HOPF MODULES AND INTEGRALS: EXAMPLES

ROLF FARNSTEINER

In this lecture we are going to put our previous results [3, 4] in perspective by looking at a few examples. Throughout, k is assumed to be a field. Let us begin with the easiest case concerning group algebras of finite groups.

Example. Let G be a finite group. We already know that the projection

$$\pi : kG \longrightarrow k \ \ ; \ \ \sum_{g\in G} \alpha_g g \mapsto \alpha_1$$

endows the group algebra kG with the structure of a symmetric algebra. The Hopf algebra structure on kG is given by the formulae

$$\Delta(g) = g \otimes g \ \ ; \ \ \varepsilon(g) = 1 \ \ ; \ \ \eta(g) = g^{-1} \qquad \forall\ g \in G.$$

Consequently, the convolution product $\psi * \pi$ is given by

$$(\psi * \pi)(g) = \psi(g)\pi(g) = \psi(1)\pi(g) \qquad \forall\ g \in G,\ \ \psi \in kG^*,$$

so that π is a left integral of kG^*. Since $x := \sum_{g\in G} g$ is a left and right integral of kG, it follows that $\zeta_\ell = \varepsilon$. As $\eta^2 = \mathrm{id}_{kG}$, [4, Thm.] shows that kG is symmetric.

Our next example concerns restricted enveloping algebras. Since their definition by Jacobson in the 1950's, these algebras, which assume the rôle of group algebras in the representation theory of restricted Lie algebras, have been compared to group algebras of finite groups.

Let $(\mathfrak{g}, [p])$ be a finite dimensional restricted Lie algebra over a field k of characteristic $p > 0$. By definition, $\mathfrak{g}$ is a Lie algebra together with a map $\mathfrak{g} \longrightarrow \mathfrak{g}\ ;\ x \mapsto x^{[p]}$ that satisfies formal properties of an associative p-th power. The restricted enveloping algebra

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]}\ ;\ x \in \mathfrak{g}\})$$

is a finite dimensional quotient of the ordinary enveloping algebra $U(\mathfrak{g})$. By work of Berkson [1], the algebra $U_0(\mathfrak{g})$ is Frobenius. Shortly thereafter, Schue [7] gave a criterion for $U_0(\mathfrak{g})$ to be symmetric.

Example. We pick a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ and use the following notation for monomials in $U_0(\mathfrak{g})$:

$$e^a := e_1^{a_1} \cdots e_n^{a_n} \qquad \forall\ a = (a_1, \ldots, a_n) \in \mathbb{N}_0^n.$$

We also define $a \le b :\Leftrightarrow a_i \le b_i$ for $1 \le i \le n$ and put $\tau := (p-1, \cdots, p-1) \in \mathbb{N}_0^n$. By the Theorem of Poincaré-Birkhoff-Witt, the monomials $\{e^a\ ;\ 0 \le a \le \tau\}$ form a basis of $U_0(\mathfrak{g})$.

The algebra $U_0(\mathfrak{g})$ inherits the structure of a Hopf algebra from $U(\mathfrak{g})$. The relevant maps are determined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \ \ ; \ \ \varepsilon(x) = 0 \ \ ; \ \ \eta(x) = -x \qquad \forall\ x \in \mathfrak{g}.$$

Date: May 31, 2006.

Writing $\binom{a}{b} := \prod_{i=1}^{n} \binom{a_i}{b_i}$ for $a \le b \in \mathbb{N}_0^n$, we obtain

$$(*) \qquad \Delta(e^a) = \sum_{0\le b\le a} \binom{a}{b} e^b \otimes e^{a-b}.$$

Thus, letting $\{\delta_a \; ; \; 0 \le a \le \tau\}$ be the basis of the commutative algebra $U_0(\mathfrak{g})^*$ that is dual to the PBW-basis, we see that

$$\delta_a \delta_b = \binom{a+b}{b} \delta_{a+b},$$

with the right-hand side being zero if $a + b \not\le \tau$. Hence every element δ_a with $a \ne 0$ is nilpotent, and $U_0(\mathfrak{g})$ is not isomorphic to the Hopf algebra kG for any group G. Since $\dim U_0(\mathfrak{g}) = p^n$, the algebras $U_0(\mathfrak{g})$ and kG are only isomorphic when G is a p-group. In that case, kG is local, which happens for $U_0(\mathfrak{g})$ if and only if the restricted Lie algebra is unipotent.

Consider the linear map

$$\pi : U_0(\mathfrak{g}) \longrightarrow k \;\; ; \;\; \sum_{0\le a\le\tau} \alpha_a e^a \mapsto \alpha_\tau.$$

Using $(*)$ we obtain

$$(\psi * \pi)(e^a) = \sum_{0\le b\le a} \binom{a}{b} \psi(e^b)\pi(e^{a-b}) = \psi(1)\pi(e^a) \qquad \forall \; \psi \in U_0(\mathfrak{g})^*,$$

proving that π is a left integral of $U_0(\mathfrak{g})^*$. Thanks to [3, Cor. 1] the bilinear form $(\; , \;)_\pi$ endows $U_0(\mathfrak{g})$ with the structure of a Frobenius algebra. This was shown by Berkson by direct computation.

A general formula for the left integral of $U_0(\mathfrak{g})$ is not known. However, we do know from [4] that the element v_π satisfying

$$\pi(hv_\pi) = \varepsilon(h) \qquad \forall \; h \in U_0(\mathfrak{g})$$

is a left integral for $U_0(\mathfrak{g})$. It follows that the basis element e^τ appears as a summand in v_π with coefficient 1.

The computation of a Nakayama automorphism proceeds as follows (cf. [9, p. 215ff] for more details): Let $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{gl}(\mathfrak{g})$ be the adjoint representation. Using the standard filtration on $U(\mathfrak{g})$ one shows for $x \in \mathfrak{g}$ and $0 \le a \le \tau$

$$[x, e^a] \equiv -\operatorname{tr}(\operatorname{ad} x)\pi(e^a)e^\tau \quad \mod(\ker \pi).$$

Consequently,

$$\pi(e^a x) = \pi(xe^a - [x, e^a]) = \pi((x + \operatorname{tr}(\operatorname{ad} x)1)e^a) \qquad 0 \le a \le \tau.$$

As a result, the Nakyama automorphism satisfies $\mu(x) = x + \operatorname{tr}(\operatorname{ad} x)1$ for every $x \in \mathfrak{g}$, and we thus have

$$\mu = \mathrm{id}_{U_0(\mathfrak{g})} * (\operatorname{tr} \circ \operatorname{ad}).$$

In particular, $U_0(\mathfrak{g})$ is symmetric if and only if $\operatorname{tr}(\operatorname{ad} x) = 0$ for every $x \in \mathfrak{g}$.

The analogue of Maschke's Theorem was first verified by Hochschild [5], who showed that $U_0(\mathfrak{g})$ is semi-simple precisely when $\mathfrak{g}$ is abelian and $\mathfrak{g} = \langle \mathfrak{g}^{[p]} \rangle$. Maschke's Theorem for Hopf algebras (cf. [3, Cor. 2]) is not useful in this context.

The foregoing examples are classical in the sense that they pertain to finite dimensional cocommutative Hopf algebras. According to general theory, the category of these algebras is equivalent to the category of finite group schemes. By contrast, the so-called quantum groups are neither commutative nor cocommutative and their antipodes do not satisfy $\eta^2 = \mathrm{id}$.

Let $d > 1$ be an odd number, $q \in \mathbb{C}$ a primitive d-th root of unity. We shall define the *restricted quantum group* $\bar{U}_q(\mathfrak{sl}(2))$. The representation theory of these algebras was studied by several authors, including [2, 8, 10].

Example. As a $\mathbb{C}$-algebra, $\bar{U}_q(\mathfrak{sl}(2))$ is generated by elements E, F, K subject to the relations

$$E^d = 0 = F^d \ , \ K^d = 1 \ , \ KEK^{-1} = q^2 E \ , \ KFK^{-1} = q^{-2} F \ , \ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

There is a PBW-Theorem, which asserts that, the monomials $E^i K^j F^\ell$, with each exponent ranging between 0 and $d-1$, form a basis of $\bar{U}_q(\mathfrak{sl}(2))$.

The Hopf algebra structure on $\bar{U}_q(\mathfrak{sl}(2))$ is defined via

$$\Delta(E) = 1 \otimes E + E \otimes K \quad , \quad \Delta(K) = K \otimes K \quad , \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1$$

and

$$\varepsilon(E) = 0 = \varepsilon(F) \quad , \quad \varepsilon(K) = 1 \quad ; \quad \eta(E) = -EK^{-1} \quad , \quad \eta(F) = -KF \quad , \quad \eta(K) = K^{-1}.$$

In this case, we have

$$\eta^2(u) = KuK^{-1} \qquad \forall\ u \in \bar{U}_q(\mathfrak{sl}(2)),$$

so that η has order $2d$. Left integrals for $\bar{U}_q(\mathfrak{sl}(2))$ and $\bar{U}_q(\mathfrak{sl}(2))^*$ seem to be unknown, yet the representation theory of this algebra is well-understood. In [2, (3.8)] the authors show that $\mathrm{Soc}(P(S)) \cong S$ for every simple $\bar{U}_q(\mathfrak{sl}(2))$-module S. Since η^2 is an inner automorphism of $\bar{U}_q(\mathfrak{sl}(2))$, [4, Cor. 2] implies that $\bar{U}_q(\mathfrak{sl}(2))$ is symmetric. By work of Xiao [10], the non-simple blocks of $\bar{U}_q(\mathfrak{sl}(2))$ are Morita equivalent to trivial extensions of the Kronecker algebra and are thus also Morita equivalent to the non-simple blocks of $U_0(\mathfrak{sl}(2))$.

References

[1] A. Berkson. *The u-algebra of a restricted Lie algebra is Frobenius*. Proc. Amer. Math. Soc. **15** (1964), 14-15

[2] V. Chari and A. Premet. *Indecomposable restricted representations of quantum sl_2*. Publ. RIMS Kyoto Univ. **30** (1994), 335-352

[3] R. Farnsteiner. *Hopf modules and integrals: The space of integrals*. Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`

[4] ———. *Hopf modules and integrals: Nakayama automorphisms*. Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`

[5] G. Hochschild. *Representations of restricted Lie algebras of characteristic p*. Proc. Amer. Math. Soc. **5** (1954), 603-605

[6] J. Humphreys. *Symmetry for finite dimensional Hopf algebras*. Proc. Amer. Math. Soc. **68** (1978), 143-146.

[7] J. Schue. *Symmetry for the enveloping algebra of a restricted Lie algebra*. Proc. Amer. Math. Soc. **16** (1965), 1123-1124

[8] R. Suter. *Modules over $U_q(sl_2)$*. Commun. in Math. Phys. **163** (1994), 359-393

[9] H. Strade and R. Farnsteiner. *Modular Lie Algebras and their Representations*. Pure and Applied Mathematics **116**. Marcel Dekker 1988

[10] J. Xiao. *Finite dimensional representations of $U_t(sl(2))$ at roots of unity*. Canad. J. Math. **49** (1997), 772-787