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# Homological mirror symmetry for compact surfaces with boundary, tame noncommutative nodal curves and spherical obj's

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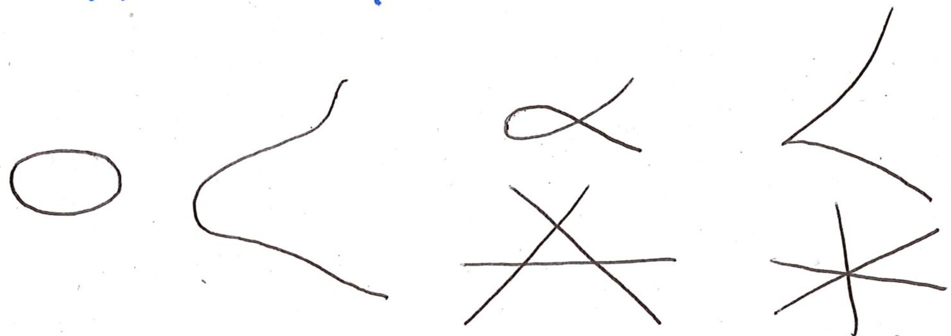
(Notes by David Fernández)

Let $\Bbbk = \bar{\Bbbk}$
$X \equiv$ Calabi-Yau curve (i.e. reduced proj. curve s.t. $\Omega_X \simeq \mathcal{O}_X$)



elliptic — cycles of projective lines

Serre duality: $\forall \mathcal{F}, \mathcal{G} \in \mathrm{Perf}(X)$; $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{G}, \mathcal{F}[1])^*$.

DEF
$\mathcal{F} \in \mathrm{Perf}(X)$ is 1-spherical if $\mathrm{Ext}^*(\mathcal{F},\mathcal{F}) \simeq \frac{\Bbbk[t]}{t^2} \left(\simeq H^1(S^1,\Bbbk)\right)$.
($\deg t = 1$).

Examples
(i) $\mathcal{F} = \Bbbk_p$, $p \in E$ smooth.
(ii) $\mathcal{F} \in \mathrm{Pic}(X)$.
(iii) $\mathcal{F} \in VB(X)$, $\mathrm{End}(\mathcal{F}) = \Bbbk$.

Applications
Let $\mathcal{F}$ be 1-spherical.
(i) $\mathcal{F} \in \mathrm{Perf}(X)$ : $\mathrm{Perf}(X) \xrightarrow[\simeq]{T_{\mathcal{F}}} \mathrm{Perf}(X)$ (Seidel-Thomas twist functor)

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathcal{F}[-n], \mathcal{G}) \otimes \mathcal{F}[-n] \xrightarrow{ev} \mathcal{G} \longrightarrow T_{\mathcal{F}}(\mathcal{G}) \xrightarrow{+}$$

(ii) Link with Yang-Baxter equations.
[Polishchuk, Burban-Kreußler, ...]

## Theorem

Let $X$ be an irreducible Calabi–Yau curve.

(i) $\mathcal{F} \in \mathrm{Perf}(X)$ spherical $\Rightarrow$ $\mathcal{F} \underset{\text{(up to shift)}}{\simeq} \begin{cases} k_p & p \text{ smooth} \\ \text{simple vector bundle.} \end{cases}$

(ii) $\mathrm{Aut}(\mathrm{Perf}(X))$ acts transitively on the set of spherical objects.

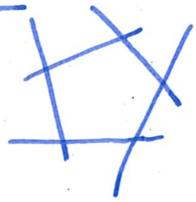
## Proof

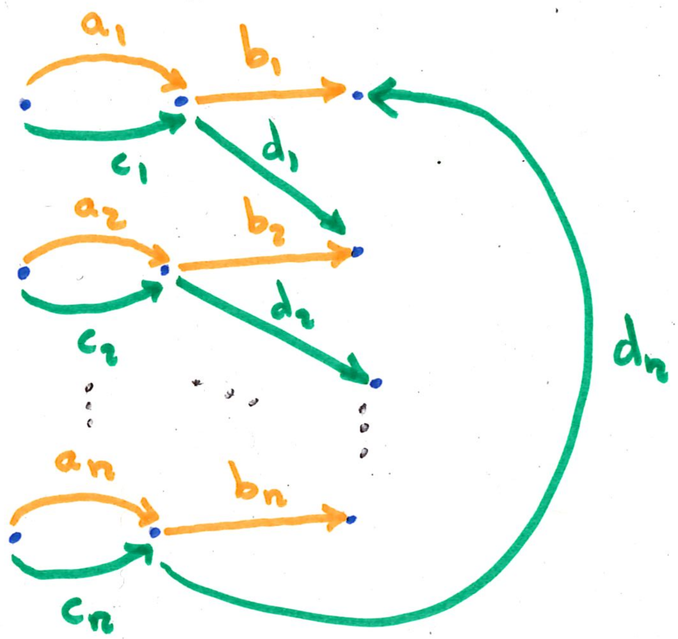
$X$ smooth $\Rightarrow$ $\begin{cases} \text{(i) obvious} \\ \text{(ii) [Lenzig–Meltzer]} \end{cases}$ ; $X$ singular [Burban–Kreußler '05]

## CONJECTURE [Polishchuk '02]

This result is true for all Calabi–Yau curves.

## Notation

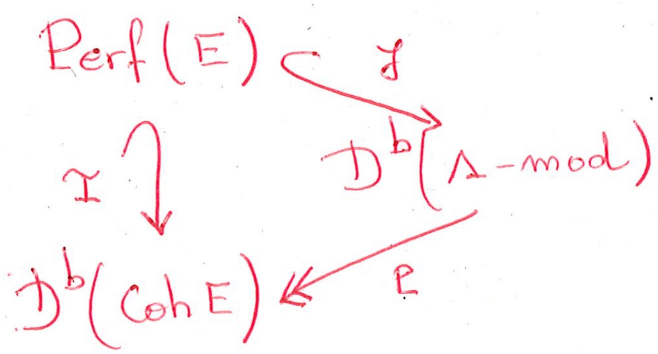
$E = E_n =$  ; $\Lambda = \Lambda_n$




(Composition of two arrows of $\neq$ colour is zero.)

**gentle algebra**

## THEOREM [Burban–Drozd '09]



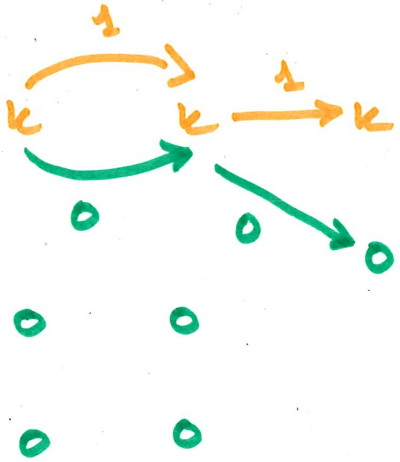

such that

(i) $\mathbb{I}, \mathbb{J}$ are fully faithful;

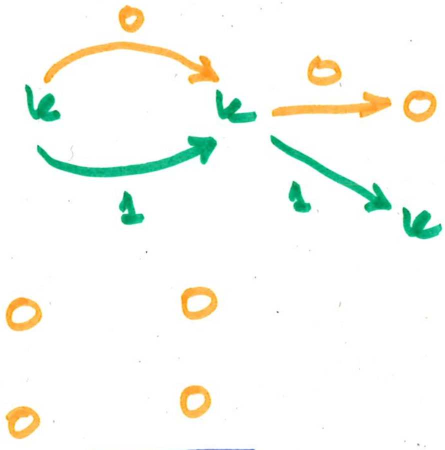
(ii) $\mathrm{Im}(\mathbb{J}) = \{ X \mid \tau(X) \simeq X \}$;

(iii) $\mathbb{P}$ is a localization with respect to $\mathcal{J} = \langle T_1^{\pm}, \ldots, T_n^{\pm} \rangle$.

Rmk




$T_1^+$




$T_1^-$

## Idea of the proof of the theorem

(i) Key intermediate notion: a noncomm. nodal curve $(\mathbb{E}, \mathcal{A})$, where

$$\mathcal{A} = \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}} \\ \mathcal{I} & \widetilde{\mathcal{O}} \end{pmatrix} \subseteq \mathrm{Mat}_{2\times 2}(\mathcal{K}).$$

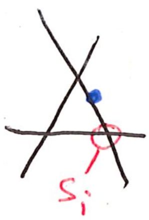
and

- $\mathcal{K} = \mathrm{Quot}(\mathcal{O})$.
- $\widetilde{E} \simeq \mathbb{P}^1 \sqcup \dots \sqcup \mathbb{P}^1 \xrightarrow{\ \nu\ } E$ "normalization".
- $\widetilde{\mathcal{O}} = \nu_*(\mathcal{O}_{\widetilde{E}}) \supseteq \mathcal{O}$; $\mathrm{Supp}(\widetilde{\mathcal{O}}/\mathcal{O}) = \mathrm{Sing}(E)$.
- $\mathcal{I} = \mathrm{Ann}_{\mathcal{O}}(\widetilde{\mathcal{O}}/\mathcal{O}) = \mathcal{I}_{\{s_1, \dots, s_n\}}$ $\ulcorner \mathcal{I}\widetilde{\mathcal{O}} = \mathcal{I} = \mathcal{I}\mathcal{O} \lrcorner$.

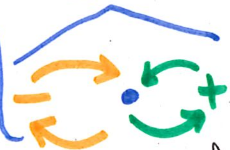
Rmk $\mathcal{K} \otimes \mathcal{A} \simeq \mathrm{Mat}_{2\times 2}(\mathcal{K})$ and $\mathcal{A}$ is a sheaf of orders on $E$.

(ii) $\mathrm{gl.dim}(\mathrm{Coh}(\mathbb{E})) = 2$

$\mathrm{gl.dim}(\mathrm{Coh}(E)) = \infty$




$\widehat{\mathcal{A}}_p \simeq \begin{cases} \mathrm{Mat}_{2\times 2}(\widehat{\mathcal{O}}_p) \overset{\text{Morita}}{\sim} \widehat{\mathcal{O}}_p, & p \text{ smooth} \\ \widehat{\quad}, & p \text{ singular.} \end{cases}$




(its global dim. is 2).

(iii) $\Gamma(E, \mathcal{A}) \ni e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\ni f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ idempotents $\begin{matrix} P = \mathcal{A}e \\ Q = \mathcal{A}f \end{matrix} \in \mathrm{VB}(\mathbb{E})$.

$$\mathrm{Coh}(E) \underset{G = \mathrm{Hom}_{\mathbb{E}}(\mathcal{P}, -)}{\overset{F = \mathcal{P} \otimes_{\mathcal{O}} -}{\rightleftarrows}} \mathrm{Coh}(\mathbb{E})$$

$GF \xrightarrow{\simeq} \mathrm{Id}_{\mathrm{Coh}(E)}$

$\Rightarrow F$ is fully faithful.

$$\mathrm{Perf}(E) \xhookrightarrow{LF} D^b(\mathrm{Coh}(\mathbb{E})) \xrightarrow[\simeq]{H} D^b(\Lambda\text{-mod})$$

($H$: tilting functor)

$$\mathrm{Perf}(E) \hookrightarrow D^b(\mathrm{Coh}(E)) \xleftarrow{DG} D^b(\mathrm{Coh}(\mathbb{E}))$$

$$D^b(\mathrm{Coh}(\mathbb{E})) \supseteq \mathrm{Ker}(DG) = \langle S_1^{\pm}, \dots, S_n^{\pm} \rangle \longrightarrow \langle T_1^{\pm}, \dots, T_m^{\pm} \rangle \subseteq D^b(\Lambda\text{-mod})$$

(iv) $\mathcal{A} f \mathcal{A} = \begin{pmatrix} \mathcal{I} & \widetilde{\mathcal{O}} \\ \mathcal{I} & \widetilde{\mathcal{O}} \end{pmatrix} \subseteq \mathcal{A} = \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}} \\ \mathcal{I} & \widetilde{\mathcal{O}} \end{pmatrix}$

$\Gamma = \Gamma(E, \mathcal{A}/\mathcal{A}f\mathcal{A}) \simeq \Gamma(E, \mathcal{O}/\mathcal{I}) = \underbrace{k \times \dots \times k}_{n}$

$\widetilde{G} = \mathrm{Hom}_{\mathbb{E}}(Q, -)$ (recall: $Q = \mathcal{A}f$)

$\widetilde{F} = Q \otimes_{\mathcal{B}} -$.

$$D^b(\Gamma\text{-mod}) \rightleftarrows D^b(\mathrm{Coh}\,\mathbb{E}) \underset{}{\overset{L\widetilde{F}}{\rightleftarrows}} D^b(\mathrm{Coh}\,\widetilde{E})$$

$\Gamma \longmapsto R \in \mathrm{Tor}(\mathbb{E})$, $R = R_1 \oplus \dots \oplus R_n$

$$0 \to \begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix} \to \begin{pmatrix} \mathcal{O} \\ \mathcal{I} \end{pmatrix} \to R \to 0.$$

We have a semi-orthogonal decomposition:

$$D^b(\mathrm{Coh}\,\mathbb{E}) = \langle R, \mathrm{Im}(L\widetilde{F}) \rangle$$

$\widetilde{E} = \mathbb{P}^1 \sqcup \dots \sqcup \mathbb{P}^1$.

$\mathrm{Coh}(\mathbb{P}^1) \ni \mathcal{C} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.

$B^\bullet = R[-1] \oplus L\widetilde{F}(\mathcal{C}_1 \oplus \dots \oplus \mathcal{C}_n)$ is a tilting complex.

$\Lambda = \mathrm{End}(B^\bullet)^{op} \simeq \Lambda_n$.

$$\Rightarrow D^b(\mathrm{Coh}\,\mathbb{E}) \xrightarrow[\text{[Keller]}]{\simeq} D^b(\Lambda\text{-mod})$$

<u>Summary</u>:

$$\begin{array}{ccccc} \mathrm{Perf}(E) & \overset{\mathbb{Y}}{\hookrightarrow} & D^b(\mathrm{Coh}\,\mathbb{E}) & \xrightarrow{\simeq} & D^b(\Lambda\text{-mod}) \\ \mathbb{I} \downarrow & \swarrow \mathbb{P} & & & \\ D^b(\mathrm{Coh}\,E) & & & & \end{array}$$

$$(v)\quad \Omega_\Lambda = \mathrm{RHom}_E(\mathcal{A}, \Omega_E) \simeq \mathrm{Hom}_E(\mathcal{A}, \mathcal{O}) \simeq \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}} \\ I & I \end{pmatrix}$$

(annotations: $\Omega_E \overset{cy/2}{=} \mathcal{O}_E$; $\mathcal{A}$ — sheaf of orders)

$$\Rightarrow X \simeq \tau(X) \Leftrightarrow X \in \mathrm{Im}(\mathbb{Y}) \quad \left(\text{recall } \tau = \Omega_\Lambda \overset{\mathbb{L}}{\otimes} -\right)$$

<u>Corollary</u>
$\mathbb{Y}$ identifies 1-spherical objects of $\mathrm{Perf}\,E$ and $D^b(\Lambda\text{-mod})$.

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$\Lambda$ is gentle $\rightsquigarrow$ derived invariant of Avella-Alaminos & Geiß.

$\mathcal{P} = \{b_1a_1, \dots, b_na_n;\ d_1c_1, \dots, d_nc_n\}$ maximal paths

$\mathcal{F} = \{c_1b_1, \dots, c_nb_n;\ a_1d_1, \dots, a_nd_n\}$ maximal forbidden threats.

<u>Claim</u>: $\mathcal{P} \underset{\sigma_+}{\overset{\sigma_-}{\rightleftarrows}} \mathcal{F}$ ; $\quad \mathcal{F} \longrightarrow \mathcal{F} \qquad \sigma = \sigma_- \sigma_+$.

<u>Example</u>

$$\underset{\mathcal{F}}{c_1b_1} \overset{\sigma_+}{\longmapsto} \underset{\mathcal{P}}{b_1a_1} \overset{\sigma_-}{\longmapsto} \underset{\mathcal{F}}{a_nd_n} \overset{\sigma_+}{\longmapsto} \underset{\mathcal{P}}{d_nc_n}$$

$$\left( (c_1b_1, a_nd_n), \dots, (-,-) \right) = \sigma$$

$\quad\downarrow (2,4)$

**AA – G function**

$\phi_\Lambda : (\mathbb{N}_0 \times \mathbb{N}_0) \to \mathbb{N}_0$ finite support, $\Phi_\Lambda(i,j) = \begin{cases} n, & (i,j) = (2,4) \\ 0 & \text{otherwise} \end{cases}$

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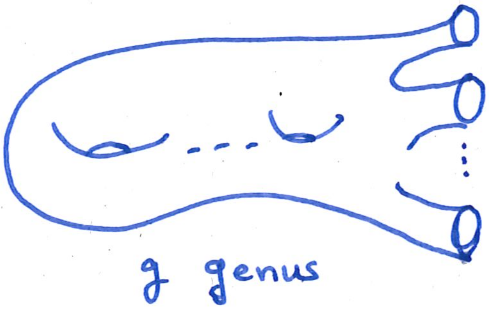
Gentle algebra $\Lambda \rightsquigarrow (\Sigma, M, [\eta])$.

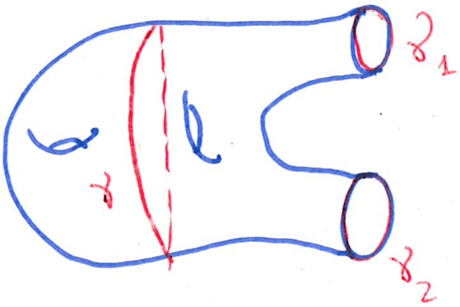
* $\Sigma \equiv$ compact oriented surface with boundary

$= \Sigma_{g,b} =$



$b := \#$ boundary components

* $M \subseteq \partial\Sigma$ finite subset $gl.dim(\Lambda) < \infty \Leftrightarrow \partial_i \Sigma \cap M \neq \emptyset, \ \forall 1 \leq i \leq b$.
* Bocklandt, Haiden–Katzarkov–Kontsevich $\rightsquigarrow$ topological Fukaya cat.
* Lekili–Polishchuk : Homological mirror symmetry.
* Opper–Plamondon–Schroll : geometric model for $D^b(\Lambda\text{-mod})$.



$\omega_\eta(\gamma), \omega_\eta(\delta_1), \dots, \omega_\eta(\delta_b) \in \mathbb{Z}$.
(winding numbers)

**Index theorem of Poincaré–Hopf**

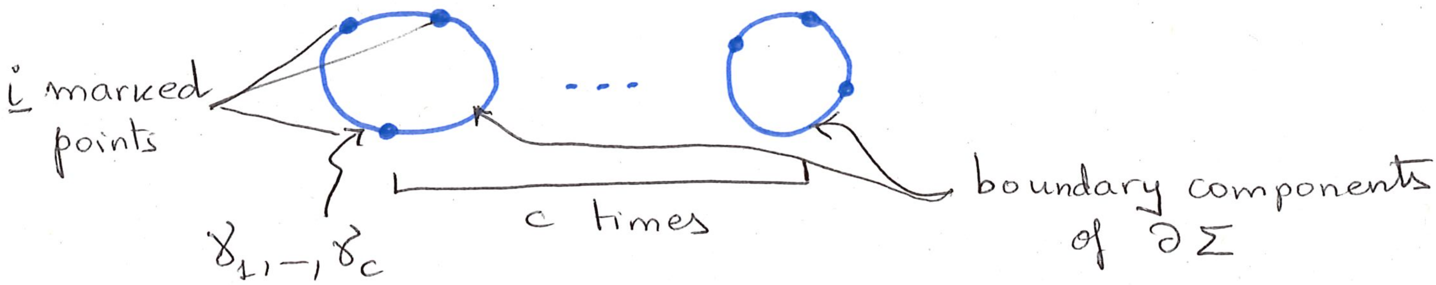
$$\sum_{i=1}^{b} \omega_\eta(\delta_i) = 2\chi(\Sigma) \quad \left(\Sigma := \Sigma_{g,b}\right)$$

$$\chi(\Sigma_{g,b}) = 2(1-g) - b.$$

**THEOREM** (Lekili–Polishchuk)
Let $\Lambda$ be a gentle algebra, $gl.dim(\Lambda) < \infty$ and $(\Sigma_{g,b}, M, \eta)$ be the corresponding geometric datum

(i) $b = \sum_{(i,j) \in \mathbb{N}_0^2} \phi_\Lambda(i,j)$; $\chi(\Sigma) = |Q_0| - |Q_1|$

(ii) $\forall (i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$ : $c = \phi_\Lambda(i,j) > 0$ we get




$$\omega_\eta(\gamma_k) = i - j, \quad \forall\, 1 \le k \le c.$$

$\Rightarrow$ * $(\Sigma_{g,b}, M)$ can be recovered from $\phi_\Lambda$.

* $|\Sigma_1|$ is a derived invariant of $\Lambda$.

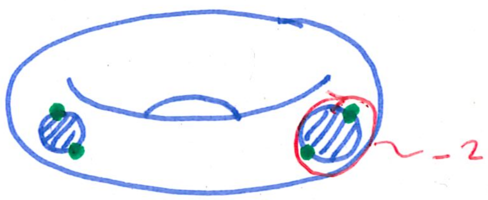
## Example

$$\phi_\Lambda(i,j) = \begin{cases} n, & (i,j) = (2,4) \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda = \Lambda_n, \quad -2n = \sum_{i=1}^{n} \omega_\eta(\delta_i) \underset{\text{Poincaré–Hopf}}{=} 2\chi(\Sigma) = 4(1-g) - 2\underset{n}{\underset{\shortparallel}{b}}$$

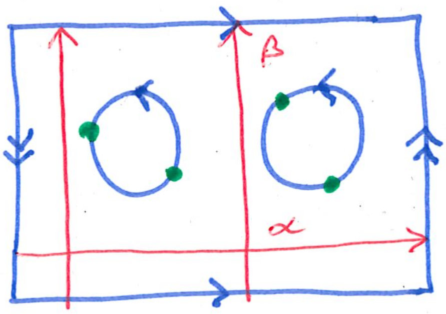
$\Rightarrow g = 1.$

$\Rightarrow \Sigma = \Sigma_{1,n} =$




$; \quad |M \cap \partial_i \Sigma| = 2, \ \forall i.$

## Extra computation




$\omega_\eta(\alpha) = 0 = \omega_\eta(\beta), \quad \Sigma = \mathbb{R}^2/\mathbb{Z}^2$

$\eta \sim \frac{\partial}{\partial x} \sim \frac{\partial}{\partial y}.$

# Homological mirror symmetry after Lekili – Polishchuk

$D^b(\mathrm{Coh}\,\mathbb{E}) \xrightarrow[\text{Burban–Drozd}]{\simeq} D^b(\Lambda\text{-mod}) \xleftarrow[\text{Haiden Katzarkov Kontsevich}]{\simeq} \mathcal{T}\mathrm{Fuk}(\Sigma, M, \eta)$

$D^b(\mathrm{Coh}\,E) \xrightarrow[\simeq]{\text{Lekili – Polishchuk}} \mathcal{W}\mathrm{Fuk}(\Sigma, \omega)$

$\mathrm{Perf}\,E \xrightarrow[\simeq]{\text{Lekili – Polishchuk}} \mathrm{Fuk}(\Sigma, \omega)$, $\omega$ = dual

<u>Q</u>: Do we have holomorphic mirrors for $\Sigma_{g,n}$, $g \geq 2$?

<u>Lekili – Polishchuk</u>: stacky chains / cycles of proj. lines $\dfrac{\mathbb{k}[[x,y]]}{xy}$

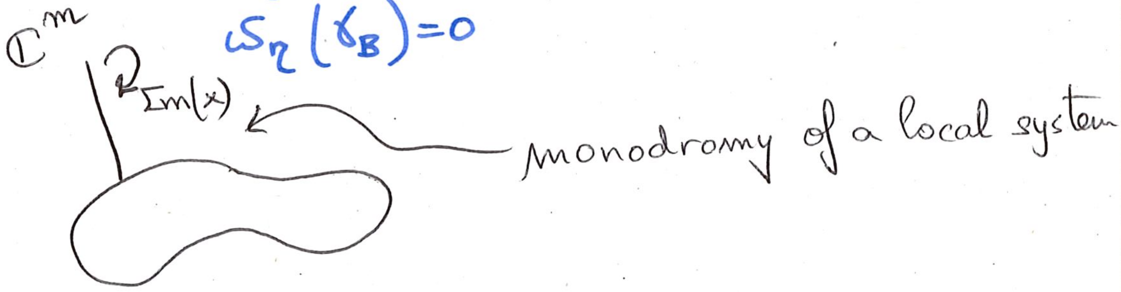
<u>Burban – Drozd</u>: tame non-comm. nodal curves $\dfrac{k[[x,y]] * \mathbb{Z}_n}{xy}$ $\cap$ nodal curves

<u>Problem</u>: $D^b(\mathrm{Coh}\,E) \simeq D^b(\Lambda\text{-mod})$ ($\Lambda$ gentle), $\boxed{\omega_\eta(\delta_i) \leq -1}$, $\forall\, 1 \leq i \leq b$.

1-spherical objects in $\mathrm{Perf}(E)$ = 1-spherical objects in $D^b(\Lambda\text{-mod})$

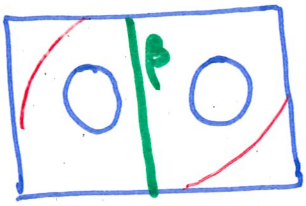
$D^b(\Lambda\text{-mod}) \ni$ bands / strings $\longleftrightarrow$ Bekkart–Merklen $B(\omega, m, \lambda)$
- $\omega$ discrete parameter
- $m \in \mathbb{N}$
- $\lambda \in k^*$

homotopy classes — closed curve $\gamma_B$ on $\Sigma$, $\omega_\eta(\gamma_B) = 0$ [OPS]



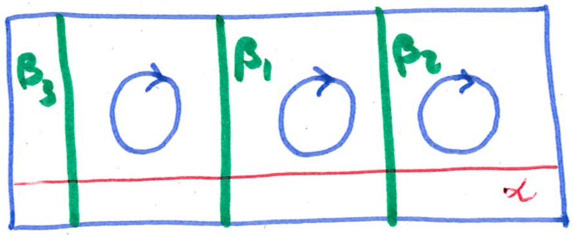
B is spherical $\Leftrightarrow$ $\gamma_B$ is simple ($\equiv$ no self-intersections).

$\omega_\eta(\gamma) = 0$ $\not\Rightarrow$ $\gamma$ is non-separating
$\Sigma \setminus \gamma$ is connected




Topological fact: $\gamma \xrightarrow{\text{diffeom.}} \alpha$ even by an element of pure mapping class gp.

Known: in the case of torus, mapping class group is generated by Dehn-twists along $\alpha, \beta_1, \dots, \beta_n$




$D^b(\Lambda\text{-mod})$

$\cup\!|$

$\text{Bands} = \{X \mid \tau(X) \simeq X\} \curvearrowleft \langle T_0, T_{k_{p_1}}, \dots, T_{k_{p_n}} \rangle$ — torus $\Sigma_{1,n}$ — $\{\text{closed curves}\} / \text{homotopy}$

$\simeq \uparrow$

Perf E

Brutal force: Dehn twists along $\alpha, \beta_1, \dots, \beta_n$.

$\Rightarrow$ transitivity of the action of the group of auto-equival. of $D^b(\text{Coh } E)$ on the set of spherical objects.

$\Longrightarrow$ POLISHCHUK'S CONJECTURE.