The dual monoid

Jean Michel

University Paris Diderot

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Invariants of complex reflection groups

Let $V = \mathbb{C}^n$ and $W \subset \mathrm{GL}(V)$ a finite group. Let SV be the symmetric algebra of V (a polynomial algebra). It is the algebra of V in terms of algebraic geometry: $V = \mathrm{Spec}\, SV$.
SV^W is the algebra of the quotient space V/W.

Theorem

SV/W is a smooth variety if and only W is a complex reflection groups. Then $SV/W \simeq V$, thus SV^W is a polynomial algebra.

In particular, if W is a complex reflection group, $(SV)^W$ is freely generated by n polynomials. These polynomials are not unique, but their degrees are unique. We write these degrees $d_1 \leq \ldots \leq d_n$.

Proposition

$|W| = d_1 \cdots d_n$ *and* $|\mathrm{Ref}(W)| = (d_1 - 1) + \cdots + (d_n - 1)$.

Further, if W is irreducible, the order of any eigenvalue of any $w \in W$ on V divides one of the d_i, and W is real iff $d_1 = 2$.

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Coxeter elements

Definition

$w \in W$ is ζ-*regular* if it has a eigenvector $v \in V$ (for the eigenvalue ζ) outside of all reflection hyperplanes.

If W is irreducible, ζ-regular elements form a single conjugacy class of W, of elements which have same order as ζ..

Proposition

If W is irreducible and well-generated, then $d_n > d_{n-1}$. $h = d_n$ is called the Coxeter number *and there exist $e^{2i\pi/h}$-regular elements; they are called* Coxeter elements.
We have $hn = |Ref(W)| + |Hyperplanes(W)|$.

The theory of regular elements further describes the eigenvalues on V of Coxeter elements. They are $\{e^{2i\pi(1-d_j)}/h\}_j$. In particular no eigenvalue is 1.

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mov

For $x \in \mathrm{GL}(V)$ we set $\mathrm{mov}(x) = \mathrm{image}(x-1)$. We have $\dim(\mathrm{mov}(x)) = \mathrm{codim}(\mathrm{fix}(x))$. In particular $\dim(\mathrm{mov}(x)) = 1$ for a reflection.

Proposition (mov subadditive)

$\dim(\mathrm{mov}(xy)) \leq \dim(\mathrm{mov}(x)) + \dim(\mathrm{mov}(y))$ *and there is equality if and only if* $\mathrm{mov}(xy) = \mathrm{mov}(x) \oplus \mathrm{mov}(y)$*;*

Proof.

We have $xy - 1 = (x-1)y + y - 1$ which implies that $\mathrm{mov}(xy) \subseteq \mathrm{mov}(x) + \mathrm{mov}(y)$, which shows the proposition. □

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If $x \leq_R c$, we have $c = xy$ where $l_R(x) + l_R(y) = n$. But from proposition $(\mathrm{mov}$ subadditive$)$ $n = \dim(\mathrm{mov}(c)) \leq \dim(\mathrm{mov}(x)) + \dim(\mathrm{mov}(y)) \leq l_R(x) + l_R(y) = n$ so there is equality everywhere. □

Note that a left divisor of such an element c for $\leq_R$ is also a right divisor since $xy = y(y^{-1}xy)$ and the lengths add also in the RHS since $\dim(\mathrm{mov}(y^{-1}xy)) = \dim(\mathrm{mov}(x))$ since $\dim(\mathrm{mov})$ is invariant by conjugacy.
So an interval $\leq_R c$ is automatically balanced.

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Unitary and orthogonal group

If W is a finite subgroup of $\mathrm{GL}(V)$, we can find a Hermitian scalar product $(,)$ invariant by W: take any scalar product and average it over W. This makes W a subgroup of the unitary group.

Such a product can be used to give a formula for a reflection with reflecting hyperplane H and non-trivial eigenvalue ζ. Let r be a vector orthogonal to H. Then (r, ζ) define a reflection s by

$$s(x) = x - (1-\zeta)\frac{(x,r)}{(r,r)}r.$$

A complex refection group defined over the reals (that is, $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and W is comes from an action on $V_{\mathbb{R}}$) is the same as a finite Coxeter group. In this case we can find on $V_{\mathbb{R}}$ an invariant scalar product and this makes W a subgroup of the orthogonal group.

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We have an action of the braid group B_n on the decompositions of an element: $c = s_1 \dots s_n$. We make σ_i act by $\sigma_i((s_1, \dots, s_n)) = (s_1, \dots, s_{i-1}, s_i s_{i+1} s_i^{-1}, s_i, s_{i+2}, \dots, s_n)$. We still get a decomposition as reflections. It turns out that the Hurwitz action is transitive on all such decompositions. This is a fundamental fact which allows to compute all simples of the dual monoid.

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The braid group

The following is easy for finite Coxeter groups but non-trivial for non-real complex reflection groups.

(Steinberg) W acts regularly on V^{reg} where $V^{reg} = V - \bigcup_{s \in R} H_s$,

given this, we can define $B_W = \Pi_1(V^{\text{reg}}/W)$ and

$$1 \to \Pi_1(V^{\text{reg}}) \to B_W := \Pi_1(V^{\text{reg}}/W) \to W \to 1.$$

(Bessis 2001) B_W needs the same number of braid reflections *to generate as W needs reflections.*

This comes from the equation for $\bigcup_{s \in R} H_s$ in V/W, the *discriminant.*

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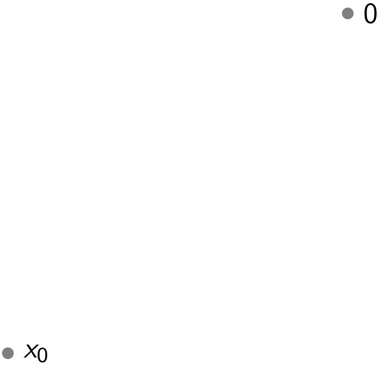
A braid reflection above a reflection s with non-trivial eigenvalue $\zeta := e^{2i\pi/e}$



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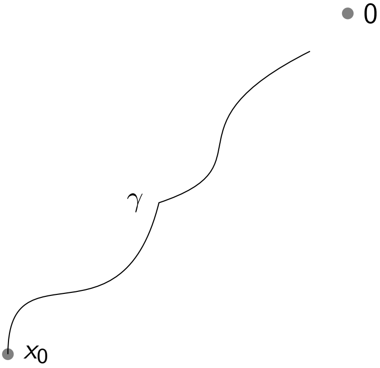

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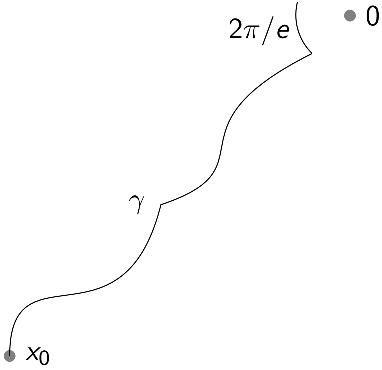

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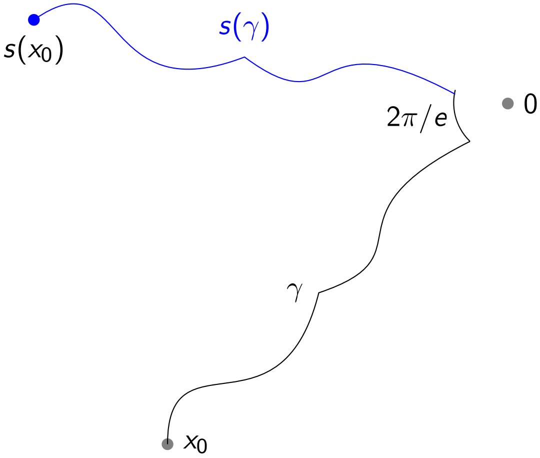

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We have $V^{\text{reg}}/W \simeq \mathbb{C}^n - \{\Delta = 0\}$ where the *discriminant* Δ is an invariant equation of $\bigcup_{r\in R} H_r$ in V/W. To express the variety $\bigcup_{r\in R} H_r$ as the zeroes of an invariant polynomial, for each reflecting hyperplane H choose a linear fom l_H of kernel H. Then $\Delta = \prod_H l_H^{|C_W(H)|}$ is an invariant equation.

Theorem (Bessis)

If W is an irreducible well generated complex reflection group, we have $\Delta = f_n^n + P_2(f_1, \ldots, f_{n-1}) f_n^{n-2} + \ldots + P_n(f_1, \ldots, f_{n-1})$ where P_i is weighted homogeneous of degree ih.

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If W is an irreducible well generated complex reflection group, we have $\Delta = f_n^n + P_2(f_1, \ldots, f_{n-1}) f_n^{n-2} + \ldots + P_n(f_1, \ldots, f_{n-1})$ where P_i is weighted homogeneous of degree ih.

It follows that if we fix the values of $f_1, \ldots, f_{n-1}$ and let f_n vary in the complex plane, the discriminant has generically n zeroes in that plane

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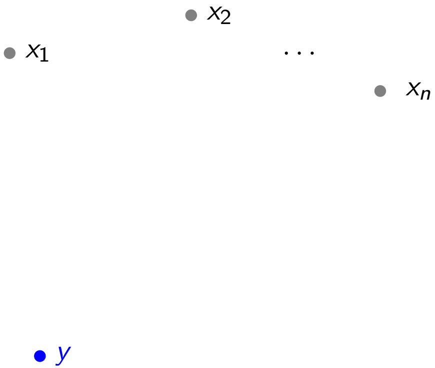

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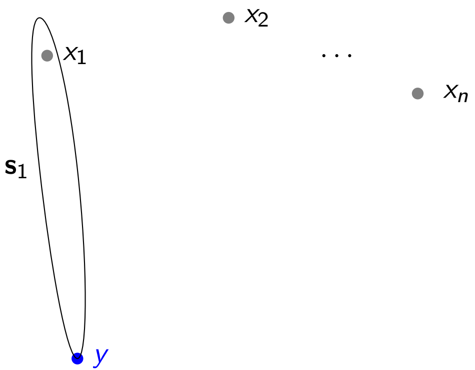

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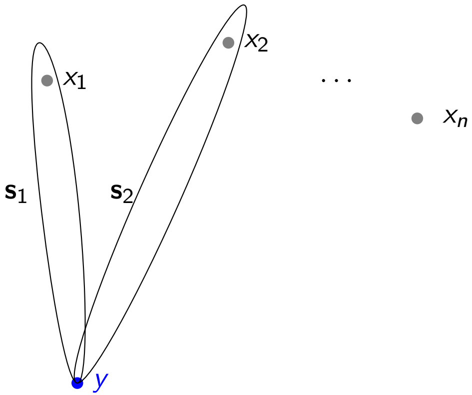

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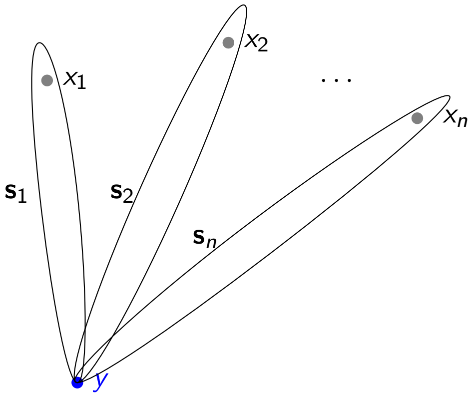
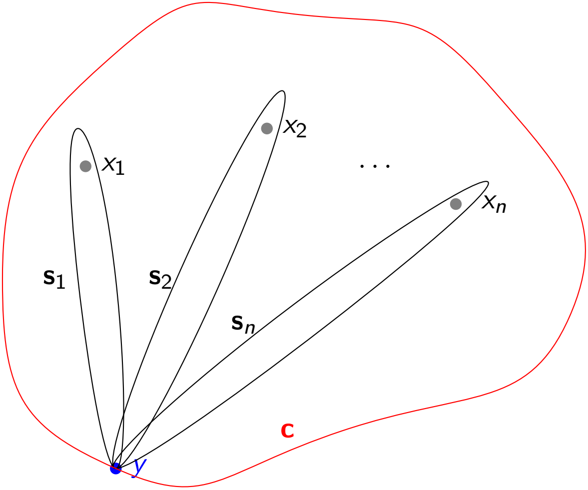


Decomposition of lift of c

Since f_n has degree h a loop in that plane maps to an element of W with eigenvalue $e^{2i\pi/h}$. We get a decomposition $\mathbf{c} = \mathbf{s}_1 \dots \mathbf{s}_n$ as braid reflections.



The Birman-Ko-Lee monoid

For the symmetric group $\mathfrak{S}_n$ the reflections are the transpositions (ij). For $c = (12 \ldots n)$ the simples dividing c correspond to non-crossing partitions: the cycles correspond to non-crossing polygons on the circle:

The picture corresponds to (134)(5689). The least common multiple of two non-crossing transposition is their product. If two transpositions cross:

their least common multiple is the cycle (1358).

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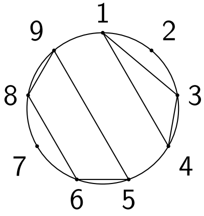

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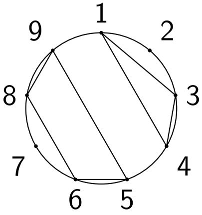

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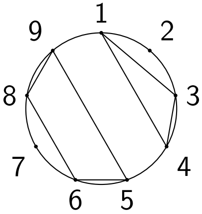

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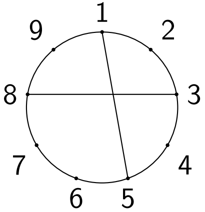

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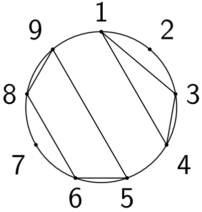

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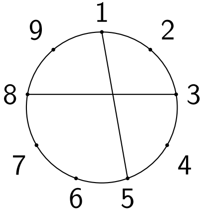

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