Talk given 11.12.2019 by Karin M. Jacobsen

Introduction to gentle algebras II: Geometric model(s)

[OPS] = Opper, Plamondon, Schroll: A geometric model for the derived category of gentle algebras arXiv:1801.09659

[LP] = Lekili, Polischuk: Derived equivalences of gentle algebras via Fukaya categories arXiv:1801.06370

This is really a plan for our lecture, too

Setup: k is a field (1)

all modules are fin. dim left modules

$ba = \cdot \xrightarrow{a} \cdot \xrightarrow{b} \cdot$

1a: Geometric model for gentle algebras [OPS]

Def: An algebra A is locally gentle if it is isomorphic to one of the form kQ/I such that:

1 Q is a finite quiver
2 I is admissible ($R^m \subseteq I \subseteq R^2$)
3 I is generated by paths of length two
4 For any arrow a there is at most one arrow
- b such that $ab \in I$
- c " " $ca \in I$
- b " " $ab \notin I$
- c " " $ca \notin I$

A is gentle if it is in addition

Example Q

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \arrow[ll, "c"] \end{tikzcd}$$

$I = \{ba, cb, ac\}$

kQ and kQ/I are both locally gentle.
However, only kQ/I is gentle

1.1 Ribbon graphs

Def: For $A = kQ/I$ locally gentle, let

$\cancel{T}$ P: Set of permitted threads, i.e. paths $w \notin I$ (in Q) such that for any arrow a we have $aw \in I$ and $wa \in I$.

F: Set of forbidden threads: paths $w = a_n \cdots a_1$ such that $a_{i+1} a_i \in I$, and there is no arrow b with either $b a_n \in I$ or $a_1 b \in I$

$\cancel{T}_0$ P_0: Set of trivial paths e_i such that $\longrightarrow i$, $i \longrightarrow$, or $\xrightarrow{a} i \xrightarrow{b}$ with $ba \notin I$

F_0: As P_0, but have $ba \in I$

$\overline{P} = P \cup P_0$ and $\overline{F} = F \cup F_0$.

Note: Each vertex appears in exactly two members of $\overline{P}$ if A is gentle

Example Q and I as before.

For kQ/I: $\overline{T} = \{a, b, c\}$

$\overline{F} = \{e_1, e_2, e_3\}$ (all others end up infinite!)

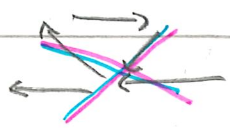
For kQ: $\overline{T}' = \overline{F}$, $\overline{F}' = \overline{T}$ (these are Koszul, cf Jan's talk)

imprecise, but good enough for our purposes.

"Def" A ribbon graph is a undirected graph with a cyclic ordering σ of the edges around each vertex

From now on (and until I say otherwise) $A = kQ/I$ will be <u>gentle</u>, in particular: fin dim.

To simplify the next definition(s) we will assume that no $w \in \overline{T}$ has a vertex occurring more than once:

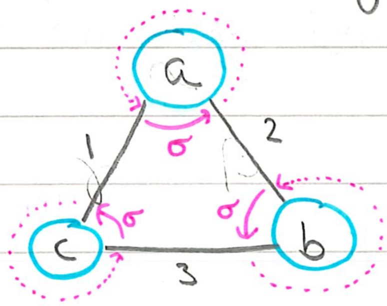


(the def. would still work, but it would be more cumbersome)

Def: Given a gentle algebra $A = kQ/I$, define a ribbon graph Γ_A:

- **vertices**: $\overline{\overline{P}}$
- **edges**: add edges whenever two $w \in \overline{\overline{P}}$ pass through the same vertex of Q
- **ordering** σ: induced by the order $w \in \overline{\overline{P}}$ passes through the vertices of Q

Ex: $A = kQ/I$ as before $\overline{\overline{P}} = \{a, b, c\}$



1.2. Ribbon Surfaces

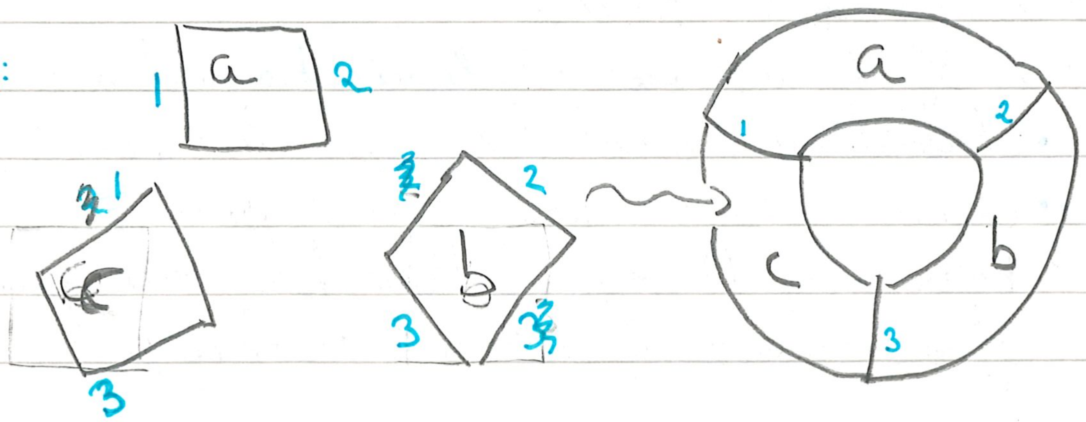
Def Let Γ be a connected ribbon graph. We associate to Γ a **ribbon surface** S_Γ through the following construction:

1 For each vertex $v \in \Gamma$, with valency $d(v)$ $= \#\text{arrows in} + \#\text{arrows out}$ associate a $2d(v)$-gon Q_v

2 Following the orientation, label every other side of Q_v with the edges incident to v.

3 Glue together sides with the same label

Ex:



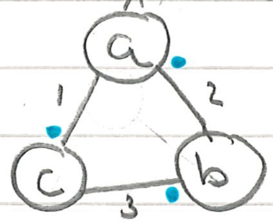
Note: * The ribbon surface S_Γ is oriented
* We can embed Γ in S_Γ: Vertices = centers of polygons
Edges = curves between ?
* If $\Gamma = \Gamma_A$, we write $S_\Gamma = S_A$. **Ex**:

1.3 Markings

Def: A **marked ribbon graph** is a ribbon graph with a map m mapping each vertex to one of its incident edges.

For Γ_A, we define $m(v) = t(v)$ (mark end of path

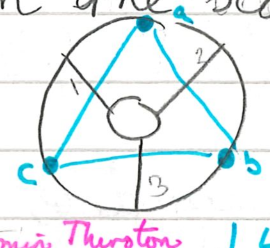
Ex



For S_Γ, we create a marking by putting a mark on the side of Q_v between $m(v)$ and $\sigma(m(v))$;

$\Rightarrow$ Marked surface (S, M)

Ex:



Note: Quite different from Marked surfaces in sense of Assem et al, cf. Bill's talk.

Fomin Thurston

1.4 Lamination

Def: Let (S, M) be a marked surface. A **lamination** $\mathcal{L}$ on (S, M) is a finite set of non-selfintersecting and pairwise non-intersecting curves, considered up to isotopy relative to M.

Elements of $\mathcal{L}$ are called **laminates** and are either

- closed curves not homotopic to a point or
- or curves between non-marked points on the boundary. Disregard those hom to a part of the boundary without marked points

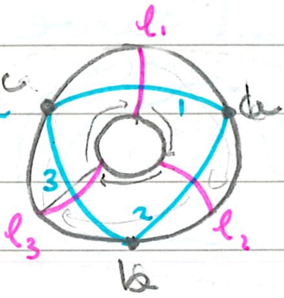
Proposition (Canonical lamination)

For the finite dimensional gentle algebra $A = kQ/I$, there is a unique lamination $\mathcal{L}_A$ on S_A s.t

1. $\mathcal{L}$ contains no closed loops
2. For every vertex $i \in Q_0$ (= edge of Γ_A) $\exists!\ \ell_i \in \mathcal{L}$ that crosses the curve labeled by i in the embedding of Γ_A once, and none other
3. (2) describes all laminates in $\mathcal{L}_A$

"Proof" Reuse the „seams" from the construction of S_A

Example



Prop: Let $A = kQ/I$ be gentle, $\mathcal{L}_A$ its canonical lamination

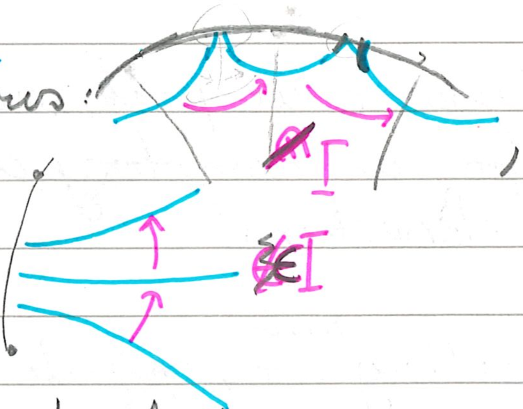
Define $Q_{\mathcal{L}}$ as follows:

vertices laminates in $\mathcal{L}_A$

arrows

Define $I_{\mathcal{L}}$ as follows:

but



Then $kQ_{\mathcal{L}}/I_{\mathcal{L}} \simeq kQ/I$ as k-algebras.

Lemma 2.6: 1 $\mathcal{L}_A$ divides S_A into polygons whose sides are laminates and boundary segments

2 Each such polygon contains exactly one marked point

(3 Each boundary segment of S_A contains endpoint of at least one lamination)

2 Derived categories [OPS]

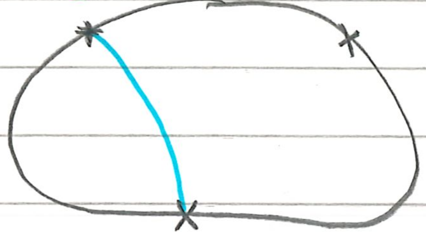
Aim: Create a model for $D^b(A)$ using the ribbon surface S_A

"Def" Let A be gentle, Γ_A ribbon graph, S_A ribbon surface with marked points $\mathcal{M}$ (from embedding), $\mathcal{L}_A$ canonical lamination.

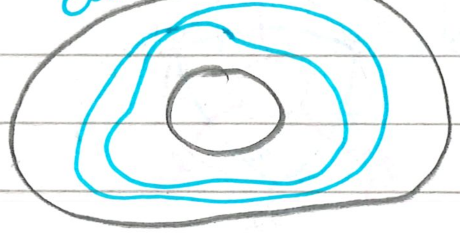
• Let all curves in the following be non-contractible; everything is up to homotopy

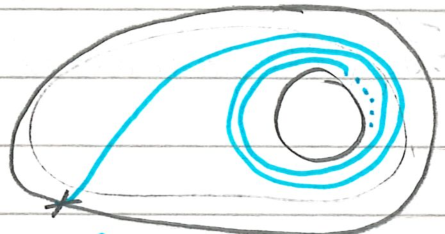
A ~~closed~~ ~~finite arc~~ =

arc:

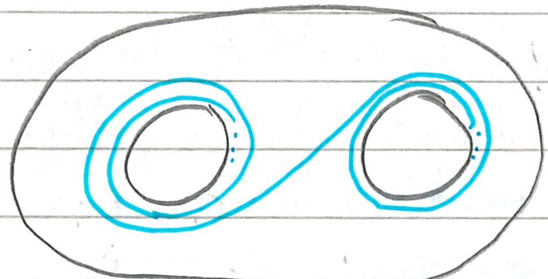


closed curve





infinite arc



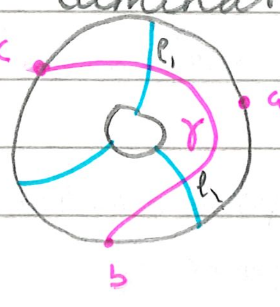
infinite arc.

Assume we pick „minimal" representations for our homotopy classes: No unnecessary intersections!

Lemma: Any finite arc, infinite arc or closed curve γ is uniquely determined by the sequence of laminates it crosses, $\mathcal{L}_\gamma$

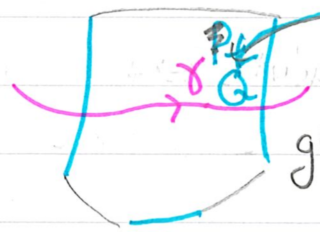
ordered multi set w/ well def. „previous" & „next" crossings

Ex:



$\mathcal{L}_A = (l_1, l_2)$

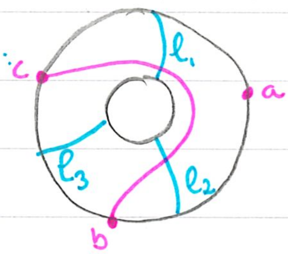
Def: γ arc/closed curve on S_A.
A **grading** on γ is a function $g : \mathcal{L}_\gamma \to \mathbb{Z}$ satisfying:
Let $\ell \in \mathcal{L}_\gamma$, and ℓ' the next crossing:



$$g(\ell') = \begin{cases} g(\ell)+1 & \text{if } X \text{ is left of } \gamma \\ g(\ell)-1 & \text{" " "right" "} \end{cases}$$

(γ, g) is a **graded curve.**

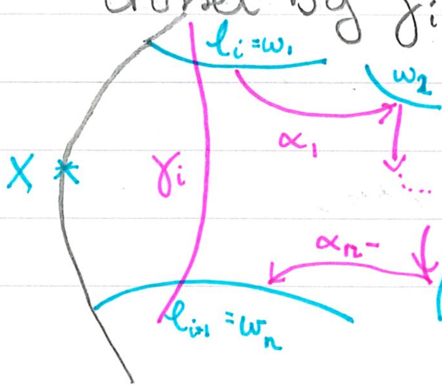
Ex:

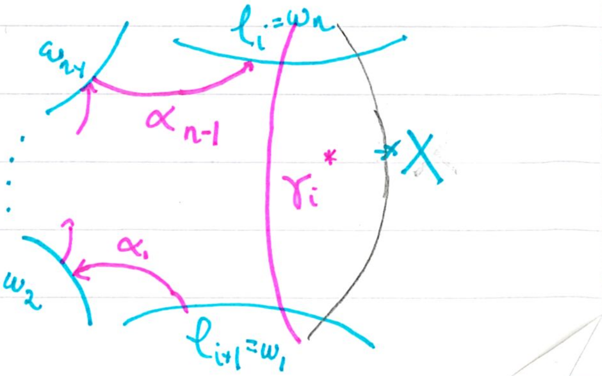


$\mathcal{L}_\gamma = (\ell_1, \ell_2)$
$g(\mathcal{L}_\gamma) = (0, 1)$

Note:
* Any grading is defined by its value on a single element of $\mathcal{L}_\gamma$
* If g is a grading of γ, then so is $g[n] : \ell \mapsto g(\ell) - n$
* All gradings of γ can be written as $g[n]$
* finite & infinite arcs always have gradings
* Closed curves don't!

Construction Let (γ, g) be a finite graded arc on S_A.
Let $\mathcal{L}_A = (\ell_1, \ell_2, \dots \ell_r)$
For $1 \le i \le r-1$, let γ_i be the part of γ between ℓ_i and ℓ_{i+1}.
Look at the laminates in the polygon Q crossed by γ_i



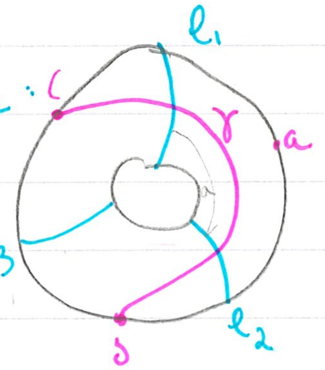


$\sigma(\gamma_i) = \gamma_{n-1}, \ldots, \gamma_1 \qquad \sigma(\gamma_i^*) = (\gamma_{n-1}, \ldots, \gamma_1)^{-1}$.

$$\sigma(\gamma) = \begin{cases} \prod \sigma(\gamma_i) & \text{if } r > 1 \\ e_{\ell_1} & \text{if } r = 1 \end{cases}$$

this is a string (you don't need to know what that means).

Example:



$\mathcal{L}_\gamma = (\ell_1, \ell_2)$
$\sigma(\gamma) = a$

Construction (continued)

Define $P^\bullet_{(\gamma,g)} = \ldots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to \ldots$ by $\forall j \in \mathbb{Z}$,

$$P^j = \bigoplus_{\substack{0 \le i < r \\ g(\ell_i) = j}} P_{\ell_i}$$ ← indec. proj. corr. to vertex labeled ℓ_i

Each direct (resp. inverse) $\sigma(\gamma_i)$ defines a map $P_{\ell_i} \to P_{\ell_{i+1}}$ (resp. $P_{\ell_{i+1}} \to P_{\ell_i}$); these maps form the differential in $P^\bullet(\gamma, g)$.

Similarly, we can create complexes of projectives for graded infinite arcs, and (somewhat less similarly) for graded closed curves.

The objects $P^\bullet_{(\gamma,g)}$ are indecomposable in $D^b(\text{mod } A)$. In fact...

Theorem [OPS] The following classes are in bijection:

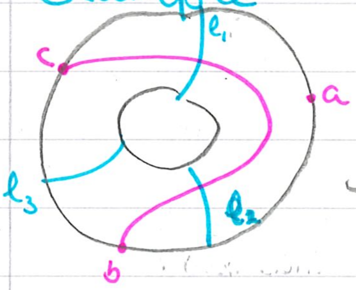
$\{\text{Indecomposable objects in } D^b(\text{mod } A)\}/\text{iso}$

and

$\left\{\begin{array}{l}\text{graded finite arcs,} \\ \text{graded infinite arcs} \\ \text{graded closed curves in } S_A\end{array}\right\}$

"Proof": Show that the latter class is in bijection with Bekkert & Merklens graded homotopy strings & bands. "□"

Example:



$L_\gamma = (\ell_1, \ell_2)$

$\sigma(\gamma) = a \quad g(\gamma) = (0,1)$

$P^\bullet_{(\gamma,g)} = 0 \to P^0 \to P^1 \to 0$

where $P^0 = P_{\ell_1} = P_1$

$P^1 = P_{\ell_2} = P_2$

$\Rightarrow P^\bullet_{(\gamma,g)} = 0 \to P_1 \xrightarrow{(a)} P_2 \to 0$

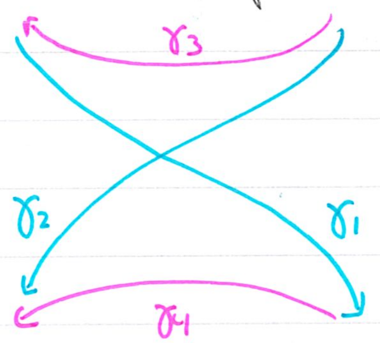
Note: The geometric model of $D^b(\text{mod } A)$ also gives us:

(1) **maps**: The basis for $\text{Hom}_{D^b(\text{mod} A)}(P^\bullet_{(\gamma_1, g_1)}, P^\bullet_{(\gamma_2, g_2)})$ is in bijection with the set of graded intersections (i.e. intersections where grading agrees) of (γ_1, g_1) and (γ_2, g_2) (unless $\gamma_1 = \gamma_2$ closed curve).

This follows from [Arnesen, Laking, Pauksztello '16], which found Hom_{D^b} for string & band objects

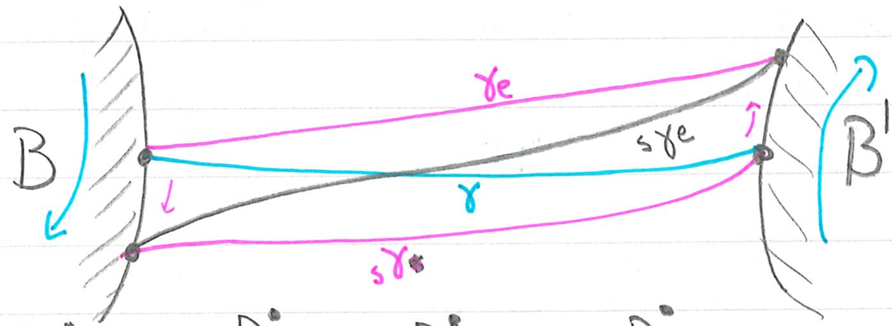
Mapping cones

(2) Suppose we have a basis map $\varphi \in \mathrm{Hom}_{D^b(\mathrm{Mod})}(P^\bullet_{(\gamma_1,g_1)}, P^\bullet_{(\gamma_2,g_2)})$
$\rightarrow$ corresponds to a crossing of γ_1, γ_2



(γ_3, g_3) and (γ_4, g_4) are arcs or closed curves given by the **resolution of the crossing**
$\rightsquigarrow M^\bullet_\varphi = P^\bullet_{(\gamma_3,f_3)} \oplus P^\bullet_{(\gamma_4,f_4)}$ is the mapping cone of φ

(3) AR-triangles: Take a (graded) arc $(\gamma; g)$.



$P^\bullet_{(\gamma,g)} \longrightarrow P^\bullet_{(\gamma e, g e)} \oplus P^\bullet_{(s\gamma, s g)} \longrightarrow P^\bullet_{(s\gamma e, s g e)} \longrightarrow P^\bullet_{(\gamma,g)}[1]$
is an AR-triangle

gradings are inherited from γ

(We can do something similar for bands)

3. Homologically smooth algebras

$\rightsquigarrow$ no infinite forbidden paths

- [Green-Zachariah]: A gentle algebra A is Koszul, and its dual $A^!$ is gentle or locally gentle.
- Recall that Jan identified $A = kQ/I$ with (Q, P, F) (permitted). Its Koszul dual $A^!$ was identified with (Q^{op}, F^{op}, P^{op}) (forbidden).
- By [Bessenrodt-Holm] if A is finite dimensional, then $A^!$ is homologically smooth.

Via a theorem of Keller, [LP] show that
$$\underline{D}_f(A) \simeq \underline{D}(A^!)$$
where:

- A is viewed as gentle, graded in degree 0
- $A^!$ has path length grading
- NB! $\underline{D}(A^!)$ is the perfect derived category of $A^!$ viewed as a dg algebra with zero differential.

However, we can do more!

Let $A = A = {}^{kQ}/_{I}$ be a graded, homologically smooth locally gentle algebra

Model of the algebra

Construct a ribbon graph, then surface, using using the forbidden paths of A (so far, ignore grading). Don't forget to get a lamination!

Model of grading:

Recall from Jan's talk: • A boundary component of A is a cyclic path in $\overline{Q} = Q \cup \overline{Q}^{op}$, consisting of alternating forbidden & opposite permitted threads avoiding "$a\bar{a}$" (a is a letter)

• Boundary components of A are in 1-1 correspondance with boundary components in S_A.

Grading of a permitted thread $p = \alpha_n \dots \alpha_1 \in kQ$:
$$|p| = -\sum_{i=1}^{n} |\alpha_i|$$
Grading of a forbidden thread $f = \alpha_n \dots \alpha_1 \in kQ$:
$$|f| = \sum_{i=1}^{n} |\alpha_i| - (n-2)$$

To each boundary component θ in S_A, corr. to a boundary component $f_n p_n \cdots f_1 p_1$ of A, we attach a winding number

$$w(\theta) = \sum_{i=1}^{n} |f_i| + |p_i|$$

With a winding number attached to each component in this way we can:

- Read off the grading (& thus all information about the algebra) from the surface.
- Define a line field η on S_A, which we will need to get the correspondence with the partially wrapped Fukaya category....

but that will be in David's talk.