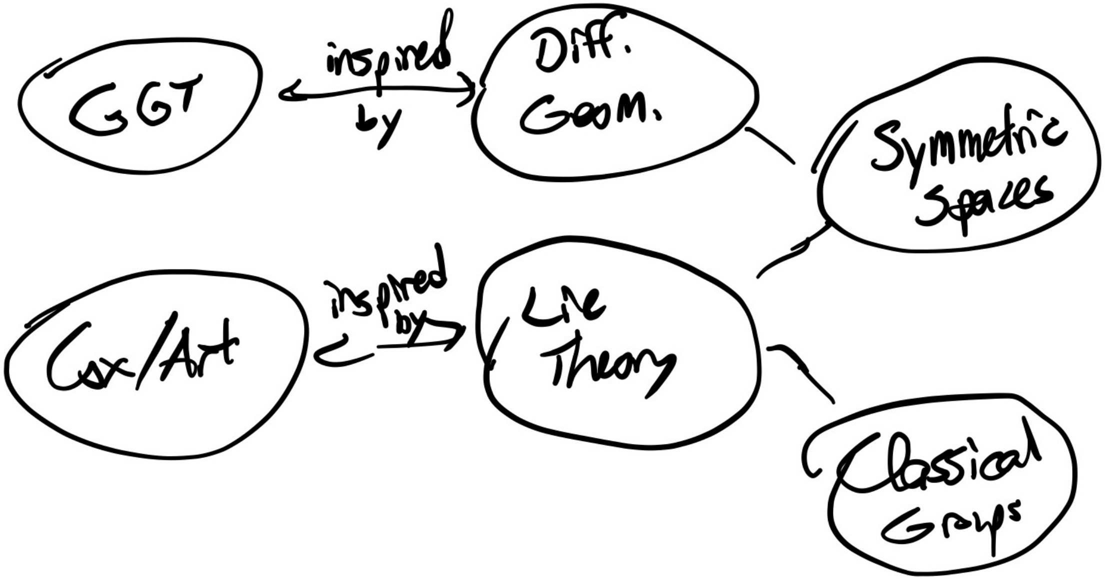
Step 2: understand the restriction to Coxeter reflections.

Step 3: check to see if the interval defines the Artin group,

Step 4: check to see if the interval is a lattice,

Step 5: deal with any problems that arise.

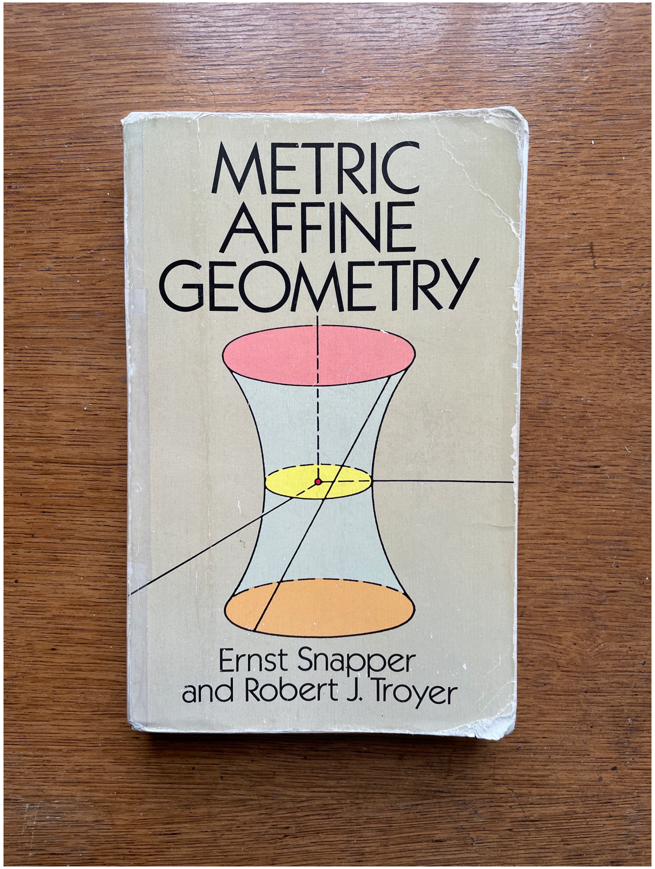
Classical Geometry

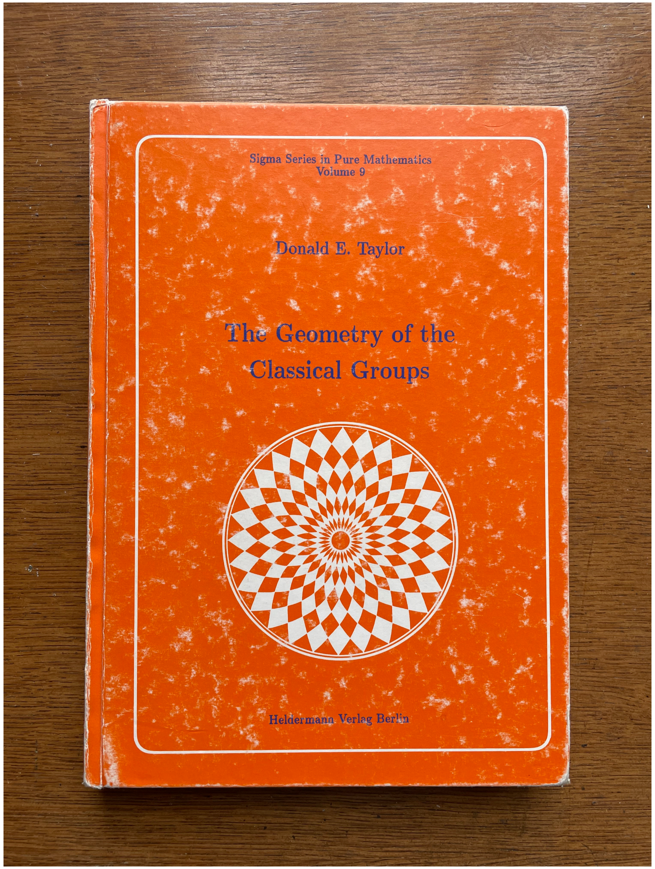


Remark

Everyone should learn classical geometry. It's fun! (and useful)

Two very good books





First Exercises

Definition (Atoms and Basic isometries)

The atoms are the reflections, with reflection length 1. The basic isometries are those with reflection length 2. These are the simplest orientation-preserving isometries.

Exercise (A Basic Question)

When is the product of two basic isometries basic (i.e. when is the product of two reflection length 2 elements also reflection length 2)? Do this in $\mathbf{S}^2$, $\mathbf{S}^n$, $\mathbf{E}^2$, $\mathbf{E}^n$, $\mathbf{H}^2$ and $\mathbf{H}^n$.

Exercise (Bonus Question just for fun)

If you move only one vertex of a triangle and keep the area constant, what curve do you trace out? You probably know the answer in $\mathbf{E}^2$, but what about $\mathbf{S}^2$ and $\mathbf{H}^2$?

Factoring spherical isometries

Definition (Orthogonal transformations)

Let $V = \mathbb{R}^n$ be an n-dimensional real vector space with its standard inner product and consider $\text{Isom}(V) = O(n, \mathbb{R})$. These are the $n \times n$ matrices A with $A^T A = I$.

Definition (Mixing operators)

Let $T\colon V \to V$ be an orthogonal transformation. We say T is a mixing operator if the only point fixed by T is the origin.

Definition (Fixed-set and Move-set)

If T is an operator, $\text{Fix}(T) = \ker(I - T)$ and $\text{Mov}(T) = \text{im}(I - T)$.

Fix and Move

Proposition (Orthogonal decomposition)

The subspaces $\textsc{Fix}(T) \oplus \textsc{Mov}(T)$ *is an orthogonal decomposition of* V*.*

If T is mixing, $\textsc{Mov}(T) = \mathbb{R}^n$ and $\textsc{Fix}(T) = \mathbb{R}^0$

Proposition (Up and Down)

The dimension of $\textsc{Fix}(T)$ *and* $\textsc{Mov}(T)$ *change by* 1 *when* T *is multiplied by a reflection - and it's easy to tell whether it goes up or down.*

So $\dim(\textsc{Mov}(T))$ is a lower bound on reflection length.

Flags and Factorizations

Definition (Flags)

A (maximal) flag is a nested chain of subspaces, one per dimension. Every ordered Basis $(v_1, v_2, \ldots, v_n)$ determines a maximum flag by letting $\mathbb{R}^i = \textsc{Span}(\{v_1, \ldots, v_i\})$.

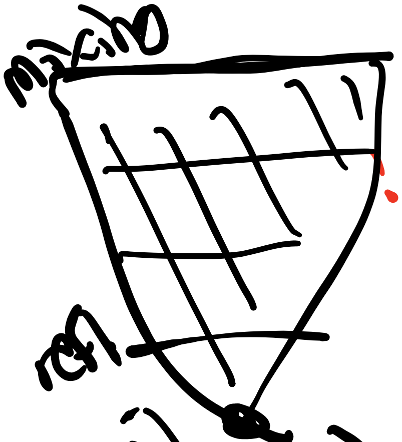
$$\{0\} = \mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^n = V$$

Proposition

For every mixing operator T and for every maximal flag, there is a reflection factorization of T so that the fixed spaces of the prefixes are the subspaces in the flag.

Proof idea: simply use the only reflection that will work.

Building a Flag



Cay(O(n), refl.) ordered by refl. length

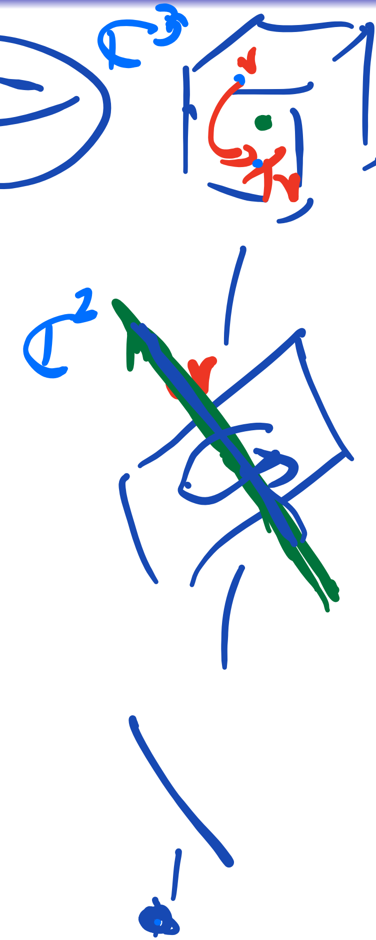
T

Mor(T) =

refl. that moves T(r) back to r.

Mor( ) =

Id



Spherical Intervals

This turns motions into spaces.

Spherical reflection factorizations are amazing!

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Proposition (Wall, Brady-Watt)

Intervals in $O(n, \mathbb{R})$ are equal to $\textsc{Lin}(V)$, the poset of linear subspaces under inclusion.

Actually, this is true more generally.

Proposition

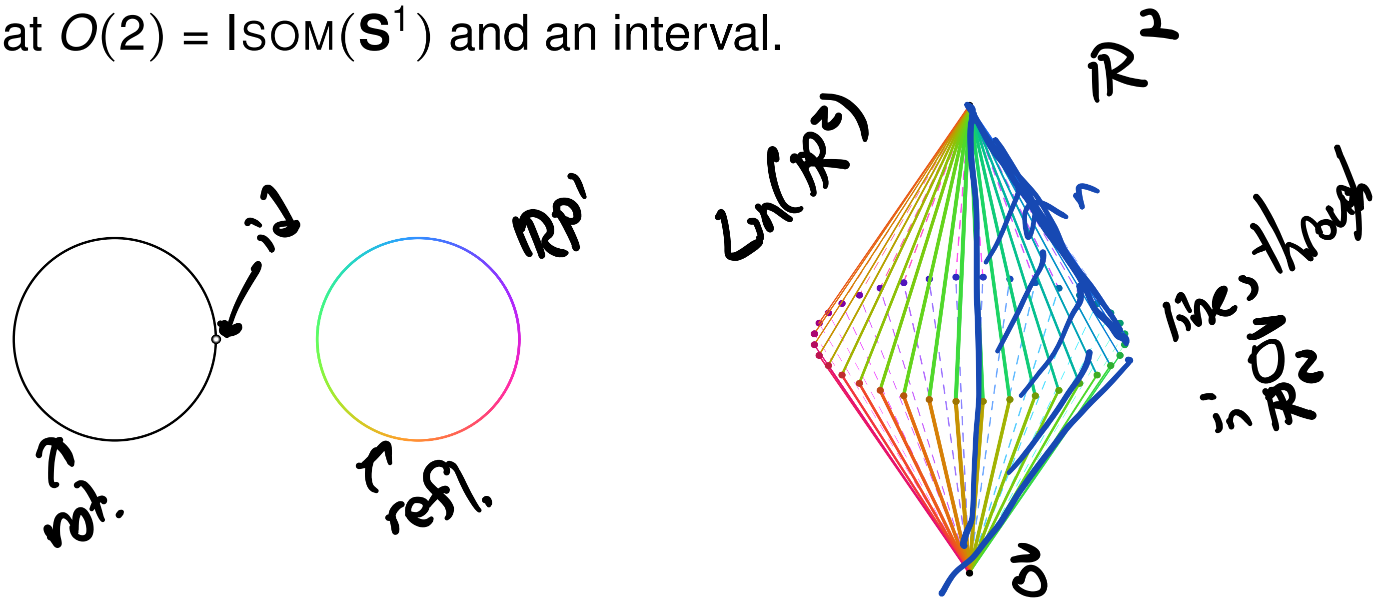
Let T be a unitary mixing operator over $\mathbb{C}^n$. There is a minimal length complex reflection factorization for T corrsponding to each maximal flag, and intervals are the poset of linear subspaces.

Orthogonal Groups

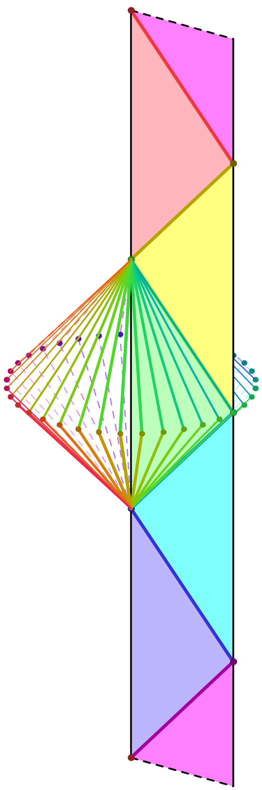
Remark

$\text{Isom}(V) = O(n, \mathbb{R})$ has an uncountably generated dual Coxeter presentation. And there is corresponding continuous "Artin" group (M - unpublished).

Look at $O(2) = \text{Isom}(\mathbf{S}^1)$ and an interval.

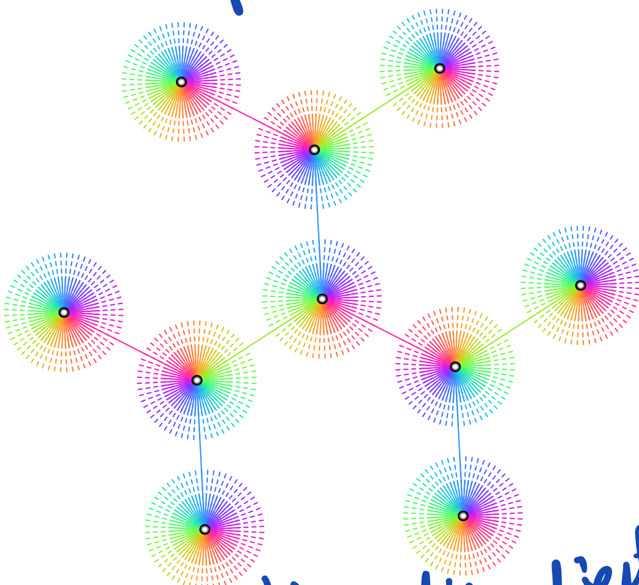


Pulled apart orthogonal group



Top View

$T_{S^1} \times \mathbb{R}$



$\tilde{A}_n$ building-like object $\times \mathbb{R}$

contractible

Restricting and the Lattice Property

The next step is restricting to the allowed factorizations.

Remark (Lattice property)

For a spherical Coxeter group (or a finite complex reflection group) you then need to restrict to those factorizations only using the appropriate reflections. Even in these cases, proving the lattice property for the restriction is still hard (Reading, Brady-Watt).

Remark (Armstrong)

There are mixing operators in $\textsc{Cox}(D_4)$ where the lattice property fails.

From spherical to Euclidean

Definition (Affine space)

Let $V = \mathbb{R}^n$ be real vector space with a well-defined origin. The corresponding affine space is a set E with a simply transitive V-action. You can think of E as a copy of V where we forget the location of the origin. Elements of V are vectors. Elements of E are points.

Remark (Coordinates)

When you need to work with E you pick an origin and fix coordinates. But, as Petra mentioned in her talk, picking an origin (and a corresponding semidirect produt structure) can sometimes get in the way.

Move-sets and Min-sets

Definition (Move-sets and Min-sets)

For every $w \in \textsc{Isom}(E)$ there are two affine subspaces: $\textsc{Mov}(w) \subset V$ and $\textsc{Min}(w) \subset E$. The first records the motions of elements under T, and the second is the elements moved a minimal distance.

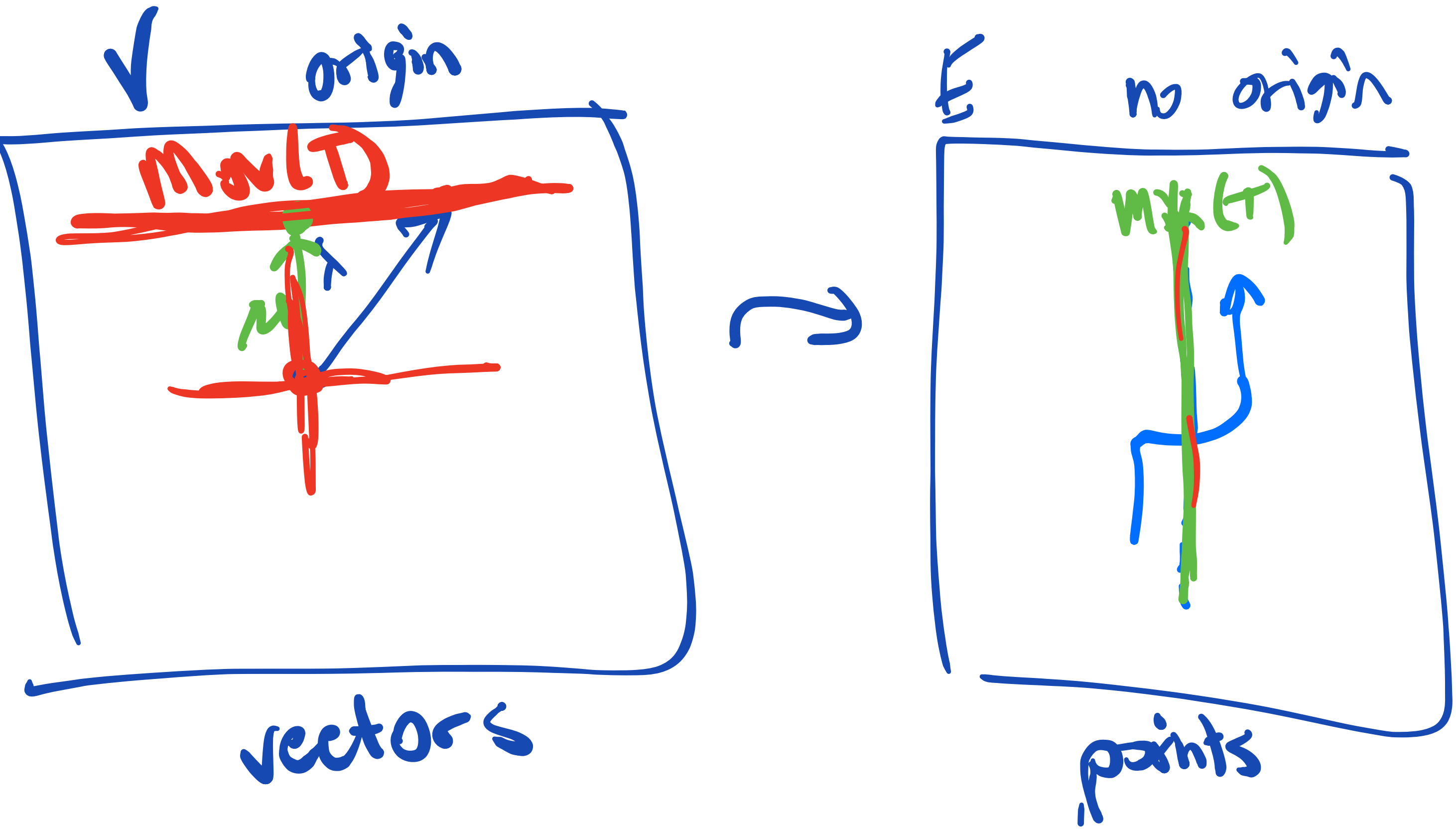
Theorem (Scherk '50)

If T is a Euclidean isometry, its reflection length is $\dim(\textsc{Mov}(T))$ is $\textsc{Mov}(T)$ is a linear subspace of V and $\dim(\textsc{Mov}(T)) + 2$ if it is an affine subspace of V.

Remark

Once you restrict to Coxeter reflection length, this changes drastically, but not inside the Coxeter element intervals.

Example: Glide Reflection

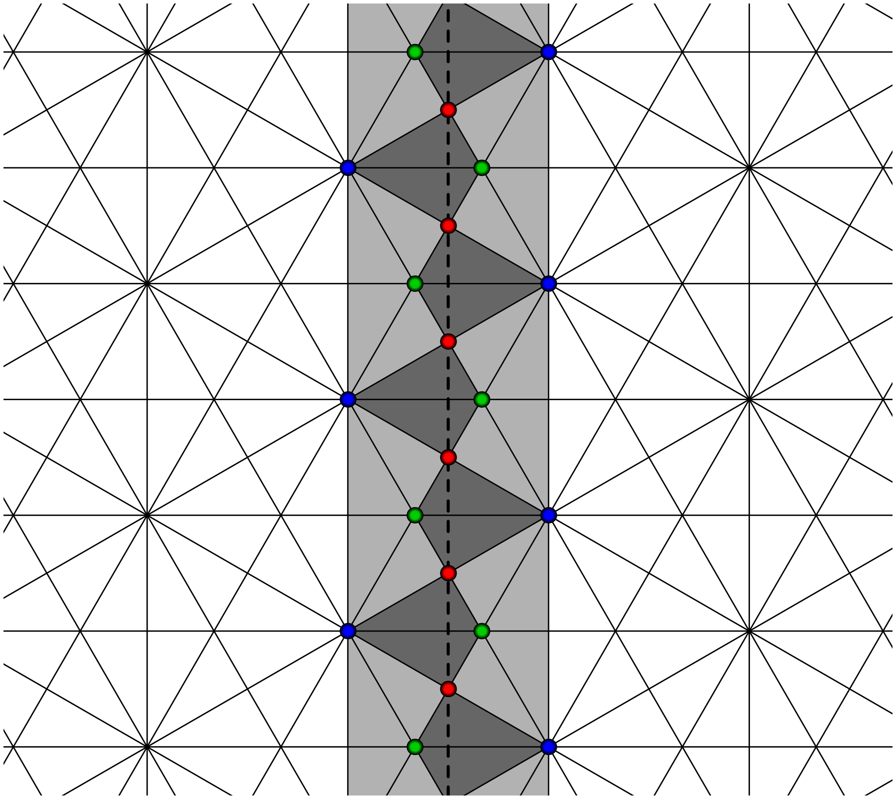


Multiplying by a reflection

We have that $\textsc{Dir}(\textsc{Mov}(T))$ and $\textsc{Dir}(\textsc{Min}(T))$ orthogonally decompose V, and it is easy to see how they change under product with a reflection.

Bipartite Coxeter Elements

The Euclidean Coxeter Group $\mathrm{Cox}(\widetilde{G}_2)$



Coarse Structure

Bowties

Thank You