REPRESENTATIONS OF THE KRONECKER QUIVER

ANDREW HUBERY

1. The Kronecker quiver

We consider the Kronecker quiver $K\colon 1 \rightrightarrows 2$.

A representation $X = (U, V; A, B)$ of K over a field k is given by a pair of (finite dimensional) vector spaces U, V and a pair of linear maps $A, B\colon A \to B$; we also write this graphically as $X\colon U \overset{A}{\underset{B}{\rightrightarrows}} V$. A morphism $X \to X'$ between two such representations if given by a pair of linear maps $f\colon U \to U'$ and $g\colon V \to V'$ such that both squares below commute

$$\begin{array}{ccc} U & \xrightarrow{A} & V \\ \downarrow f & & \downarrow g \\ U' & \xrightarrow{A'} & V' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} U & \xrightarrow{B} & V \\ \downarrow f & & \downarrow g \\ U' & \xrightarrow{B'} & V' \end{array}$$

These form an abelian category denoted $\operatorname{rep}_k K$, which is equivalent to the category of left modules over the path algebra

$$kK := \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$$

There are two simple objects: the simple injective $S_1 = I(0) = (k, 0; 0, 0)$ and the simple projective $S_2 = P(0) = (0, k; 0, 0)$. The Grothendieck group of the category is therefore isomorphic to $\mathbb{Z}^2$, with basis $e_1 = \underline{\dim}\, S_1$ and $e_2 = \underline{\dim}\, S_2$. Given a representation $X = (U, V; A, B)$ we write $\underline{\dim}\, X = (\dim U, \dim V)$ for its image in the Grothendieck group.

The category $\operatorname{rep}_k K$ is hereditary, so $\operatorname{Ext}^i(X, Y) = 0$ for all X, Y and all $i \geq 2$. Thus the Euler form of the category is given by

$$\langle X, Y\rangle := \dim \operatorname{Hom}(X, Y) - \dim \operatorname{Ext}^1(X, Y),$$

and this descends to a bilinear form on the Grothendieck group. With respect to the standard basis this is represented by the matrix

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

We also have the symmetric bilinear form $(x, y) = \langle x, y\rangle + \langle y, x\rangle$ on $\mathbb{Z}^2$. Thus

$$x = (a, b) \quad \text{implies} \quad (x, x) = s(a - b)^2,$$

so this symmetric bilinear form is positive semi-definite, with radical generated by $\delta := (1, 1)$.

A useful concept for a representation is then the defect ∂, where

$$\partial(X) = \langle \delta, \underline{\dim}\, X\rangle.$$

1.1. **Duality.** The vector space duality $D = \operatorname{Hom}_k(-, k)$ induces a duality on the category $\operatorname{rep}_k K$

$$D\colon\quad U \underset{B}{\overset{A}{\rightrightarrows}} V \quad\mapsto\quad D(V) \underset{D(B)}{\overset{D(A)}{\rightrightarrows}} D(U)$$

This swaps the entries of the dimension vector, and hence changes the sign of the defect

$$\underline{\dim}\, X = (a, b) \Rightarrow \underline{\dim}\, D(X) = (b, a), \qquad \text{and} \qquad \partial(D(X)) = -\partial(X).$$

1.2. **Reflection functors.** We introduce two endofunctors $S^{\pm}$ of the category of representations. The functor S^+ is given by a pull-back construction:

$$S^+\colon\quad U \underset{B}{\overset{A}{\rightrightarrows}} V \quad\mapsto\quad T \underset{B'}{\overset{A'}{\rightrightarrows}} U \qquad \text{given by the pull-back} \qquad \begin{array}{ccc} T & \xrightarrow{A'} & U \\ \downarrow{\scriptstyle B'} & & \downarrow{\scriptstyle B} \\ U & \xrightarrow{A} & V \end{array}$$

Dually the functor S^- is given by a push-out construction:

$$S^-\colon\quad U \underset{B}{\overset{A}{\rightrightarrows}} V \quad\mapsto\quad V \underset{B'}{\overset{A'}{\rightrightarrows}} W \qquad \text{given by the push-out} \qquad \begin{array}{ccc} U & \xrightarrow{A} & V \\ \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B'} \\ V & \xrightarrow{A'} & W \end{array}$$

Lemma 1.1. *We have $DS^+ \cong S^-D$.* □

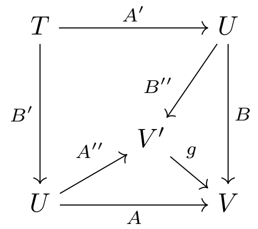
Theorem 1.2. *The reflection functors form an adjoint pair, so we have an isomorphism*

$$\operatorname{Hom}(S^-X, Y) \cong \operatorname{Hom}(X, S^+Y)$$

which is natural in both X and Y.

Moreover, the unit of the adjunction $X \to S^+S^-X$ is an epimorphism, with kernel some number of copies of the simple injective $I(0)$, and the counit of the adjunction $S^-S^+X \to X$ is a monomorphism, with cokernel some number of copies of the simple projective $P(0)$.

Proof. That we have an adjoint pair follows immediately from the universal properties of the pull-back and push-out. To compute the unit, consider a representation $X\colon U \underset{B}{\overset{A}{\rightrightarrows}} V$. Then $S^+S-(X) = \ U \underset{B''}{\overset{A''}{\rightrightarrows}} V'$ is given by the commutative diagram



The map g is necessarily injective, so $(1, g)$ defines a monomorphism $X \to S^+S^-X$. The cokernel is given by

$$0 \rightrightarrows V/V'$$

which is easily seen to be isomorphic to $P(0)^n$ where $n = \dim V/V'$. The result for the counit is dual. □

Corollary 1.3. *Let X be indecomposable and set $\underline{\dim}\, X = (a, b)$.*

(1a) *If X is simple projective, then $S^+(X) = 0$.*
(1b) *If X is not simple projective, then*
 (i) *$S^+(X)$ is indecomposable.*
 (ii) *$\underline{\dim}\, S^+(X) = (2a - b, a)$.*
 (iii) *$X \cong S^- S^+ X$.*

Dually we have

(2a) *If X is simple injective, then $S^-(X) = 0$.*
(2b) *If X is not simple injective, then*
 (i) *$S^-(X)$ is indecomposable.*
 (ii) *$\underline{\dim}\, S^-(X) = (b, 2b - a)$.*
 (iii) *$X \cong S^+ S^- X$.*

We consider the full additive subcategory $\underline{\mathrm{rep}}_k\, K$ generated by all indecomposables except the simple projective $P(0)$. Dually let $\overline{\mathrm{rep}}_k\, K$ be the full additive subcategory generated by all indecomposables except the simple injective $I(0)$.

Proposition 1.4. *The subcategory $\underline{\mathrm{rep}}_k\, K$ is closed under extensions and taking quotients; dually $\overline{\mathrm{rep}}_k\, K$ is closed under extensions and taking subobjects. Moreover, the reflection functors restrict to give a mutually inverse equivalances*

$$S^+ \colon \underline{\mathrm{rep}}_k\, K \xrightarrow{\sim} \overline{\mathrm{rep}}_k\, K \quad \text{and} \quad S^- \colon \overline{\mathrm{rep}}_k\, K \xrightarrow{\sim} \underline{\mathrm{rep}}_k\, K.$$

Proof. Since $P(0)$ is simple projective, we have

$$\underline{\mathrm{rep}}_k\, K = \{X : \mathrm{Hom}(X, P(0)) = 0\}.$$

It is then clear that $\underline{\mathrm{rep}}_k\, K$ is closed under extensions and quotients. Moreover, $\underline{\mathrm{rep}}_k\, K$ is the essential image of S^-. For, given $Y \in \mathrm{rep}_k\, K$ we have

$$\mathrm{Hom}(S^- Y, P(0)) \cong \mathrm{Hom}(Y, S^+ P(0)) = 0,$$

so $S^- Y \in \underline{\mathrm{rep}}_k\, k$, and conversely if $X \in \underline{\mathrm{rep}}_k\, K$, then $S^- S^+ X \cong X$.

Dually, $X \in \overline{\mathrm{rep}}_k\, K$ if and only if $\mathrm{Hom}(I(0), X) = 0$, so this subcategory is closed under extensions and subrepresentations, and is the essential image of S^+.

It is now clear that $S^\pm$ induce mutually inverse equivalences between $\underline{\mathrm{rep}}_k\, K$ and $\overline{\mathrm{rep}}_k\, K$, and so in particular preserve exact sequences. □

2. Classification of representations

By the Krull-Remak-Schmidt Theorem, every representation is a finite direct sum of indecomposable representations in an essentially unique way. We therefore wish to classify the indecomposable representations over a field k.

Consider the ring of polynomials $k[s,t] = \bigoplus_{d \geq 0} V_d$, graded according to total degree. For convenience we also set $V_{-1} = 0$.

For $d \geq 0$ we define

$$P(d) \colon V_{d-1} \overset{s}{\underset{t}{\rightrightarrows}} V_d \qquad \underline{\dim}\, P(d) = (d, d+1), \qquad \partial(P(d)) = -1$$

where the two maps are multiplication by s and t; we also set $I(d) := D(P(d))$, so

$$I(d) \colon D(V_d) \overset{D(s)}{\underset{D(t)}{\rightrightarrows}} D(V_{d-1}) \qquad \underline{\dim}\, I(d) = (d+1, d), \qquad \partial(I(d)) = 1$$

finally for $0 \neq f \in V_d$ we define

$$R(f) \colon V_{d-1} \overset{s}{\underset{t}{\rightrightarrows}} V_d/(f) \qquad \underline{\dim}\, R(f) = (d, d), \qquad \partial(R(f)) = 0$$

We will see that the $P(d)$ and $I(d)$ are all indecomposable, as are the $R(f)$ for f a power of an irreducible polynomial, and that these yield a classification of all indecomposable representations.

2.1. Non-zero defect.

Proposition 2.1. *We have $S^{-d}(P(0)) \cong P(d)$. Moreover, if X is indecomposable of negative defect, then $X \cong P(d)$ for some $d \geq 0$.*

Dually, $S^{+d}(I(0)) \cong I(d)$. Moreover, if X is indecomposable of positive defect, then $X \cong I(d)$ for some $d \geq 0$.

Proof. It is easy to see that we have a push-out diagram

$$\begin{array}{ccc} V_{d-1} & \xrightarrow{s} & V_d \\ \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle t} \\ V_d & \xrightarrow{s} & V_{d+1} \end{array}$$

so we have $S^-(P(d)) = P(d+1)$. Dually $S^+(I(d)) = I(d+1)$.

Now suppose that X is indecomposable, and set $\underline{\dim}\, X = (a, b)$. Assume $\partial(X) < 0$, so $a < b$. If X is simple projective, then $X \cong P(0)$. Otherwise $S^+(X)$ is again indecomposable and $\underline{\dim}\, S^+(X) = (2a - b, a)$. Thus $2a - b < a < b$, so by induction on the dimension vector we must have $S^+(X) \cong P(d)$ for some $d \geq 0$. It follows that $X \cong S^- S^+(X) \cong S^-(P(d)) = P(d + 1)$. Dually for $\partial(X) > 0$. □

Proposition 2.2. *For $e \geq 0$ we have $\mathrm{Hom}(P(d), P(d + e)) \cong V_e$, and every non-zero homomorphism is injective.*

Proof. The result is clear when $d = 0$, and using the reflection functor S^{-d} we have that

$$\mathrm{Hom}(P(d), P(d + e)) \cong \mathrm{Hom}(P(0), P(e)) \cong V_e.$$

Moreover, if $f \in V_e$, the corresponding homomorphism $P(0) \to P(e)$ is given by multiplication by f; applying S^{-d} we then see that the induced homomorphism $P(d) \to P(d+e)$ is also given by multiplication by f. In particular, if f is non-zero, then this homomorphism is injective. □

Corollary 2.3. *We have*

$$\mathrm{Hom}(P(e + 1), P(d)) = 0 \quad \textit{and} \quad \mathrm{Ext}^1(P(d), P(e)) = 0 \quad \textit{for all } e \geq d.$$

Proof. Take a homomorphism $\theta\colon P(e) \to P(d)$. Composing with an injective map $f\colon P(d) \to P(e)$ we obtain an endomorphism $f\theta$ of $P(e)$, so necessarily a scalar. It cannot be an isomorphism for dimension reasons, so has to be zero. Thus $\theta = 0$.

For the second statement we know that $\dim \mathrm{Ext}^1(P(d), P(e))$ equals

$$\dim \mathrm{Hom}(P(d), P(e)) - \langle P(d), P(e)\rangle = (e - d + 1) - (e - d + 1) = 0. \qquad \square$$

We next want to compute homomorphisms and extensions between the representations $P(d)$ and $R(f)$. For this, we first observe that the representations $R(f)$ are unchanged by the reflection functors.

Lemma 2.4. *Let $f \in V_n$ be non-zero. Then $S^{\pm}R(f) \cong R(f)$, and hence we have a short exact sequence*

$$0 \to P(d) \xrightarrow{f} P(d + n) \to R(f) \to 0.$$

Proof. Fix any vector space isomorphism $\theta\colon V_{d-1} \xrightarrow{\sim} V_d/(f)$, and set $\alpha := \theta^{-1} \circ s$ and $\beta := \theta^{-1} \circ t$. It follows that $R(f)$ is isomorphic to the representation $V_{d-1} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V_{d-1}$, and hence $S^{\pm}(R(f)) \cong R(f)$ follows from the push-out (and pull-back) diagram

$$\begin{array}{ccc} V_{d-1} & \xrightarrow{\alpha} & V_{d-1} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ V_{d-1} & \xrightarrow{\alpha} & V_{d-1} \end{array}$$ □

It follows that we can realise $R(f)$ as the representation

$$R(f) \cong V_{d+n-1}/(fV_{d-1}) \overset{s}{\underset{t}{\rightrightarrows}} V_{d+n}/(fV_d)$$

Corollary 2.5. *For $f \in V_n$ non-zero and $m \geq 0$ we have*

$$\mathrm{Hom}(P(d), R(f)) \cong V_{n+m}/(fV_m),$$

whereas

$$\mathrm{Hom}(R(f), P(d)) = 0 \quad \textit{and} \quad \mathrm{Ext}^1(P(d), R(f)) = 0.$$

Proof. Consider the short exact sequence

$$0 \to P(d+m) \xrightarrow{f} P(d+m+n) \to R(f) \to 0.$$

Applying $\mathrm{Hom}(P(d), -)$ and using that $\mathrm{Ext}^1(P(d), P(d+m)) = 0$ yields the short exact sequence

$$0 \to V_m \xrightarrow{f} V_{m+n} \to \mathrm{Hom}(P(d), R(f)) \to 0.$$

Now use the Euler form to compute that $\mathrm{Ext}^1(P(d), R(f)) = 0$.

On the other hand, applying $\mathrm{Hom}(-, P(d))$ yields that

$$\mathrm{Hom}(R(f), P(d)) \leq \mathrm{Hom}(P(d+n), P(d)) = 0.$$ □

Finally we compute homomorphisms and extensions between $P(d)$ and $I(e)$.

Lemma 2.6. *We have*

$$\mathrm{Hom}(P(d), I(e)) \cong D(V_{d+e-1}) \quad \textit{and} \quad \mathrm{Ext}^1(P(d), I(e)) = 0.$$

Also $\mathrm{Hom}(I(e), P(d)) = 0$ *and* $\dim \mathrm{Ext}^1(I(e), P(d)) = d + e + 2$.

Proof. Applying S^{-e} we have

$$\mathrm{Hom}(P(d), I(e)) \cong \mathrm{Hom}(P(d+e), I(0)) \cong D(V_{d+e-1}).$$

Similarly, it is easy to check that $\mathrm{Hom}(I(0), P(d)) = 0$ for all d. Then by the Euler form we have

$$\dim \mathrm{Ext}^1(P(d), I(e)) = \dim \mathrm{Hom}(P(d), I(e)) - \langle P(d), I(e) \rangle = (d+e) - (d+e) = 0,$$

and similarly $\dim \mathrm{Ext}^1(I(e), P(d)) = d + e + 2$. □

2.2. **Zero defect.** We now want to describe the indecomposable representations of defect zero. We will show that if f is irreducible, then $R(f^m)$ is indecomposable for all $m \geq 1$; moreover any indecomposable representation of defect zero is isomorphic to one of this form (with the irreducible polynomial uniquely determined up to scalar). More generally, if f factors up to scalar as $p_1^{m_1} \cdots p_r^{m_r}$, where the p_i are irreducible and pairwise coprime, then

$$R(f) \cong R(p_1^{m_1}) \oplus \cdots \oplus R(p_r^{m_r}).$$

Denote by $\operatorname{rep}_k^0 K$ the full additive subcategory generated by the indecomposable representations of defect zero, and by $\operatorname{rep}_k^{0,f} K$ for f irreducible the full additive subcategory generated by all the $R(f^m)$. We will show that $\operatorname{rep}_k^0 K$ and each $\operatorname{rep}_k^{0,f} K$ are thick abelian subcategories[1] of $\operatorname{rep}_k K$, and

$$\operatorname{rep}_k^0 K \cong \coprod \operatorname{rep}_k^{0,f} K,$$

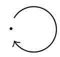
the coproduct being taken over all irreducible homogeneous polynomials up to scalar.

Finally we will show that for f irreducible we have

$$\operatorname{rep}_k^{0,f} K \cong \operatorname{mod} \widehat{\mathcal{O}}_f,$$

where $\widehat{\mathcal{O}}_f$ is a complete DVR over k, depending on f. We can write $\mathcal{O}_f$ as the degree zero part of the graded localisation $k[s,t]_{(f)}$. Explicitly, if s does not divide f, then this is isomorphic to $k[u]_{(f(1,u))}$, whereas if t does not divide f, then it is isomorphic to $k[u]_{f(u,1)}$.

2.2.1. *Modules over* $k[u]$. We begin by reviewing the module theory for the principal ideal domain $k[u]$. A finite-dimensional module is determined by a pair $(V;\phi)$, where V is a finite-dimensional vector space, and $\phi \in \operatorname{End}(V)$ gives the action on u. We can regard such pairs as a k-representation for the Jordan quiver

Q: 

and in fact we obtain an equivalence (even an isomorphism) of categories

$$\operatorname{mod} k[u] \cong \operatorname{rep}_k Q.$$

On the other hand, the structure theorem for finitely-generated modules over a principal ideal domain implies that every finite-dimensional indecomposable $k[u]$-module is isomorphic to $k[u]/(p^n)$ for some monic irreducible polynomial p. In this case the corresponding representation of the Jordan quiver has vector space $k[u]/(p^n)$ and endomorphism corresponding to multiplication by u. More generally, if f is any monic polynomial, then we can factorise $f = p_1^{m_1} \cdots p_r^{m_r}$ into a product of distinct monic irreducible polynomials $p_1, \dots, p_r$, in which case the cyclic module $k[u]/(f)$ is isomorphic to the direct sum

$$k[u]/(f) \cong \big(k[u]/(p_1^{m_1})\big) \oplus \cdots \oplus \big(k[u]/(p_r^{m_r})\big).$$

[1] A thick abelian subcategory is one which is closed under kernels, cokernels and extensions.

If we take the basis $\{1, u, u^2, \dots, u^{d-1}\}$ where $\deg f = d$, then multiplication by u is represented by the companion matrix $C(f)$.
If $f = g^m$ with $\deg g = r$, then we may take the basis
$$\{u^i g^j : 0 \leq i < r, 0 \leq j < m\},$$
in which case multiplication by u is represented by a matrix in block form, with $C(g)$ on the diagonal and the elementary matrix E_{1r} on the lower diagonal. In particular, if $g = u - \lambda$ is linear, then this specialises to the (transpose of the) Jordan matrix $J_m(\lambda)$.

Thus the structure theorem can be expressed in the following form.

Theorem 2.7. *We have an equivalence of categories*
$$\operatorname{mod} k[u] \cong \coprod \operatorname{mod} \widehat{\mathcal{O}}_f$$
where the product is taken over all monic irreducible polynomials $f \in k[u]$, and $\mathcal{O}_f := k[u]_{(f)}$ is a DVR.

If the residue field $\kappa(f) := k[u]/(f)$ is separable over k, then the Cohen Structure Theorem tells us that $\widehat{\mathcal{O}}_f \cong \kappa(f)[[u]]$. In general we always have such an isomorphism as rings, but when the residue field is inseparable over k, the k-algebra structure is not the obvious one coming from $k \to \kappa(f) \to \kappa(f)[[u]]$.

2.2.2. *Two exact embeddings.*

Proposition 2.8. *We have exact embeddings*
$$F_0\colon \operatorname{mod} k[u] \to \operatorname{rep}_k^0 K, \quad (V;\phi) \mapsto \quad V \overset{\phi}{\underset{1}{\rightrightarrows}} V$$
and
$$F_\infty\colon \operatorname{mod} k[u] \to \operatorname{rep}_k^0 K, \quad (V;\phi) \mapsto \quad V \overset{1}{\underset{\phi}{\rightrightarrows}} V$$
In particular, $F_0(k;0) \cong R(s)$ and $F_\infty(k;0) \cong R(t)$.

Proof. Write F for either F_0 or F_∞. It is straightforward to check that F is a fully-faithful and exact functor from $\operatorname{mod} k[u]$ to $\operatorname{rep}_k K$. It remains to show that its image lies in the subcategory $\operatorname{rep}_k^0 K$.

Clearly $F(V;\phi)$ cannot have any $I(d)$ as a direct summand, since the two linear maps $D(s)$ and $D(t)$ used in $I(d)$ have non-trivial kernel. Since $F(V;\phi)$ has zero defect and has no summand of positive defect, it also cannot have a summand of negative defect, so $F(V;\phi) \in \operatorname{rep}_k^0 K$. $\square$

It follows from the proposition that the essential image of F_0 is the full subcategory consisting of those representations $U \overset{A}{\underset{B}{\rightrightarrows}} V$ such that B is an isomorphism; it is then clear that this is a thick abelian subcategory of $\operatorname{rep}_k K$. Similarly for F_∞.

Lemma 2.9. *Let $f \in V_n$ be non-zero, and assume that t does not divide f. Set $\bar{f} := f(u,1) \in k[u]$. Then the evaluation map $\operatorname{ev}_{(u,1)}\colon k[s,t] \to k[u]$ induces an* *isomorphism*

$$\begin{array}{ccc} R(f) & & V_{n-1} \overset{s}{\underset{t}{\rightrightarrows}} V_n/(f) \\ \downarrow\wr & & \downarrow \mathrm{ev}_{(u,1)} \qquad\qquad \downarrow \mathrm{ev}_{(u,1)} \\ F_0(k[u]/(\bar f)) & & k[u]/(\bar f) \overset{u}{\underset{1}{\rightrightarrows}} k[u]/(\bar f) \end{array}$$

Similarly, if s does not divide f, then the evaluation $\mathrm{ev}_{(1,u)}$ induces an isomorphism $R(f) \cong F_\infty(k[u]/(f(1,u)))$. □

Lemma 2.10. *Let X be indecomposable of defect zero. Then X lies in the essential image of either F_0 or F_∞.*

Proof. Write the representation X as $V \overset{A}{\underset{B}{\rightrightarrows}} V$. We need to show that one of A or B is an isomorphism. Assume therefore that A is not an isomorphism, and take $0 \neq v \in \mathrm{Ker}(A)$. If $Bv = 0$, then we can use v to define a monomorphism $S_1 \to X$. Since S_1 is injective, this must be a split monomorphism, contradicting the fact that X is indecomposable of defect zero. Thus $Bv \neq 0$ and we have a monomorphism

$$\begin{array}{ccc} R(t) & & k \overset{0}{\underset{1}{\rightrightarrows}} k \\ \downarrow & & \downarrow v \qquad \downarrow Bv \\ X & & V \overset{A}{\underset{B}{\rightrightarrows}} V \end{array}$$

Let Y be the cokernel, which again has zero defect. Moreover, every indecomposable summand of Y has zero defect, so $Y \in \mathrm{rep}_k^0 K$. For, we know that $\mathrm{Ext}^1(P(d), R(t)) = 0$, so X indecomposable implies that Y has no summand of negative defect, and hence also no summand of positive defect.

Consider an indecomposable summand of the cokernel, say $X' \colon V' \overset{A'}{\underset{B'}{\rightrightarrows}} V'$. Suppose that A' is an isomorphism. Then it is clear that $\mathrm{Hom}(X', R(t)) = 0$, and so by the Euler form also $\mathrm{Ext}^1(X', R(t)) = 0$. Thus the split monomorphism $X' \to Y$ lifts to a monomorphism $X' \to X$, and hence X' is a direct summand of X, a contradiction. By induction on dimension vector we deduce that X' lies in the essential image of F_0. Since this is true for every direct summand of Y, we deduce that Y, and hence also X, lies in the essential image of F_0. □

Theorem 2.11. *We have*

$$\mathrm{rep}_k^0 K \cong \coprod \mathrm{mod}\, \widehat{\mathcal{O}}_f$$

where the product is taken over all irreducible homogeneous polynomials up to scalar, and that this is a thick abelian subcategory of $\mathrm{rep}_k K$.

2.2.3. *Duality.* We next show that $D(R(f)) \cong R(f)$ for all homogeneous polynomials f. To do this, we recall that a finite-dimensional commutative k-algebra A is called a Frobenius algebra provided there is a linear functional $\pi \colon A \to k$ whose kernel contains no non-zero ideal of A. Each Frobenius algebra is self-injective; in fact, the map π induces an isomorphism of A-modules

$$A \xrightarrow{\sim} D(A), \quad a \mapsto (b \mapsto \pi(ab)).$$

Lemma 2.12. *Let $f \in k[u]$ be a monic irreducible polynomial. Then $k[u]/(f^{m+1})$ is a Frobenius algebra.*

Proof. Consider the basis A given by $e_{a,b} := f^a u^b$ for $0 \le a \le m$ and $0 \le b < \deg f$. We take $\pi = \delta_{m,0}$ to be the dual basis element corresponding to $e_{m,0}$. Now take $0 \ne g \in k[u]/(f^{m+1})$. Write $g = f^a \bar{g}$ with $\bar{g} \notin (f)$. Since $k[u]/(f)$ is a field, we can find $\bar{h}$ such that $\bar{g}\bar{h} = 1 \in k[u]/(f)$. Now set $h = f^{m-a}\bar{h}$, so that $gh = f^m$ and hence $\pi(gh) = 1$. Thus $\mathrm{Ker}(\pi)$ cannot contain any non-zero ideal. □

Proposition 2.13. *We have $D(R) \cong R$ for all $R \in \mathrm{rep}_k^0 K$.*

Proof. It is enough to prove this for every indecomposable. By Lemma 2.9 we can pass to the indecomposable $k[u]$-module $k[u]/(f^m)$, where $f \in k[u]$ is monic irreducible, and by the previous lemma we know that this is isomorphic to its dual. □

Corollary 2.14. *For each non-zero homogeneous polynomial $f \in V_d$ we have a short exact sequence*

$$0 \to R(f) \to I(d+e) \xrightarrow{D(f)} I(e) \to 0.$$

Proof. Apply the duality to the short exact sequence

$$0 \to P(e) \xrightarrow{f} P(d+e) \to R(f) \to 0. \qquad \square$$

2.2.4. *Computation of homomorphisms.* Since we have decomposed $\mathrm{rep}_k^0 K$ into a coproduct, it is easy to compute homomorphisms between indecomposables, and hence between arbitrary representations. The following version is still useful, however.

Lemma 2.15. *Let f, g be homogeneous, and write $h = \gcd(f,g) \in V_n$. Then*

$$\mathrm{Hom}(R(f), R(g)) \cong V_n/(h) \cong \mathrm{Ext}^1(R(f), R(g)).$$

Proof. Let f and g have degrees d and e respectively, giving the short exact sequence

$$0 \to P(0) \xrightarrow{f} P(d) \to R(f) \to 0.$$

Applying $\mathrm{Hom}(-, R(g))$ and using Corollary 2.5 we have the map

$$V_e/(g) \cong \mathrm{Hom}(P(d), R(g)) \xrightarrow{f} \mathrm{Hom}(P(0), R(g)) \cong V_{d+e}/(gV_d),$$

whose kernel is $\mathrm{Hom}(R(f), R(g))$ and whose cokernel is $\mathrm{Ext}^1(R(f), R(g))$.

Now write $f = \bar{f}h$ and $g = \bar{g}h$. Then the kernel is $\bar{g}V_{e-n}/(g)$ and the cokernel is $V_{d+e}/(h(\bar{f}V_{e-n} + \bar{g}V_{d-n})) \cong V_{d+e}/(hV_{d+e-n})$, where we have used that $\bar{f}$ and $\bar{g}$ are coprime. Finally, for all m we have an isomorphism $V_n/(h) \cong V_{m+n}/(hV_m)$. □

2.3. **Computations of some extensions.**

Lemma 2.16. *Let $f \in V_d$ and $g \in V_e$ be coprime. If $n \ge d+e$, then we have a short exact sequence*

$$0 \longrightarrow P(n-d-e) \xrightarrow{\binom{g}{-f}} P(n-d) \oplus P(n-e) \xrightarrow{(f,g)} P(n) \longrightarrow 0$$

whereas if $n < d+e$, then we have a short exact sequence

$$0 \longrightarrow P(n-d) \oplus P(n-e) \xrightarrow{(f,g)} P(n) \longrightarrow I(d+e-n-1) \longrightarrow 0$$

Proof. Suppose first that $n = d+e$. Since f and g are coprime, the map $(f,g)\colon P(e) \oplus P(d) \to P(d+e)$ has kernel spanned by $\binom{g}{-f}$, as required. Applying S^- yields the first sequence when $n \ge d+e$.

Suppose instead that $d+e = n+1$. Then the map $(f,g)\colon P(e-1) \oplus P(d-1) \to P(d+e-1)$ is injective, and the cokernel has dimension vector $(1,0)$, so must be isomorphic to $I(0)$. Applying S^+ yields the second sequence when $n < d+e$. □

Proposition 2.17. *We have*

$$\mathrm{Ext}^1(I(e), P(d)) \cong V_{d+e+1}.$$

In fact, for each non-zero $f \in V_{d+e+1}$, the corresponding extension is of the form

$$0 \to P(d) \to R(f) \to I(e) \to 0.$$

Proof. Applying S^{-e}, we may assume that $e = 0$. Now consider the projective resolution

$$0 \longrightarrow P(0)^2 \xrightarrow{(s,t)} P(1) \longrightarrow I(0) \longrightarrow 0.$$

Applying $\mathrm{Hom}(-, P(d))$ yields the short exact sequence

$$0 \longrightarrow V_{d-1} \xrightarrow{\binom{s}{t}} V_d^2 \longrightarrow \mathrm{Ext}^1(I(0), P(d)) \longrightarrow 0,$$

so that $\mathrm{Ext}^1(I(0), P(d)) \cong V_{d+1}$, and this isomorphism is given explicitly by forming the push-out along $P(0)^2 \to P(d)$.

Suppose first that the field k is infinite. Given $g, h \in V_d$, set $f := gt - hs$. Take $p \in V_1$ coprime to f, write $p = at - bs$ with $a, b \in k$ and set $q := bg - ah$. If $f \neq 0$, then we have a commutative square

$$\begin{array}{ccc} P(0)^2 & \xrightarrow{\left(\begin{smallmatrix} s & t \\ g & h \end{smallmatrix}\right)} & P(1) \oplus P(d) \\ \big\downarrow{\scriptstyle (a,b)} & & \big\downarrow{\scriptstyle (q,p)} \\ P(0) & \xrightarrow{f} & P(d+1) \end{array}$$

Note that if p divides q, then it necessarily divides af and bf, so divides f, a contradiction. Thus each vertical map is surjective by the previous lemma, and has kernel $P(0)$. Moreover, the induced endomorphism of $P(0)$ is non-zero, so is an automorphism. It follows that the cokernel of the top horizontal map is isomorphic to the cokernel $R(f)$ of the bottom horizontal map.

Suppose instead that the field k is finite, so there are irreducible polynomials of arbitrarily high degree. Take a homogeneous irreducible polynomial $p \in V_{n+1}$ with $n > d$ and write $p = at - bs$ for some $a, b \in V_d$. Given $g, h \in V_d$, set $q := bg - ah$ and $f := gt - hs$. If $f \neq 0$, then we have a commutative square

$$\begin{array}{ccc} P(0)^2 & \xrightarrow{\left(\begin{smallmatrix} s & t \\ g & h \end{smallmatrix}\right)} & P(1) \oplus P(d) \\ \big\downarrow{\scriptstyle (a,b)} & & \big\downarrow{\scriptstyle (q,p)} \\ P(n) & \xrightarrow{f} & P(d+n+1) \end{array}$$

Moreover, since p is irreducible, a and b must be coprime. Also, if p divides q, then it necessarily divides af and bf, so divides f, a contradiction since $n > d$. Thus p and q are also coprime. From the previous lemma we deduce that the cokernel of each vertical map is $I(n-1)$. Moreover, the composition $P(n) \xrightarrow{f} P(n+d+1) \to I(n-1)$ does not vanish, since it cannot factor through $(q, p)\colon P(1) \oplus P(d) \to P(n+d+1)$. Thus the induced endomorphism of $I(n-1)$ is non-zero, so is an automorphism. We conclude that the cokernel of the top horizontal map is isomorphic to the cokernel $R(f)$ of the bottom horizontal map.

In all cases we have shown that the push-out of the projective resolution for $I(0)$ along the map $(g, h)\colon P(0)^2 \to P(d)$ is isomorphic to $R(gt - hs)$. □

This result also yields the important factorisation result.

Corollary 2.18. *Let f be irreducible. Then every homomorphism $P(d) \to I(e)$ factors through some $R(f^m)$.*

Proof. Consider the short exact sequence

$$0 \to P(d) \to R(f^m) \to I(e') \to 0,$$

where e' depends on m. For m, and hence e', sufficiently large we know that $\mathrm{Ext}^1(I(e'), I(e)) = 0$, and so the map

$$\mathrm{Hom}(R(f^m), I(e)) \to \mathrm{Hom}(P(d), I(e))$$

is surjective. □