Reflection Length in Coxeter groups

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(W,S) Coxeter system

$R = \bigcup_{w \in W} w S w^{-1}$ the set of reflections

How long is a given $w \in W$?

Can we compute or estimate its reflection length?

Some properties:

- l_R is constant on conjugacy classes
- $l_R(wr) = l_R(w) \pm 1 \quad \forall w, \forall r \in R$
- Δ- inequality:
  $$l_R(w_1 w_2) \leq l_R(w_1) + l_R(w_2)$$
- [McCP '11] one can reduce to irreducible Coxeter groups
  $W = W_1 \times W_2 \ni w = w_1 \cdot w_2$ then $l_R(w) = l_R(w_1) + l_R(w_2)$

Thm (Dyer '01)

(W,S) arbitrary, $w \in W$, then

$\ell_R(w) =$ min # letters deleted from any reduced expression for w s.th. the resulting element equals $\mathbb{1}$.

root systems & move sets

$W \curvearrowright \mathbb{R}$-VS V

define the <u>move-set</u> of $w \in W$ as

$$\operatorname{Mov}(w) := \operatorname{im}(w - \mathbb{1})$$
$$= \{\mu \in V \mid \exists \lambda \in V \text{ with } w(\lambda) = \lambda + \mu\}$$

root systems & move sets

W sph.es / affine

$W \circlearrowright \mathbb{R}$-VS V

Define the **move-set** of $w \in W$ as

$$\mathrm{Mov}(w) := \operatorname{im}(w - \mathbb{1})$$
$$= \{\mu \in V \mid \exists \lambda \in V \text{ with } w(\lambda) = \lambda + \mu\}$$

vector in V

Let r be a reflection, then $\mathrm{Mov}(r) = \mathbb{R} \cdot \alpha_r$

a **root** of r

Roots can be organized in root systems $\Phi = \Phi(W,S) \subset V$

s.th. Φ is finite, contains pairs $\pm\alpha$, and $W \circlearrowright \Phi$

ℓ_R in the spherical case

Thm (Carter 1972)

$$\ell_R(w) = \dim(\mathrm{Mov}(w))$$

A given reflection presentation $w = r_1 \cdots r_k$ is minimal (and $\ell_R(w) = k$) $\Longleftrightarrow$ the roots α_{r_i} are l.i.

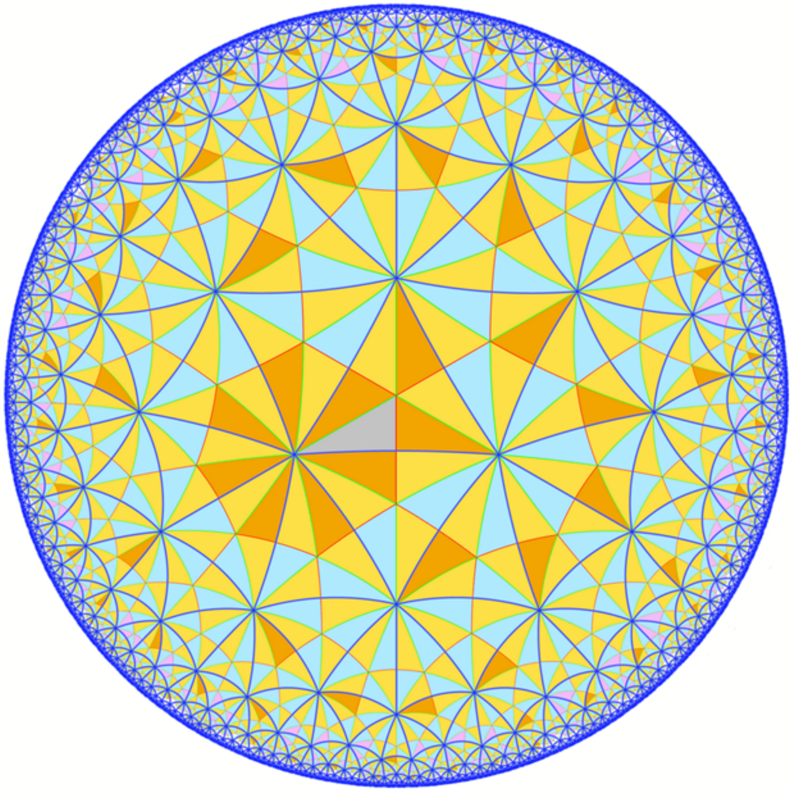
$\rightsquigarrow$ $\ell_R(w) \leq |S|$

ℓ_R in the hyperbolic case

Thm (Duszenko '12)

ℓ_R is unbounded

i.e. $\forall n \in \mathbb{N} \; \exists w \in W$ s.th. $\ell_R(w) \geq n$.

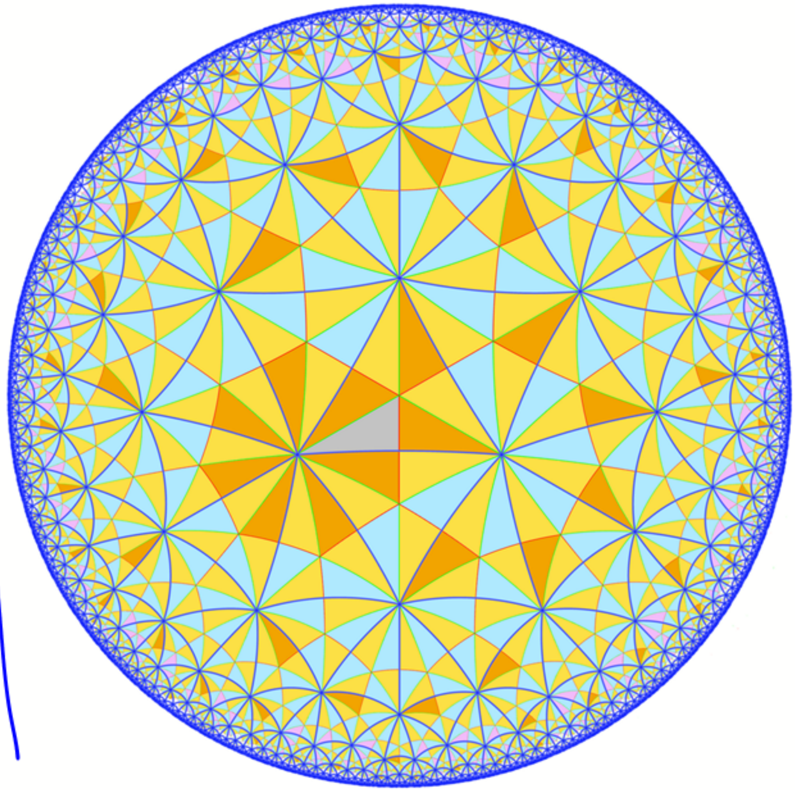


l_R in the hyperbolic case

Thm (Duszenko '12)

l_R is unbounded

i.e. $\forall n \in \mathbb{N}\ \exists w \in W$ s.th. $l_R(w) \geq n$.



? How much structure is left?

Upper bounds in l_S?

One has $l_R(w) \leq l_S(w)$ ⇝ Can one do better?

better bounds:

recall: all S-reduced minimal presentations of $w \in W$ contain the same letters

and define: $nd(w) :=$ # distinct generators in w

Thm (Drake-Peters 2021)

w hyperbolic, $w \in W$ then

$\lceil x \rceil$, least integer greater or equal to x

$$l_R(w) \leq l_S(w) - 2 \cdot \left\lceil \frac{l_S(w)}{nd(w)} \right\rceil + 2.$$

this bound is sharp.

Universal Coxeter groups

$W_n = \langle s_1, \ldots, s_n \mid s_i^2 = \mathbb{1} \rangle$

- Drake-Peters / Lotz

  "=" holds for elements $w \in W_n$ of the form

  $$w = (s_1 \cdot s_2 \cdot \ldots \cdot s_n)^j s_1 \ldots s_r \qquad j \in \mathbb{N}, r \leq j$$

  where $\ell_S(w) = n \cdot j + r$

  and $\ell_R(w) = (n-2) \cdot j + 2 + r$

- Lotz:
  - estimates on how often each generator appears in any S-reduced expression with $\ell_R(w) \geq 3$
  - $\mathrm{Aut}(W_n)$ preserves $\ell_R(w)$

Now to the affine case

W affine, W_0 associated spherical

$p : W \twoheadrightarrow W_0$ natural projection

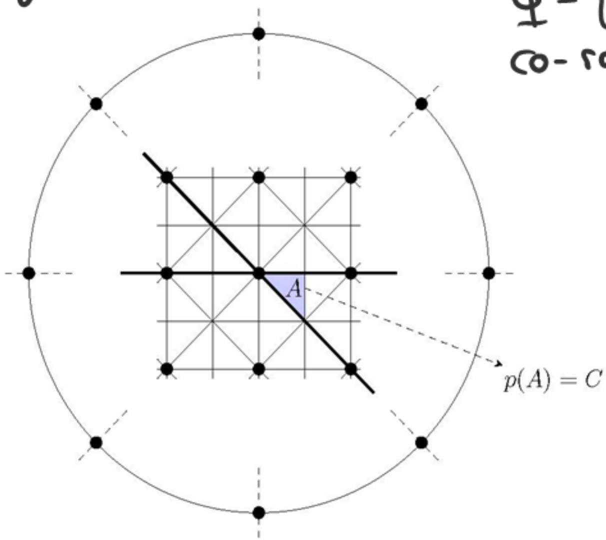
$\ker(p) := T$ = translations in W

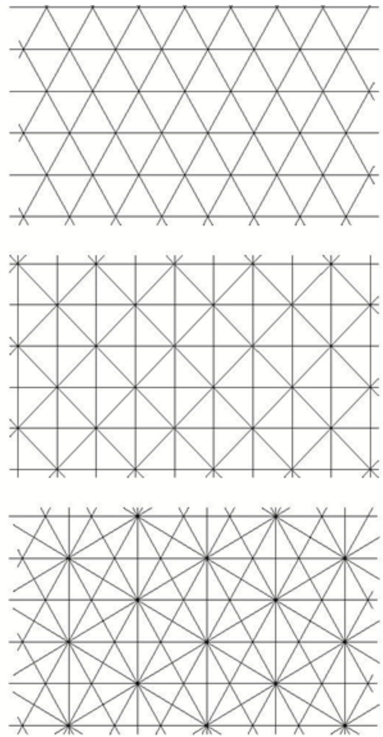
$\cong \langle \Phi^\vee \rangle_{\mathbb{Z}}$ where

$W \cong W_0 \ltimes T$

$\Phi^\vee = \left\{ \frac{\alpha}{(\alpha,\alpha)} \,\middle|\, \alpha \in \Phi \right\}$

co-roots





The **dimension** of $w \in W$ ← affine is

$$\dim(w) := \min\{\text{dim of a root-subsp. of } V \text{ containing } \mathrm{Mov}(w)\}$$

<u>Thm</u> (Lewis-McCammond-Petersen-S. 2018)

(W,S) affine, $w \in W$, then

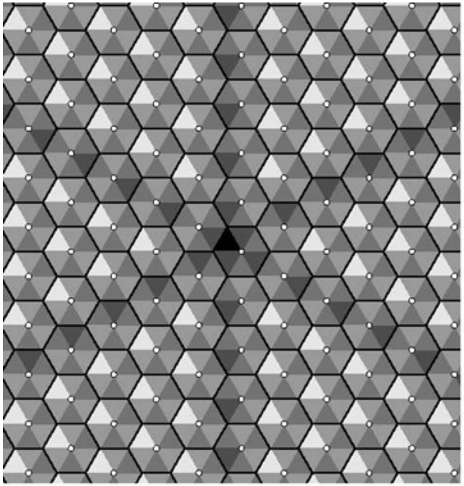
$$\boxed{\ell_R(w) = 2 \cdot \dim(w) - \dim(p(w))}$$

$$= 2d + e$$

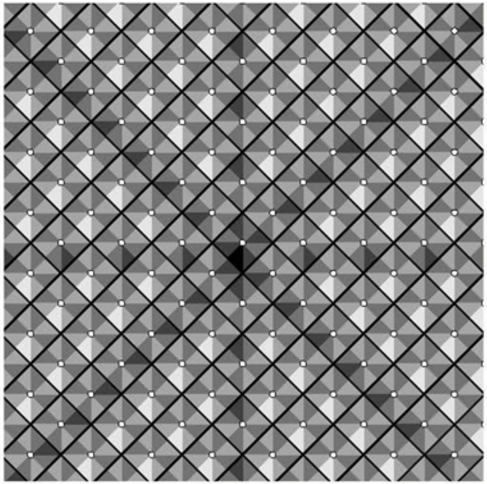
<u>elliptic dimension</u> $\dim(p(w))$

<u>differential dimension</u> $\dim(w) - \dim(p(w))$

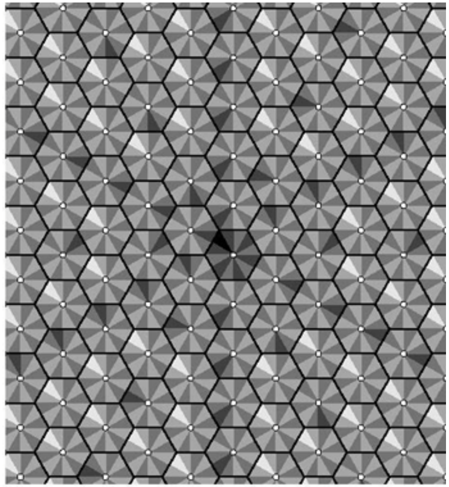
reflection length distribution in dim 2



type $\tilde{A}_2$

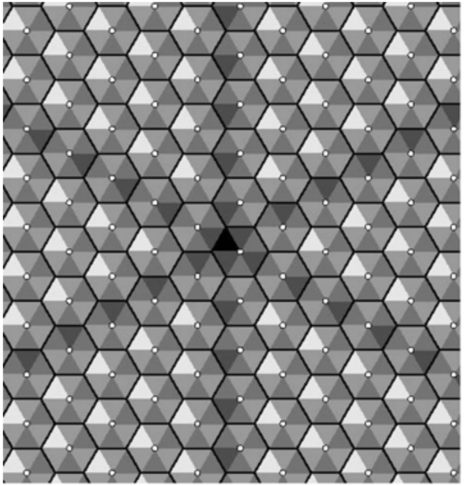


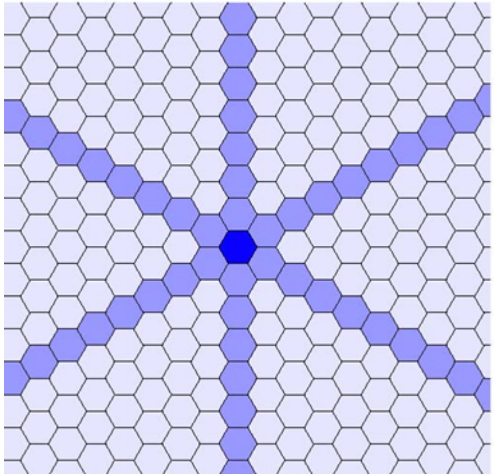
$\tilde{B}_2$



$\tilde{G}_2$

type $\tilde{A}_2$:





refl. length distribution & local length patterns

⤳ the patterns on the right can be made precise using generating functions

Open problem:

recall: W spherical, then $w = r_1 \cdots r_k$ is minimal if and only if $\alpha_{r_1}, \ldots, \alpha_{r_k}$ are l.i. roots.

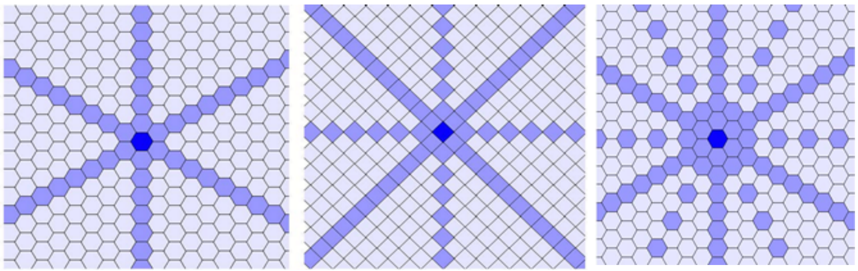
? What is the corresponding criterion in the affine case?

The best we have:

Brady-McC. 2015: W affine, $w = r_1 \cdots r_k$

$$\left[\ w \text{ elliptic } \& \ \ell_R(w) = k \iff \alpha_{r_1}, \ldots, \alpha_{r_k} \text{ are l.i.} \ \right]$$

Thank you!



Questions are welcome.

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