# Sylvester rank functions for rings and universal localization

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This is a slightly revised version of the slides for a talk in the BIREP working seminar on June 2, 2021. Further revised June 15, mainly to give proper credit to Bergman and Dicks, and June 17, to include an argument of Henning Krause on the existence of universal localizations. My aim was to discuss Sylvester rank functions, universal localization and Schofield's Theorem giving a 1:1 correspondence between Sylvester rank functions on a ring $R$ and equivalence classes of homomorphisms from $R$ to a simple artinian ring.

## 1 Sylvester rank functions

### 1.1 Definition

**Proposition/Definition** (Schofield [6, §7], Malcolmson).
Let $R$ be a ring. A *Sylvester rank function* for $R$ can be defined in two ways:

(1) A function $\rho$ : {maps between f.g. projective left $R$-modules $P \xrightarrow{\alpha} Q$ } $\to$ $\mathbb{R}$ with

(1a) $\rho\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) = \rho(\alpha) + \rho(\beta)$ for $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) : P_0 \oplus P_0' \to P_1 \oplus P_1'$.
(1b) $\rho\left(\begin{smallmatrix} \alpha & 0 \\ \gamma & \beta \end{smallmatrix}\right) \geq \rho(\alpha) + \rho(\beta)$.
(1c) $\rho(\beta\alpha) \leq \min\{\rho(\alpha), \rho(\beta)\}$ for $\alpha : P_0 \to P_1$ and $\beta : P_1 \to P_2$.
(1d) Normalized by $\rho(1 : R \to R) = 1$.

(2) A function $\rho$ : {finitely presented (f.p.) left $R$-modules} $\to \mathbb{R}$ satisfying

(2a) $\rho(X \oplus Y) = \rho(X) + \rho(Y)$.
(2b) If $X \to Y \to Z \to 0$ is exact, then $\rho(Z) \leq \rho(Y) \leq \rho(X) + \rho(Z)$, or
(2c) Normalized by $\rho(R) = 1$.

A function on modules defines one on maps by $\rho(\alpha) = \rho(Q) - \rho(\operatorname{Coker} \alpha)$

for $\alpha : P \to Q$. A function on maps defines one on modules by $\rho(X) = \rho(1_Q) - \rho(\alpha)$ for a presentation $P \xrightarrow{\alpha} Q \to X \to 0$.

## 1.2 Remarks

(i) One can also define a Sylvester rank function by specifying it on maps between f.g. free modules, so on rectangular matrices over $R$.

(ii) One can swap between right and left modules by using the duality $P^* = \mathrm{Hom}_R(P, R)$ in a Sylvester map rank function.

(iii) It is automatic that $\rho$ takes only non-negative values.

(iv) A simple artinian ring $R = M_n(D)$ has a unique Sylvester rank function $\rho_R$ with $\rho_R(X) = \frac{1}{n}\,\mathrm{length}(X)$.

(v) If $R \to S$ is a ring homomorphism, any Sylvester rank function $\rho_S$ on $S$ restricts to one on $R$ via $\rho_R({}_R X) = \rho_S(S \otimes_R X)$.

(vi) Malcolmson showed that $\mathbb{Z}$-valued Sylvester rank functions are in 1:1 correspondence with Cohn's 'prime matrix ideals'. Schofield is mainly interested in the case when $\rho$ takes values in $\frac{1}{n}\mathbb{Z}$ for some $n$. We shall call such $\rho$ 'discrete'.

(vii) The name perhaps comes from the following observation.

**Lemma.** Let $\rho$ : {maps between f.g. projective left $R$-modules} $\to \mathbb{R}$ satisfy conditions (1a), (1c), (1d). Then $\rho$ satisfies (1b), so is a Sylvester rank function, if and only if

(*) For $P_0 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_2$ we have $\rho(\beta\alpha) \geq \rho(\alpha) + \rho(\beta) - \rho(1_{P_1})$.

Compare this with Sylvester's law of nullity, which says that $\mathrm{nullity}(BA) \leq \mathrm{nullity}(A) + \mathrm{nullity}(B)$ for $n \times n$ matrices $A, B$ over a field, so $\mathrm{rank}(BA) \geq \mathrm{rank}(A) + \mathrm{rank}(B) - n$.

**Proof.** If $\alpha : P \to Q$, then condition (1c) ensures that $\rho(\alpha) = \rho(\alpha\phi) = \rho(\psi\alpha)$ for isomorphisms $\phi, \psi$.

Suppose (1b) holds. As homomorphisms from $P_1 \oplus P_0$ to $P_1 \oplus P_2$ we have an equality

$$\begin{pmatrix} 1_{P_1} & 0 \\ 0 & \beta\alpha \end{pmatrix} \begin{pmatrix} 1_{P_1} & \alpha \\ 0 & -1_{P_0} \end{pmatrix} = \begin{pmatrix} 0 & 1_{P_1} \\ 1_{P_2} & -\beta \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 1_{P_1} & \alpha \end{pmatrix}.$$

The inner two matrices are invertible, so the outer two have the same rank, so

$$\rho(\alpha)+\rho(\beta) \leq \rho\begin{pmatrix}\beta & 0\\ 1_{P_1} & \alpha\end{pmatrix} = \rho\begin{pmatrix}1_{P_1} & 0\\ 0 & \beta\alpha\end{pmatrix} = \rho(1_{P_1})+\rho(\beta\alpha),$$

giving (*).

Conversely suppose that (*) holds. We can factorize

$$\begin{pmatrix}\alpha & 0\\ \gamma & \beta\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & 1_{P'_1}\end{pmatrix}\begin{pmatrix}1_{P_0} & 0\\ \gamma & 1_{P'_1}\end{pmatrix}\begin{pmatrix}1_{P_0} & 0\\ 0 & \beta\end{pmatrix}.$$

Now the middle matrix is invertible, so if $\phi$ is the product of the first two matrices, then

$$\rho(\phi) = \rho\begin{pmatrix}\alpha & 0\\ 0 & 1_{P'_1}\end{pmatrix},$$

and (*) gives

$$\rho\begin{pmatrix}\alpha & 0\\ \gamma & \beta\end{pmatrix} \geq \rho(\phi)+\rho\begin{pmatrix}1_{P_0} & 0\\ 0 & \beta\end{pmatrix} - \rho(1_{P_0\oplus P'_1}) = \rho(\alpha)+\rho(\beta),$$

using (1a).

## 1.3 Characters

As a variation of this, by a *character* $\chi$ for $R$ we mean a $\mathbb{Z}$-valued unnormalized Sylvester module rank function, so a function satisfying (2a) and (2b).

**Theorem** [5].
(i) Any character can be written uniquely as a sum of 'irreducible' characters
(ii) The assigment $M \mapsto \chi_M$ with

$$\chi_M(X) = \mathrm{length}_{\mathrm{End}_R(M)} \mathrm{Hom}(X, M)$$

gives a 1:1 correspondence between isoclasses of indecomposable left $R$-modules $M$ such that $M$ has finite length as an $\mathrm{End}_R(M)$-module (finite 'endolength') and irreducible characters for $R$.

**Idea of proof.** We explain only how, starting with a character, one can come up with a finite endolenth module. Let $D(R)$ be the category of covariant additive functors from the category of f.p. left $R$-modules to abelian groups. Recall that any coherent functor $\mathcal{F}$ has a projective resolution $0 \to \mathrm{Hom}(Z,-) \to \mathrm{Hom}(Y,-) \to \mathrm{Hom}(X,-) \to \mathcal{F} \to 0$ for some right exact sequence $X \to Y \to Z \to 0$.

Defining $\chi(\mathcal{F}) = \chi(X) - \chi(Y) + \chi(Z)$ for $\mathcal{F}$ as above, any character $\chi$ gives a function from the set of coherent functors to $\mathbb{Z}_{\geq 0}$. With a little work this can be extended to a function from all functors to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ which is additive on short exact sequences.

Given a character $\chi$, one can find a functor $\mathcal{F}$ with $0 < \chi(\mathcal{F}) < \infty$, such that every non-zero subfunctor $\mathcal{G}$ of $\mathcal{F}$ has $\chi(\mathcal{G}) = \chi(\mathcal{F})$. Now $D(R)$ is a Grothendieck category, and the injective objects are functors of the form $M \otimes_R -$ with $M$ a pure-injective module. The condition on $\mathcal{F}$ ensures that its injective envelope is of the form $M \otimes_R -$ with $M$ of finite endolength.

## 1.4 Inner projective rank functions

**Definition.** By a *projective rank function* we mean a function

$$\rho : \{\text{f.g. projective left } R\text{-modules}\} \to \mathbb{R}_{\geq 0}$$

satisfying $\rho(P \oplus Q) = \rho(P) + \rho(Q)$ and normalized by $\rho(R) = 1$. Equivalently it is given by a group homomorphism $\rho : K_0(R\text{-proj}) \to \mathbb{R}$ with $\rho([P]) \geq 0$ for all $P$ and $\rho([R]) = 1$.

The *inner projective rank function* given by a projective rank function $\rho$ is the mapping

$$\rho : \{\text{maps between f.g. projective left } R\text{-modules } \alpha : P \to Q\} \to \mathbb{R}$$

defined by

$$\rho(P \xrightarrow{\alpha} Q) = \inf\{\rho(P') : \alpha \text{ factors as } P \to P' \to Q\}.$$

Observe that $\rho(1_P) = \rho(P)$, for if $1_P$ factors through $P'$ then the map $P \to P'$ is split mono, so $\rho(P) \leq \rho(P')$.

Clearly conditions (1c) and (1d) hold for any inner projective rank function.

**Theorem.** An inner projective rank function is a Sylvester rank function if and only if

(**) For $P_0 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_2$ with $\beta\alpha = 0$ we have $\rho(\alpha) + \rho(\beta) \leq \rho(1_{P_1})$.

If so, one says the projective rank function is a *Sylvester projective rank function*.

**Proof.** If $\rho$ is a Sylvester rank function, then (**) is a special case of the lemma above. For the other direction, if (**) holds, then condition (1b) follows from [6, Lemma 1.14] and (1a) from [6, Lemma 1.15], so $\rho$ is a Sylvester rank function.

**Note.** (**) is always true over a hereditary (or 'weakly semihereditary') ring, since $\operatorname{Im}\beta$ is a submodule of $P_2$, so projective. Then $P_1 \to \operatorname{Im}\beta$ is split epi, so there is a decomposition $P_1 = \operatorname{Ker}\beta \oplus P'$. Now $\alpha$ factors through $\operatorname{Ker}\beta$ and $\beta$ through $P'$, so $\rho(\alpha) + \rho(\beta) \leq \rho(1_{\operatorname{Ker}\beta}) + \rho(1_{P'}) = \rho(1_{P_1})$.

# 2 Universal localization

## 2.1 Inverting things

Let $R$ be a ring and $S$ an $R$-ring, so a ring equipped with a homomorphism $f : R \to S$.

(i) One can ask to invert elements of a ring. If $\Sigma$ is a subset of $R$, we say that $S$ *inverts* $\Sigma$ if $f(a)$ is invertible in $S$ for all $a \in \Sigma$.

(ii) More generally, one can ask to invert matrices over a ring (see Cohn [4, §7.2]). If $\Sigma$ is a set of matrices over $R$, we say that $S$ *inverts* $\Sigma$ if the matrix $f(a) := (f(a_{ij}))$ is invertible over $S$ for all $a = (a_{ij}) \in \Sigma$.

(iii) More generally still, one can ask to invert maps between f.g. projective modules ([1, 3] and [6, §4]). If $\Sigma$ is a set of maps $\alpha : P \to Q$ between f.g. projective left $R$-modules, we say that $S$ *inverts* $\Sigma$ if $S \otimes \alpha : S \otimes_R P \to S \otimes_R Q$ is an isomorphism for all $\alpha \in \Sigma$.

This last version makes sense for $S$ any right $R$-module.

Perpendicular categories can arise this way. For example if

$$0 \to P_1 \xrightarrow{f} P_0 \to Y \to 0$$

is a projective resolution of a right $R$-module $Y$ with $P_0$ and $P_1$ finitely generated, then $S$ inverts $f^* : P_1^* \to P_0^*$ if and only if $S \in Y^\perp := \{M : \operatorname{Hom}(Y, M) = \operatorname{Ext}^1(Y, M) = 0\}$.

## 2.2 Definition and existence

The *universal localization* of $R$ with respect to $\Sigma$, if it exists, is an $R$-ring $R_\Sigma$ which inverts $\Sigma$, and with the property that for any $R$-ring $S$ which inverts

$\Sigma$, there is a unique $\theta : R_\Sigma \to S$ giving a commutative triangle

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[dr, "f"'] & R_\Sigma \arrow[d, "\theta"] \\ & S \end{tikzcd}$$

**Existence.** (i) To invert elements, if $\Sigma$ is a multiplicative set with the 'left Ore condition' one can consider the Ore localization $\Sigma^{-1}R$.

In general the best we can do is to consider the ring $R\langle a' : a \in \Sigma\rangle$ obtained from $R$ by adjoining indeterminates $a'$ for each $a \in \Sigma$, not necessarily commuting with each other or with $R$, and set $R_\Sigma = R\langle a' : a \in \Sigma\rangle/(\{aa' - 1, a'a - 1 : a \in \Sigma\})$.

Then $a'$ is an inverse for $a$ in $R_\Sigma$, and given $S$ inverting $\Sigma$, $\theta$ exists and is unique because we can and must have $\theta(a') = f(a)^{-1}$ in $S$.

(ii) [4, Theorem 7.2.4]. To invert matrices, for each matrix $a \in \Sigma$ we want an inverse matrix $a'$, and so we adjoin its entries $a'_{ij}$ as indeterminates, and then impose relations forcing $a'$ to be an inverse to $a$. Thus

$$R_\Sigma = R\langle a'_{ji} : a = (a_{ij}) \in \Sigma\rangle/(\{aa' - I, a'a - I : a \in \Sigma\}).$$

Since we have insured the existence of an inverse to each $a \in \Sigma$, if $S$ inverts $\Sigma$, we obtain a map $\theta$. Moreover $\theta$ is unique by the uniqueness of inverses.

(iii) [3, Construction 2.1]. To invert maps between projectives, for each $\alpha : P \to Q$ in $\Sigma$, choose idempotent endomorphisms $e, f$ of a free module $R^n$ with images $P, Q$. Then $a = \alpha e$ is an endomorphism of $R^n$ with $ae = a = fa$. If $a'$ is an endomorphism of $R^n$, the following are equivalent

(1) $ea' = a' = a'f$, $a'a = e$ and $aa' = f$, and
(2) $\alpha$ is invertible and $a' = \alpha^{-1}f$.

Now the equations in (1) are matrix equations, so take $R_\Sigma = R\langle a'_{ji} : \alpha \in \Sigma\rangle/(\text{relations (1)})$. Again, the fact that $a'$ is uniquely determined by $\alpha$ gives the uniqueness of $\theta$.

**Alternative construction of $R_\Sigma$ in case (iii).** Schofield [6, Theorem 4.1] considers the category $R$-proj of f.g. projective left $R$-modules as a ring with several objects. One has a functor

$$R\text{-proj} \xrightarrow{G} R\text{-proj}_\Sigma.$$

This gives a homomorphism $R = \mathrm{End}_R(R)^{op} \xrightarrow{g} \mathrm{End}_{R\text{-proj}_\Sigma}(G(R))^{op} =: R_\Sigma$.

Given any $f : R \to S$ which inverts $\Sigma$, the functor $F = S \otimes_R -$ factors as

$$\begin{tikzcd} R\text{-proj} \arrow[r, "G"] \arrow[dr, "F"'] & R\text{-proj}_\Sigma \arrow[d, "\Theta"] \\ & S\text{-proj} \end{tikzcd}$$

for a unique $\Theta$. This gives

$$\begin{tikzcd} R = \mathrm{End}_R(R)^{op} \arrow[r, "g"] \arrow[dr, "f"'] & \mathrm{End}_{R\text{-proj}_\Sigma}(G(R))^{op} = R_\Sigma \arrow[d, "\theta"] \\ & \mathrm{End}_{S\text{-proj}}(F(R))^{op} = S \end{tikzcd}$$

so the existence of $\theta$. But uniqueness of $\theta$ is not obvious.

I am grateful to Henning Krause for the following argument for uniqueness.

Recall that for an additive category $\mathcal{C}$, we write $\mathcal{C}$-Mod for the category of contravariant additive functors from $\mathcal{C}$ to abelian groups. Then ($R$-proj)-Mod is equivalent to $R$-Mod, by the functors sending a functor $F$ in ($R$-proj)-Mod to $F(R)$, and sending an $R$-module $M$ to $\mathrm{Hom}_R(-, M)$.

For the uniqueness of $\theta$, it suffices to show that $g$ is an epimorphism of rings, or equivalently that the restriction functor $R_\Sigma$-Mod $\to$ $R$-Mod is fully faithful. Now this functor identifies with the composition

$$(R_\Sigma\text{-proj})\text{-Mod} \to (R\text{-proj}_\Sigma)\text{-Mod} \to (R\text{-proj})\text{-Mod}.$$

The right hand functor is restriction via $G$, and it is fully faithful by the uniqueness of inverses for elements of $\Sigma$. The left hand functor is restriction via the functor $H : (R\text{-proj})_\Sigma \to R_\Sigma\text{-proj}$ sending an object of $(R\text{-proj})_\Sigma$, say $G(P)$ with $P$ a projective $R$-module, to $\mathrm{Hom}_{(R\text{-proj})_\Sigma}(G(R), G(P))$. Now every object in $(R\text{-proj})_\Sigma$ is a direct summand of a finite direct sum of copies of $G(R)$, and every object in $R_\Sigma$-proj is a direct summand of a finite direct sum of copies of $R_\Sigma$. It follows that $H$ is an equivalence 'up to direct summands', so the left hand functor is an equivalence. Thus the composition of the two functors is fully faithful, as wanted.

## 2.3 Properties of universal localization

(a) Without the Ore condition there is no nice canonical form for elements of $R_\Sigma$. But there are weaker results, 'Malcolmson's criterion' [6, Theorem 4.2] and 'Cramer's rule' [6, Theorem 4.3].

(b) Any f.p. left $R_\Sigma$-module $X$ is induced from some f.p. left $R$-module, so of the form $X \cong R_\Sigma \otimes_R Y$ for some f.p. module $Y$ [6, Corollary 4.5].

(c) It follows from the definition that $g : R \to R_\Sigma$ is an epimorphism of rings. Thus the category of right $R_\Sigma$-modules can be identified with the full subcategory of Mod-$R$ consisting of the modules $M$ which invert $\Sigma$. The category of right $R_\Sigma$-modules is closed under extensions in Mod-$R$ [6, Theorem 4.7].

(d) The epimorphism $g : R \to R_\Sigma$ is a 'pseudo-flat', meaning that for $R_\Sigma$-modules $M, N$, we have $\mathrm{Ext}^1_R(M,N) = \mathrm{Ext}^1_{R_\Sigma}(M,N)$, or equivalently $\mathrm{Tor}^R_1(R_\Sigma, R_\Sigma) = 0$ [6, Theorem 4.8]. If $R$ is hereditary, so is $R_\Sigma$.

## 2.4 Another approach

This is due to Schofield [7]. Recall that the category $R$-fpmod of f.p. left $R$-modules is additive with cokernels, so finite coproducts, but is not necessarily abelian.

A *severe left* Ore set* is a set $\sigma$ of maps in $R$-fpmod satisfying
(i) $\sigma$ contains all isomorphisms.
(ii) $\sigma$ is closed under compositions.
(iii) If $s : X \to Y$ is in $\sigma$ and $f : X \to Z$, then the pushout $s'$ of $s$ along $f$ is in $\sigma$,
(iv) If $s : X \to Y$ is in $\sigma$ and $g : Y \to Z$ satisfies $gs = 0$, then $Z \to \operatorname{Coker} g$ is in $\sigma$.

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Z & \xrightarrow{s'} & W. \end{array} \qquad X \xrightarrow{s} Y \xrightarrow{g} Z \xrightarrow{\text{nat}} \operatorname{Coker} g$$

[(*) Schofield says 'right', but composes morphisms as if written on the right, which is the opposite to the notation we use.]

Given $\Sigma$, let $\sigma$ be the severe left Ore set generated by $\Sigma$. (Perhaps it is the set of all maps $X \to Y$ such that $M \otimes_R X \to M \otimes_R Y$ is an isomorphism for all right $R$-modules $M$ which invert $\Sigma$.)

**Theorem** [S 2007, Thm 4.1]. There is an equivalence $R_\Sigma$-fpmod $\cong \sigma^{-1}(R\text{-fpmod})$.

The category on the right denotes an Ore localization. It might be nice to have a direct proof of this, and then to derive other properties of universal localization from it.

## 2.5 Example

Let $A, B$ be $R$-rings. Recall that there is a pushout in the category of rings

$$\begin{tikzcd} & A \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & A \coprod_R B \\ & B \arrow[ur] & \end{tikzcd}$$

otherwise known as a *free product with amalgamation.*

Let $T = \begin{pmatrix} A & A \otimes_R B \\ 0 & B \end{pmatrix}$.

We have a map of projective left $T$-modules $\mu : Te_{11} \to Te_{22}$, $t \mapsto t \cdot \left(\begin{smallmatrix} 0 & 1 \otimes_R 1 \\ 0 & 0 \end{smallmatrix}\right)$.

**Theorem** [6, Theorem 4.10]. We have $T_{\{\mu\}} \cong M_2(A \coprod_R B)$.

**Proof.** A right $T$-module $M$ is given by an $A$-module $U = Me_{11}$, a $B$-module $V = Me_{22}$ and an $B$-module map $U \otimes_A A \otimes_R B \to V$, or equivalently an $R$-module map $f : U \to V$.

Then $T_{\{\mu\}}$-modules are the same as $T$-modules $M$ which invert $\mu$, so with the map $M \otimes_T Te_{11} \to M \otimes_T Te_{22}$ an isomorphism, or equivalently with $f$ is invertible.

Such a module is given by an $A$-module and a $B$-module with the same underlying $R$-module structure, so by an $A \coprod_R B$-module.

## 2.6 Localization with a Sylvester rank function

Let $R$ be a ring with a Sylvester rank function $\rho$.

A map $\alpha : P \to Q$ is said to be *$\rho$-full* if $\rho(\alpha) = \rho(P) = \rho(Q)$.

**Theorem** [6, Theorem 7.4]. If $\Sigma$ is a collection of maps between projectives which are $\rho$-full, then $\rho$ is the restriction of a Sylvester rank function on $R_\Sigma$ taking values in the same subgroup of $\mathbb{R}$ as $\rho$.

We define $R_\rho$ to be the universal localization of $R$ with respect to the set of all $\rho$-full maps $P \to Q$.

**Theorem** [6, Theorem 5.3]. If $\rho$ is discrete and $\rho$ is the inner projective rank function given by a Sylvester projective rank function, then $R_\rho$ is a perfect ring.

I don't know whether this works for arbitrary discrete Sylvester rank functions. But we have the following.

**Theorem** [6, Theorem 7.5]. If $\rho$ takes values in $\mathbb{Z}$, then $R_\rho$ is a local ring whose residue ring is a division ring.

# 3 Homomorphisms to simple artinian rings

Let $R$ be a ring.

## 3.1 Epimorphisms to a division ring

**Theorem** (P. M. Cohn, see [6, Theorem 7.5]). There is a 1:1 correspondence between $\mathbb{Z}$-valued Sylvester rank functions for $R$ and epimorphisms from $R$ to a division ring, up to isomorphism.

**Sketch.** Given $R \to D$, consider the restriction $\rho$ of $\rho_D$.

Given $\rho$, consider $R \to R_\rho \to R_\rho / \operatorname{Rad} R_\rho$.

Starting from $\rho$, we get $R \to R_\rho / \operatorname{Rad} R_\rho$, and we need that the restriction to $R$ of the canonical rank function on $R_\rho / \operatorname{Rad} R_\rho$ is $\rho$. Schofield says this is clear.

Starting from $R \to D$, we get $\rho$, and clearly $D$ inverts all $\rho$-full maps between projectives. This induces $R_\rho \to D$. Then we need that this map kills $\operatorname{Rad} R_\rho$. Schofield uses Cramer's rule for this. Thus get a homomorphism $R \to R_\rho / \operatorname{Rad} R_\rho \to D$. But $R \to D$ is an epimorphism, hence so is $R_\rho / \operatorname{Rad} R_\rho \to D$. But this is a map of division rings, so it is an isomorphism.

Schofield gives another proof using severe left Ore sets [7, Theorem 3.3].

Here is another argument using characters. A $\mathbb{Z}$-valued Sylvester rank function is the same thing as a character $\chi$ of degree 1, meaning that $\chi(R) = 1$. It follows that $\chi$ is irreducible, so comes from some indecomposable $R$-module $M$ of endolength 1. Thus if $E = \operatorname{End}_R(M)$, then $M$ is a simple $E$-module, and hence $D = \operatorname{End}_E(M)^{op}$ is a division ring. Now $\operatorname{End}_R(M) = \operatorname{End}_D(M)$, so since every $D$-module is semisimple, the restriction functor $\operatorname{Mod} D \to \operatorname{Mod} R$ is fully faithful. This implies that the natural map $R \to D$ is an epimorphism.

## 3.2 A key lemma

**Theorem.** Given a pushout diagram and compatible Sylvester rank functions $\rho_R$, $\rho_A$, $\rho_B$,

$$\begin{tikzcd} & A \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & A \coprod_R B \\ & B \arrow[ur] & \end{tikzcd}$$

if any two of $R, A, B$ are simple artinian, then $A \coprod_R B$ has a Sylvester rank function $\rho$ which is compatible with the others. If $\rho_R$, $\rho_A$, $\rho_B$ take values in a given subgroup of $\mathbb{R}$, then so does $\rho$.

This statement actually combines two theorems. The first is [6, Theorem 7.3]. The second is [8, Theorem 3.5], which generalizes [6, Theorem 7.10], Both are long and difficult, and unfortunately there are many misprints and ambiguities in [8], so I have not checked the details.

**Idea.** Consider $T = \begin{pmatrix} A & A \otimes_R B \\ 0 & B \end{pmatrix}$.

Using $\rho_A$ and $\rho_B$, one defines a Sylvester rank function $\rho_T$ on $T$.

Then one needs to show that the map $\mu : Te_{11} \to Te_{22}$ is $\rho_T$-full. In the case when $A, B$ are simple artinian, which is [6, Theorem 7.3], there is a simplification of the argument in [2].

Then one gets a Sylvester rank function on $R_{\{\mu\}} \cong M_2(A \coprod_R B)$, so also on $A \coprod_R B$.

## 3.3 Schofield's Theorem

Let $R$ be a ring. We say that homomorphisms to simple artinian rings $A, B$ are *equivalent* if there is a commutative diagram

$$\begin{tikzcd} & A \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & S \\ & B \arrow[ur] & \end{tikzcd}$$

with $S$ simple artinian.

**Theorem** [6, Theorem 7.12]. Assume that $R$ is a $K$-algebra for some field $K$. There is a 1:1 correspondence between discrete Sylvester rank functions on $R$ and equivalence classes of homomorphisms from $R$ to a simple artinian ring.

More generally if $R$ is not a $K$-algebra, the theorem still holds provided one only considers Sylvester rank functions for which the rank of any integer is 0 or 1, see [6, Theorem 7.14].

**Sketch.** Given a homomorphism $R \to A$, the restriction of $\rho_A$ is a discrete Sylvester rank function on $R$. If $R \to A$ and $R \to B$ are equivalent, then $\rho_A$ and $\rho_B$ are restrictions of $\rho_S$, and so they have the same restriction to $R$.

Given a discrete Sylvester rank function on $R$, taking values in $\frac{1}{n}\mathbb{Z}$, apply the key lemma to the pushout diagram

$$\begin{tikzcd} & A \arrow[dr] & \\ K \arrow[ur] \arrow[dr] & & A \coprod_K M_n(K) = C \\ & M_n(K) \arrow[ur] & \end{tikzcd}$$

Now $C$ contains $n \times n$ matrix units, so is of the form $M_n(E)$ for some $E$. Then $\rho_C$ takes values in $\frac{1}{n}\mathbb{Z}$, so corresponds to a $\mathbb{Z}$-valued Sylvester rank function on $E$. Thus by Cohn's Theorem, there is a corresponding homomorphism to a division ring $E \to D$. This gives a homomorphism $C \to M_n(D)$, so a homomorphism $A \to M_n(D)$.

If $R \to A$ and $R \to B$ are homomorphisms to simple artinian rings inducing the same Sylvester rank function $\rho$ on $R$, then we have a diagram

$$\begin{tikzcd} & A \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & A \coprod_R B \\ & B \arrow[ur] & \end{tikzcd}$$

By the key lemma, $A \coprod_R B$ has a discrete Sylvester rank function, and by what we have just proved, it is induced by a homomorphism to a simple artinian ring $S$. Now the diagram

$$
\begin{tikzcd}
 & A \arrow[dr] \arrow[drr] & & \\
R \arrow[ur] \arrow[dr] & & A \coprod_R B \arrow[r] & S \\
 & B \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

shows that $R \to A$ and $R \to B$ are equivalent.

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