Selected Topics: Exact structures on exact categories

summary/aims: We summarize known results on exact structures on exact categories. For the corespondence to subfunctors of Ext we follow [14] and [4].

Exact categories

We start with the definition of an exact category in the sense of Quillen using Keller's axioms ([14], Appendix by Keller or [21], Appendix A). Let $\mathcal{A}$ be an additive category. We call a pair of morphism (i, d) in $\mathcal{A}$ a **kernel-cokernel pair** if i is a kernel of d and d is a cokernel of i. Now, let $\mathcal{S}$ be a class of kernel-cokernel pairs on $\mathcal{A}$ which is stable under isomorphisms. If (i, d) is in $\mathcal{S}$ we will call i an **inflation**, d a **deflation** and (i, d) an **exact sequence**. We call $\mathcal{S}$ an **exact structure** on $\mathcal{A}$ (and $\mathcal{A}$ an **exact category**) if it fulfills

(Ex0) The identity morphisms of the zero object is a deflation.

(Ex1) The composition of two inflations (resp. deflations) is an inflation (resp. deflation).

(Ex2) Deflations pull back along any morphism to deflations (resp.: Inflations push out along any morphism to inflations). This means, for each $f \in \mathrm{Hom}_{\mathcal{A}}(Z', Z)$ and each deflation $d \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$, there is a cartesian square (also called pullback)

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{d} & Z \end{array}$$

where d' is a deflation. Inflations push out along any morphism to inflations.

We say an additive category $\mathcal{A}$ is **weakly idempotent complete** if every retraction has a kernel and every section has a cokernel. In the case that the category $\mathcal{A}$ is weakly idempotent complete, these axioms are equivalent to the set of axioms from [14], Appendix by Keller, first considered by Gabriel and Roiter in [16].

Historical remark: Following Bühler [10], we remark that there is a long list of predecessors to Quillen's definition [23] of an exact structure on an additive category. Buchsbaum [9], Butler-Horrocks [11] and MacLane [22], XII.4, had earlier definitions using so-called proper classes of morphisms (in abelian categories), for a survey of this theory one can consult [17]. In their definition the important pullback/pushout axiom is missing. On the other hand Heller [18] and Yoneda [25] already had foreseen Quillen's definition of an exact category if the additive category is idempotent complete.

In [10], Prop. 2.9, Cor. 2.18 it is shown: If $(\mathcal{A}, \mathcal{S})$ is an exact category, then $\mathcal{S}$ is closed under direct sums and summands of short exact sequences.
Let $(\mathcal{A}, \mathcal{S})$ be an exact category and X, Y in $\mathcal{A}$. We will write $\mathrm{Ext}^1_{\mathcal{S}}(X, Y)$ for the class of exact sequences $Y \to E \to X$ in $\mathcal{S}$ modulo the usual equivalence relation of short exact sequences. Assuming that $\mathrm{Ext}^1_{\mathcal{S}}(X, Y)$ is a set for all objects X and Y, using Baer sum this defines an additive bifunctor

$$\mathrm{Ext}^1_{\mathcal{S}} \colon \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Ab}$$

where Ab denotes the category of abelian groups. It seems common in the literature to ignore the set-theoretic issue that $\mathrm{Ext}^1_{\mathcal{S}}(X, Y)$ is only a class and not a set.
A morphism is called **strict** if it factors as a deflation followed by an inflation. A sequence of strict morphisms is called exact, if at every object, it induces a short exact sequence in $\mathcal{S}$.
Similarly, the usual definition of higher Extension groups (with Yoneda products) as equivalence classes of longer exact sequences can be given as for abelian categories. Every short exact sequence in $\mathcal{S}$ gives rise to long exact sequences (of abelian groups if the Extension groups are sets) of Extension groups (with $\mathrm{Ext}^0 = \mathrm{Hom}$). For a detailed account of this, see e.g. [15], chapter 6.

The lattice of exact structures

Here we consider the class $\mathrm{Ex}(\mathcal{A})$ of all exact structures on an additive category $\mathcal{A}$. If $\mathcal{A}$ is svelte (i.e. skelletary small) then $\mathrm{Ex}(\mathcal{A})$ is a set. By abuse of language we will call it a poset with respect to inclusion even if it is not a set.
This is only a short discussion of the poset structure, for more details see [7]. In loc. cit Lemma 5.2, they prove that the intersection of a family of exact structures gives an exact structure. This poset is a lattice, cf. [7], Theorem 5.3. The join of two exact structures is the intersection of them and the meet is the intersection of all exact structures which contain both.
It always has a unique minimum given by the so-called split exact structure [10], Lemma 2.7. This is the exact structure given by split exact sequences.
There also exists a unique maximal exact structure on $\mathcal{A}$ (cf. [24], Corollary 2) but in general an explicit description is not known. If $\mathcal{A}$ has an exact structure which is abelian (or more generally quasi-abelian), then this is the unique maximal exact structure. If $\mathcal{A}$ is weakly idempotent complete, then the maximal exact structure can be described by the so-called **stable** exact structure, cf. [13].

Enough projectives in an exact category

Let now $\mathcal{A}$ be an exact category (with respect to an exact structure $\mathcal{S}$). An object P in $\mathcal{A}$ is called **projective** (or $\mathcal{S}$-projective) if $\mathrm{Hom}_{\mathcal{A}}(P, d)$ is surjective for every deflation d (resp. an object I is called **injective** (or $\mathcal{S}$-injective) if $\mathrm{Hom}_{\mathcal{A}}(i, I)$ is injective for every inflation i. We denote by $\mathcal{P}(\mathcal{S})$ (resp. $\mathcal{I}(\mathcal{S})$) the full subcategory of projectives (resp. injectives) in $(\mathcal{A}, \mathcal{S})$. The exact category $\mathcal{A}$ **has enough projectives** if for every object X there is an exact sequence $0 \to Y \xrightarrow{i} P \xrightarrow{d} X \to 0$ with P projective. (resp. $\mathcal{A}$ **has enough injectives** if for every object Y there is an exact sequence $0 \to Y \xrightarrow{i} I \xrightarrow{d} X \to 0$ with I injective).

Remark: If $(\mathcal{A}, \mathcal{S})$ has enough projectives or injectives or if it is essentially small, then $\mathrm{Ext}^n_{\mathcal{S}}(X, Y)$ is a set for all objects X, Y in $\mathcal{A}$, $n \geq 1$.

1 Exact structures and subfunctors of Ext

Lemma 1.1 *([14], Section 1.2) Let $(\mathcal{A}, \mathcal{S})$ be an exact categeory. We have an obvious bijection between the following two sets*

(a) (additive) subfunctors $F \subset \mathrm{Ext}^1_{\mathcal{S}}$

(b) subclasses $\mathcal{S}'$ of $\mathcal{S}$ closed under isomorphisms (and direct sums of short exact sequences), pullback and pushout of short exact sequences, i.e. (Ex2) holds for $\mathcal{S}'$.

given by $F \mapsto \mathcal{S}_F$ where $\mathcal{S}_F$ consists of all exact pairs $Y \to E \to X$ in $\mathcal{S}$ such that its equivalence class is in $F(X, Y)$. Conversely, $\mathcal{S}' \mapsto F'$ with $F'(X, Y)$ consists of all equivalence classes of exact sequences in $\mathcal{S}'$.

As indicated by the brackets, the property of being an additive subfunctor translates into the property that the short exact sequences are closed under direct sums. To study the structures corresponding to additive sub(bi)functors the notion of **weakly exact structure** (i.e. those classes of kernel-cokernel pairs which fulfill (b) in the previous theorem) has been introduced and studied by [6].

Since exact structures are always closed under direct sums of short exact sequences, we will restrict to consider additive functors.

Definition 1.2 *Given an exact category $(\mathcal{A}, \mathcal{S})$ and a sub(-bi)functors $F \subset \mathrm{Ext}^1_{\mathcal{S}}$. We call F **closed** if it is additive and $F(X, -)$ and $F(-, Y)$ are half exact for all objects X and Y in $\mathcal{A}$ (*here*: A functor is half exact if applied to a short exact sequence it gives a sequence which is exact in the middle).*

Definition 1.3 *We say an exact sequence* $0 \to X \xrightarrow{i} E \xrightarrow{d} Y \to 0$ *is* ***F-exact*** *if the equivalence class of* (i, d) *in* $\mathrm{Ext}^1_{\mathcal{S}}(Y, X)$ *lies in* $F(Y, X)$*. So* $\mathcal{S}_F$ *in Lemma 1.1 consists of* F*-exact sequences.*

Then we have

Theorem 1.4 *([14, Prop.1.4]) Let* $(\mathcal{A}, \mathcal{S})$ *be an exact category. The assignment* $F \mapsto \mathcal{S}_F$ *from Lemma 1.1 is a bijective map from*

(1) closed sub(bi)functors of $\mathrm{Ext}^1_{\mathcal{S}}$ *to*

(2) exact structures $\mathcal{S}'$ *on the additive category* $\mathcal{A}$ *with* $\mathcal{S}' \subset \mathcal{S}$*.*

Remark 1.5 *Theorem 1.4 has been generalized to* n*-exangulated categories in [19], section 3.2. One can also assume that it was part of the inspiration to the definition of an extriangulated category.*

Corollary 1.6 *If* $\mathcal{A}$ *is an additive category. Let* $\mathcal{S}_{max}$ *be its maximal exact structure. Then, the bijection of the Theorem 1.4 gives a* $1-1$ *correspondence between*

(1) closed sub(bi)functors of $\mathrm{Ext}^1_{\mathcal{S}_{max}}$ *and*

(2) exact structures on $\mathcal{A}$*.*

Definition 1.7 *Let* F *be an additive closed sub(bi)functor* F *of* $\mathrm{Ext}^1_{\mathcal{S}}$*. We write* $\mathcal{P}(F)$ *(resp.* $\mathcal{I}(F)$*) for the category of projectives (resp. injectives) in* $(\mathcal{A}, \mathcal{S}_F)$ *We will say that a closed sub(bi)functor* F *of* $\mathrm{Ext}^1_{\mathcal{S}}$ ***has enough projectives*** *(resp.* ***has enough injectives****) whenever* $\mathcal{S}_F$ *has. Instead of the index* $\mathcal{S}_F$ *we write just* F*, e.g.* $\mathrm{Ext}^1_F := \mathrm{Ext}^1_{\mathcal{S}_F}$ *etc.*

Lemma 1.8 *Let* $(\mathcal{A}, \mathcal{S})$ *be an exact category. If* $F \subset \mathrm{Ext}^1_{\mathcal{S}}$ *is closed and has enough projectives, then an exact sequence* (i, d) *is* F*-exact if and only if* $\mathrm{Hom}_{\mathcal{A}}(P, -)$ *applied to it gives a short exact sequence in abelian groups for every* $P \in \mathcal{P}(F)$*.*

For a proof, look at [4], Prop. 1.5, which also works for exact categories.

Subfunctors from subcategories

We continue to look at an exact category $(\mathcal{A}, \mathcal{S})$. Let $\mathcal{X} \subseteq \mathcal{A}$ be a full subcategory of $\mathcal{A}$. We define two subfunctors $F_{\mathcal{X}}$ and $F^{\mathcal{X}}$ of $\mathrm{Ext}^1_{\mathcal{S}}$ for X, Z in $\mathcal{A}$

$$F_{\mathcal{X}}(Y, Z) := \{0 \to Z \to E \to Y \to 0 \text{ in } \mathrm{Ext}^1_{\mathcal{S}}(Y, Z) \mid \mathrm{Hom}_{\mathcal{A}}(X, -) \text{ exact on it for all } X \text{ in } \mathcal{X}\}$$
$$F^{\mathcal{X}}(Y, Z) := \{0 \to Z \to E \to Y \to 0 \text{ in } \mathrm{Ext}^1_{\mathcal{S}}(Y, Z) \mid \mathrm{Hom}_{\mathcal{A}}(-, X) \text{ exact on it for all } X \text{ in } \mathcal{X}\}$$

These are (the standard examples of) closed sub(bi)functors (closedness is proven in [14, Prop. 1.7]). The generalization of these functors to n-exangulated categories can be found in [19], Def. 3.16.

Definition 1.9 *For two additive subcategories* $\mathcal{C}$ *and* $\mathcal{D}$ *of* $\mathcal{A}$ *we write* $\mathcal{C} \vee \mathcal{D}$ *for the smallest additive subcategory containing* $\mathcal{C}$ *and* $\mathcal{D}$*. We call this the* ***join*** *of* $\mathcal{C}$ *and* $\mathcal{D}$*.*

Remark 1.10 *We remark that we have the obvious inclusions:* $\mathcal{X} \vee \mathcal{P}(\mathcal{S}) \subset \mathcal{P}(F_{\mathcal{X}})$ *(resp. dually* $\mathcal{X} \vee \mathcal{I}(\mathcal{S}) \subset \mathcal{I}(F^{\mathcal{X}})$*). Furthermore, it is clear that* $F_{\mathcal{X}} = F_{\mathcal{X} \vee \mathcal{P}(\mathcal{S})}$ *(resp.* $F^{\mathcal{X}} = F^{\mathcal{X} \vee \mathcal{I}(\mathcal{S})}$*). Also, one can see easily that any sub(bi)functor* F *of* $\mathrm{Ext}^1_{\mathcal{S}}$ *is also a sub(bi)functor of* $F_{\mathcal{P}(F)}$ *(resp. of* $F^{\mathcal{I}(F)}$*) since an* F*-exact sequence* η *fulfills that* $\mathrm{Hom}_{\mathcal{A}}(P, \eta)$ *is exact for any* $P \in \mathcal{P}(F)$*.*

Remark 1.11 *Let $(\mathcal{A}, \mathcal{S})$ be an exact category. It is obvious that the inclusion of two additive subcategories $\mathcal{X} \subset \mathcal{X}'$ of $\mathcal{A}$ implies $F_{\mathcal{X}} \supset F_{\mathcal{X}'}$ and $F^{\mathcal{X}} \supset F^{\mathcal{X}'}$.*

In [8], section 5 one can find an example of an exact structure on category of finite-dimensional modules over the Kronecker algebra which is not of the form $\mathcal{S}_{\mathcal{X}}$ for any subcategory $\mathcal{X}$.

Exact structures with enough projectives

Definition 1.12 *Let $\mathcal{A}$ be an additive category. We call a subcategory $\mathcal{M}$ **contravariantly finite** in $\mathcal{A}$ if every object X in $\mathcal{A}$ admits a **right $\mathcal{M}$-approximation**, that is a morphism $\alpha\colon M \to X$ with $M \in \mathcal{M}$ such that every $f\colon M' \to X$ with M' in $\mathcal{M}$ factors over α. Dually, one defined **covariantly finite** subcategory.*

We remark that intersections of two contravariantly finite subcategories do not necessarily have this property. Also, let $\mathcal{A}$ be an additive category and $\mathcal{B}, \mathcal{C}$ two additive subcategories. If $\mathcal{B}$ and $\mathcal{C}$ are contravariantly finite, then $\mathcal{B} \vee \mathcal{C}$ too.

For later we need to understand what it means that right approximations of a contravariantly finite subcategory are deflations. So we look at this special situation.

Lemma 1.13 *Let $(\mathcal{A}, \mathcal{S})$ be an exact category with enough projectives (resp. enough injectives). Let $\mathcal{X}$ be a contravariantly finite (resp. covariantly finite) additive subcategory. Then the following are equivalent:*

(a) Any right (resp. left) $\mathcal{X}$-approximation is a deflation (resp. inflation).

(b) $\mathcal{P}(\mathcal{S}) \subset \mathcal{X}$ (resp. $\mathcal{I}(\mathcal{S}) \subset \mathcal{X}$)

Remark 1.14 *If $\mathcal{A}$ is weakly idempotent complete and $(\mathcal{A}, \mathcal{S})$ an exact category. Then $\mathcal{P}(\mathcal{S})$ is closed under direct sums and summands (cf. [10], Rem 11.5, Cor 11.6).*

Theorem 1.15 *Let $\mathcal{A}$ be weakly idempotent complete additive category and $(\mathcal{A}, \mathcal{S})$ be an exact category. The assignments $\mathcal{X} \mapsto S_{F_{\mathcal{X}}} =: \mathcal{S}_{\mathcal{X}}$ gives a bijections between*

(1) additive contravariantly finite subcategories $\mathcal{X}$ of $\mathcal{A}$, closed under direct summands and whose right approximations are deflations and

(2) exact structures $\mathcal{S}' \subset \mathcal{S}$ which have enough projectives.

We consider the dual statement of the previous Proposition as obvious and leave it to the reader.

A classical situation

Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between exact categories $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$. Then we have maps natural in X and Y

$$\varphi_{X,Y}\colon \operatorname{Ext}^1_{\mathcal{S}}(X, Y) \to \operatorname{Ext}^1_{\mathcal{T}}(\varphi(X), \varphi(Y)).$$

This gives an additive sub(bi)functor $F := \ker \varphi_{*,*} \subset \operatorname{Ext}^1_{\mathcal{S}}$. It is closed by [14], Prop. 1.10. The F-exact sequences are the exact sequences in $(\mathcal{A}, \mathcal{S})$ which are split exact once we apply φ.

Remark 1.16 *If λ is a left adjoint functor to φ, then the counit $\lambda\varphi(X) \to X$ for an object X in $\mathcal{A}$ provides a right $\lambda(\mathcal{B})$-approximation of X. In particular, $\lambda(\mathcal{B})$ is contravariantly finite in $\mathcal{A}$.*

Lemma 1.17 *If the functor φ has a left adjoint λ then*

(1) $F = F_{\lambda(\mathcal{B})} = F_{\lambda(\mathcal{B})\vee\mathcal{P}(\mathcal{S})}$.

(2) *If all counits* $\lambda\varphi(X) \to X$ *are deflations in* $(\mathcal{A}, \mathcal{S})$*, then* F *has enough projectives and furthermore,* $\mathcal{P}(F)$ *consists of all direct summands of objects in* $\lambda(\mathcal{B})$*.*

(3) *If* $\mathcal{A}$ *is weakly idempotent complete and* $(\mathcal{A}, \mathcal{S})$ *has enough projectives, then* F *has enough projectives and* $\mathcal{P}(F)$ *consists of direct summands of* $\lambda(\mathcal{B}) \vee \mathcal{P}(\mathcal{S})$*.*

There is a dual version if the functor φ *has a right adjoint.*

Example 1.18 *Let* $f\colon B \to A$ *a ring homomorphism and* $\varphi\colon \text{A-Mod} \to \text{B-Mod}, X \mapsto {}_BX$ *the functor given by restriction of scalars along* f*.*
Then, there is a left adjoint given by the following tensor functor $\lambda(X) := A \otimes_B X$ *called the* ***induced module*** *and a right adjoint given by the following Hom-functor* $\rho(X) := \mathrm{Hom}_B(A, X)$ *called the* ***co-induced module****.* *The counits* $\lambda\varphi(X) = A \otimes_B X \to X$ *are epimorphisms since their restrictions of scalars are surjective maps by the triangle identity. The units* $X \to \mathrm{Hom}_B(A, {}_BX)$ *are monomorphisms since their restrictions of scalars are injective maps by the triangle identity. Therefore, by the previous lemma we have for* $F = \ker \varphi_{*,*}$ *the following*

(1) $F = F_{A\otimes_B \text{B-Mod}} = F^{\mathrm{Hom}_B(A,\text{B-Mod})}$

(2) F *has enough projectives and enough injectives. The* F*-projectives are the direct summands of* $A \otimes_B \text{B-Mod}$*, the* F*-injectives are the direct summands of* $\mathrm{Hom}_B(A, \text{B-Mod})$*.*

This exact structure on A-Mod *has been introduced by Hochschild in* [20] *in 1956. In loc. cit. this has been used to define relative Hochschild homology, a Tor and Ext functor have been defined for this setup. A very nice application of the classical situation is the finite representation type classification for group algebras, cf.* [5]*, chapter III, section 3. A recent application to Han's conjecture can be found in* [12]*.*

Example 1.19 *Let* Γ *be a ring and* $e \in \Gamma$ *an idempotent, we define* $\Lambda := e\Gamma e$*. Then, the restriction functor* $e\colon \Gamma\text{-Mod} \to \Lambda\text{-Mod}, X \mapsto eX$ *has a left adjoint* $\ell = \Gamma e \otimes_\Lambda (-)$ *and right adjoint* $r = \mathrm{Hom}_\Lambda(e\Gamma, -)$*. Therefore, we have for* $F = \ker e_{*,*}$ *the following description (numbered by the parts of the lemma 1.17 that are used)*

(1) $F = F_{\Gamma e\otimes_\Lambda \Lambda\text{-Mod}} = F^{\mathrm{Hom}_\Lambda(e\Gamma,\Lambda\text{-Mod})}$.

(3) *Since* Γ-Mod *is abelian, it is weakly idempotent complete. It has enough projectives and enough injectives. So, it follows that* F *has enough projectives and enough injectives. We have* $\mathcal{P}(F)$ *consists of direct summands of* $(\Gamma e \otimes_\Lambda \Lambda\text{-Mod}) \vee \mathrm{Add}(\Gamma)$ *and* $\mathcal{I}(F)$ *consists of direct summands of* $\mathrm{Hom}_\Lambda(e\Gamma, \Lambda\text{-Mod}) \vee \mathcal{I}(\Gamma\text{-Mod})$*.*
If we take a noetherian ring Γ *and consider the abelian* Γ-mod *category given by finitely generated* Γ*-modules, then this category has not in general enough injectives but it has enough projectives given by* $\mathrm{add}(\Gamma)$*. Assume that* $\Lambda = e\Gamma e$ *is again noetherian, then the restriction functor* $e\colon \Gamma\text{-mod} \to \Lambda\text{-mod}$ *has a well-defined left adjoint functor* $\ell = \Gamma e \otimes_\Lambda (-)$*. We conclude that in this case* F *has enough projectives given by the direct summands of* $(\Gamma e \otimes_\Lambda \Lambda\text{-Mod}) \vee \mathrm{add}(\Gamma)$*.*

2 Literature

For the basics of exact categories one should consult [10]. It is a very good source for diagram chasing in exact categories analogously as it is known in abelian categories. Nevertheless, it does not treat the higher extension groups and their long exact sequences, for this consult [15], chapter 6. For the lattice of exact structures, see [7], section 5. The correspondence between closed subfunctors of Ext and exact structures is explained in [14]. The functors of the form $F_{\mathcal{X}}$ for a subcategory $\mathcal{X}$ are defined by Auslander and Solberg [4], [2], [3], [1] if the underlying exact category is the module category of an artin algebra and more generally in [14].

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