SELF-INJECTIVE ALGEBRAS: FROBENIUS ALGEBRAS AND COALGEBRAS

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Throughout, we shall be working over a field k. All k-vector spaces are assumed to be finite-dimensional.

Let Λ be a k-algebra. The objective of this lecture is to characterize non-degenerate associative forms on Λ in terms of coalgebra structures. Such an interrelation apparently plays a rôle within topological quantum field theory, which exclusively deals with commutative Frobenius algebras.

Definition. A *coalgebra* (C, Δ, ε) is a triple, consisting of a k-vector space C and two linear maps $\Delta : C \longrightarrow C \otimes_k C$ and $\varepsilon : C \longrightarrow k$ such that

(1) $(\mathrm{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_C) \circ \Delta$ (co-associativity), and

(2) $(\varepsilon \bar{\otimes} \mathrm{id}_C) \circ \Delta = \mathrm{id}_C = (\mathrm{id}_C \bar{\otimes} \varepsilon) \circ \Delta$ (counit).

Here $\bar{\otimes}$ refers to the ordinary tensor product of maps followed by scalar multiplication.

We require two simple observations:

- If (C, Δ, ε) is a coalgebra and $f : V \longrightarrow C$ is an isomorphism of k-vector spaces, then V obtains the structure of a coalgebra via
$$\Delta_V := (f^{-1} \otimes f^{-1}) \circ \Delta \circ f \quad \text{and} \quad \varepsilon_V := \varepsilon \circ f.$$
- If Λ is a k-algebra with multiplication $m : \Lambda \otimes_k \Lambda \longrightarrow \Lambda$, then the dual space Λ^* carries a coalgebra structure, with comultiplication $m^* : \Lambda^* \longrightarrow \Lambda^* \otimes_k \Lambda^*$ and counit $\underline{\mathbf{1}} : \Lambda \longrightarrow k$ given by
$$m^*(f) = \sum_{i=1}^{n} f_i \otimes g_i \quad \Leftrightarrow \quad f(ab) = \sum_{i=1}^{n} f_i(a) g_i(b) \quad \forall\ a, b \in \Lambda$$
and
$$\underline{\mathbf{1}}(f) = f(1) \quad \forall\ f \in \Lambda^*.$$

Recall that Λ is a Frobenius algebra if Λ possesses a non-degenerate associative form
$$(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k,$$
that is,
$$(ax, b) = (a, xb) \qquad \forall\ a, b, x \in \Lambda$$
If $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k$ is such a form, then there exists an automorphism $\mu : \Lambda \longrightarrow \Lambda$ such that
$$(y, x) = (\mu(x), y) \quad \forall\ x, y \in \Lambda.$$
This automorphism is referred to as the Nakayama automorphism of the Frobenius algebra $(\Lambda, (\ ,\ ))$. As explained in [1], μ corresponds to the Nakayama permutation ν of the self-injective algebra Λ.

The associative form $(\ ,\ )$ induces an isomorphism
$$\Theta : \Lambda \longrightarrow \Lambda^* \quad ; \quad a \mapsto (a, -)$$

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of k-vector spaces. In view of the above observations, Θ in turn gives rise to the following coalgebra structure on Λ:

$$\Delta := (\Theta^{-1} \otimes \Theta^{-1}) \circ m^* \circ \Theta \;\; ; \;\; \varepsilon := \underline{\mathbf{1}} \circ \Theta.$$

We can re-write this in the following form:

$$(*) \qquad \Delta(a) = \sum_{i=1}^{n} a_i \otimes b_i \;\; \Leftrightarrow \;\; (a, xy) = \sum_{i=1}^{n} (a_i, x)(b_i, y) \quad \forall \; x, y \in \Lambda$$

and

$$(**) \qquad \varepsilon(a) = (a, 1) \quad \forall \; a \in \Lambda.$$

Proposition 1. *Let Λ be a k-algebra. Then the following statements are equivalent:*

(1) *Λ is a Frobenius algebra.*

(2) *There exists a coalgebra structure $(\Lambda, \Delta, \varepsilon)$ such that*

$$\Delta(x) = \sum_{i=1}^{n} a_i x \otimes b_i = \sum_{i=1}^{n} a_i \otimes x b_i \quad \forall \; x \in \Lambda.$$

Proof. (2) $\Rightarrow$ (1). We define $(\, , \,) : \Lambda \times \Lambda \longrightarrow k$ via

$$(x, y) := \varepsilon(xy) \quad \forall \; x, y \in \Lambda.$$

Then $(\, , \,)$ is an associative form. If $x \in \Lambda$ satisfies $(a, x) = 0$ for every $a \in \Lambda$, then

$$x = (\varepsilon \bar{\otimes} \mathrm{id}_\Lambda) \circ \Delta(x) = \sum_{i=1}^{n} \varepsilon(a_i x) b_i = \sum_{i=1}^{n} (a_i, x) b_i = 0,$$

so that $(\, , \,)$ is non-degenerate.

(1) $\Rightarrow$ (2). Given a non-degenerate associative form $(\, , \,)$ with Nakayama automorphism μ, we define Δ and ε via $(*)$ and $(**)$. We write $\Delta(1) = \sum_{i=1}^{n} a_i \otimes b_i$ and obtain for $x, y, z \in \Lambda$

$$\begin{aligned} \sum_{i=1}^{n} (a_i, y)(x b_i, z) &= \sum_{i=1}^{n} (a_i, y)(x, b_i z) = \sum_{i=1}^{n} (a_i, y)(b_i z, \mu^{-1}(x)) = \sum_{i=1}^{n} (a_i, y)(b_i, z\mu^{-1}(x)) \\ &= (1, yz\mu^{-1}(x)) = (yz, \mu^{-1}(x)) = (x, yz), \end{aligned}$$

so that $\Delta(x) = \sum_{i=1}^{n} a_i \otimes x b_i$. The other identity follows similarly. □

In order to construct examples, we need to identify $\Delta(1)$. We call an ordered pair $(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})$ of ordered bases of Λ a *dual pair* for $(\, , \,)$ if

$$(x_i, y_j) = \delta_{ij} \qquad 1 \leq i, j \leq n.$$

Lemma 2. *Let $(\Lambda, (\, , \,))$ be a Frobenius algebra with Nakayama automorphism μ and associated coalgebra $(\Lambda, \Delta, \varepsilon)$. If $(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})$ is a dual pair, then the following identities hold:*

(1) $\Delta(1) = \sum_{i=1}^{n} x_i \otimes y_i$.

(2) $\sum_{i=1}^{n} \mu(a) x_i \otimes y_i = \sum_{i=1}^{n} x_i \otimes y_i a \quad \forall \; a \in \Lambda.$

Proof. Writing $\Delta(1) = \sum_{j=1}^{m} a_j \otimes b_j$, Proposition 1 implies

$$x = (\mathrm{id}_\Lambda \bar{\otimes} \varepsilon) \circ \Delta(x) = \sum_{j=1}^{m} a_j \varepsilon(x b_j) = \sum_{j=1}^{m} (x, b_j) a_j$$

for every $x \in \Lambda$. Consequently, $\{a_1, \ldots, a_m\}$ generates the vector space Λ. After suitable renumbering we may thus assume that $\{a_1, \ldots, a_n\}$ is a basis of Λ. If $\Delta(1) = \sum_{i=1}^{n} a_i \otimes b'_i$, the above identity now implies that $(\{a_1, \ldots, a_n\}, \{b'_1, \ldots, b'_n\})$ is a dual pair.

Consider the isomorphism $\Omega : \Lambda \otimes_k \Lambda \longrightarrow \mathrm{Hom}_k(\Lambda, \Lambda)$, given by

$$\Omega(a \otimes b)(z) := (a, z)b \quad \forall\ a, b, z \in \Lambda.$$

If $(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})$ is a dual pair, then $\Omega(\sum_{i=1}^{n} x_i \otimes y_i) = \mathrm{id}_\Lambda$, so that

$$\Delta(1) = \sum_{i=1}^{n} a_i \otimes b'_i = \sum_{i=1}^{n} x_i \otimes y_i.$$

It remains to verify (2). Note that

$$\Omega(\mu(a)x \otimes y)(z) = (\mu(a)x, z)y = (\mu(a), xz)y = (xz, a)y = (x, za)y = \Omega(x \otimes y)(za),$$

whence

$$\Omega(\sum_{i=1}^{n} \mu(a)x_i \otimes y_i)(z) = \Omega(\sum_{i=1}^{n} x_i \otimes y_i)(za) = za = \mathrm{id}_\Lambda(z)a = \sum_{i=1}^{n} (x_i, z)y_i a = \Omega(\sum_{i=1}^{n} x_i \otimes y_i a)(z),$$

as desired. □

Examples. (1) Let G be a finite group. The group algebra kG is a Frobenius algebra with associative form given by

$$(g, h) = \delta_{gh,1} \quad \forall\ g, h \in G.$$

Writing $G = \{g_1, \ldots, g_n\}$, we obtain a dual pair $(\{g_1, \ldots, g_n\}, \{g_1^{-1}, \ldots, g_n^{-1}\})$. In view of Proposition 1 and Lemma 2, the associated coalgebra structure on kG satisfies

$$\Delta(x) = \sum_{g \in G} gx \otimes g^{-1} = \sum_{i=1}^{n} g \otimes xg^{-1} \quad \forall\ x \in kG.$$

This markedly differs from the usual caoalgebra structure on kG, reflecting the fact that the counit of a Hopf algebra H defines a non-degenerate form on H if and only if $\dim_k H = 1$.

Let $m : kG \otimes_k kG \longrightarrow kG$ be the multiplication. Then $m \circ \Delta(x) \in \mathfrak{Z}(kG)$, the center of kG, so that $m \circ \Delta$ is invertible only if G is abelian. Moreover, $m \circ \delta(1) = \mathrm{ord}(G)1$, rendering the separability of kG another necessary condition for invertiblity. Clearly, both conditions together are also sufficient.

(2) Let $\Lambda := k[X]/(X^n)$ be a truncated polynomial ring with canonical basis $\{1, \ldots, x^{n-1}\}$. Then

$$(x^i, x^j) = \delta_{n-1,i+j}$$

defines a non-degenerate associative form on Λ such that $(\{1, \ldots, x^{n-1}\}, \{x^{n-1}, \ldots, 1\})$ is a dual pair. Consequently,

$$\Delta(a) = \sum_{i=0}^{n-1} x^i a \otimes x^{n-1-i} \quad \forall\ a \in \Lambda.$$

Since $(m \circ \delta)(a) = nx^{n-1}a$, the map $m \circ \delta$ is invertible if and only if $n = 1$.

(3) Let $\Lambda := \mathrm{Mat}_n(k)$ be the algebra of $(n \times n)$-matrices. Then

$$(x, y) = \mathrm{tr}(xy) \quad \forall\ x, y \in \Lambda$$

endows Λ with the structure of a Frobenius algebra such that the standard basis of Λ defines a dual pair (E_{ij}, E_{ji}). Direct computation yields

$$m \circ \Delta(x) = (\sum_{j=1}^{n} x_{jj}) I_n,$$

so that $m \circ \Delta$ is invertible if and only if $n = 1$. (Here I_n denotes the identity matrix.)

The passage between non-degenerate associative forms on Λ is effected by invertible elements of Λ. Let $u \in \Lambda^{\times}$ be invertible. If

$$\{a, b\} = (ua, b) \quad \forall\, a, b \in \Lambda,$$

then a dual pair $(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})$ of Λ relative to $(\,,\,)$ induces a dual pair $(\{u^{-1}x_1, \ldots, u^{-1}x_n\}, \{y_1, \ldots, y_n\})$ relative to $\{\,,\,\}$. Letting Λ act on $\Lambda \otimes_k \Lambda$ via left multiplication on the first factor, we may thus write

$$\Delta_{\{\,,\,\}} = u^{-1}\Delta_{(\,,\,)}.$$

Consequently, $m \circ \Delta_{\{\,,\,\}} = u^{-1} m \circ \Delta_{(\,,\,)}$, showing that the property of $m \circ \Delta : \Lambda \longrightarrow \Lambda$ being an invertible homomorphism of right Λ-modules does not depend on the choice of our associative form.

Recall that Λ is referred to as *separable* if the algebra $\Lambda_K := \Lambda \otimes_k K$ is semi-simple for every field extension $K\!:\!k$.

Proposition 3. *Let $(\Lambda, (\,,\,))$ be a Frobenius algebra with multiplication $m : \Lambda \otimes_k \Lambda \longrightarrow \Lambda$ and associated coalgebra $(\Lambda, \Delta, \varepsilon)$. Then the following statements are equivalent.*

(1) *The map $m \circ \Delta$ is an invertible homomorphism of right Λ-modules.*
(2) *The algebra Λ is commutative and separable.*

Proof. (1) $\Rightarrow$ (2). By assumption, there exists an invertible element $u \in \Lambda$ such that $m \circ \Delta = \ell_u$, the left multiplication effected by u. In view of the above, passage to another associative form allows us to assume that $m \circ \Delta = \mathrm{id}_\Lambda$.

Let $(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})$ be a dual pair of $(\,,\,)$. Given $a \in \Lambda$, Lemma 2(2) yields

$$\mu(a) = \mu(a)(m \circ \Delta)(1) = \mu(a)(\sum_{i=1}^{n} x_i y_i) = m(\sum_{i=1}^{n} \mu(a) x_i \otimes y_i) = m(\sum_{i=1}^{n} x_i \otimes y_i a) = a.$$

In view of Lemma 2, the map $a \mapsto \sum_{i=1} a x_i \otimes y_i$ is a (Λ, Λ)-linear splitting of the multiplication of Λ, and Lemma 2 of [2] ensures that Λ is separable.

Let K be an algebraic closure of k. Then

$$(x \otimes \alpha, y \otimes \beta)_K := (x, y)\alpha\beta \quad \forall\, x, y \in \Lambda,\ \alpha, \beta \in K$$

defines an associative form on Λ_K with dual pair $(\{x_1 \otimes 1, \ldots, x_n \otimes 1\}, \{y_1 \otimes 1, \ldots, y_n \otimes 1\})$. Consequently, the associated comultiplication Δ_K satisfies

$$m_K \circ \Delta_K = \mathrm{id}_{\Lambda_K}.$$

Since Λ_K is semi-simple, Wedderburn's theorem provides a decomposition

$$\Lambda = \bigoplus_{i=1}^{r} \mathrm{Mat}_{n_i}(K),$$

with the sumands being mutually orthogonal. Consequently, each summand satisfies (1), and the trace form of our example (3) also enjoys this property. This, however, implies $n_i = 1$, so that Λ is commutative.

(2) $\Rightarrow$ (1). Let K be an algebraic closure of k. Then $\Lambda_K \cong K^n$ for some $n \in \mathbb{N}$. As before, we consider the extended form and coalgebra structure on Λ_K. Let $\{e_1, \ldots, e_n\}$ be the primitive idempotents of Λ_K. Then $(e_i, e_j)_K = 0$ for $i \neq j$, so that $\alpha_i := (e_i, e_i)_K \neq 0$. Setting $\beta_i := \alpha_i^{-1}$, we obtain a dual pair $(\{e_1, \ldots, e_n\}, \{\beta_1 e_1, \ldots, \beta_n e_n\})$. The element $u' := \sum_{i=1}^n \beta_i e_i \in \Lambda_K$ is invertible and

$$(m_K \circ \Delta_K)(x) = \sum_{i=1}^{n} \beta_i x e_i = u'x \quad \forall\ x \in \Lambda_K.$$

Since $(m_K \circ \Delta_K)(\Lambda \otimes 1) \subseteq \Lambda \otimes 1$, it follows that $u' \in \Lambda \otimes 1$. Hence there exists an invertible element $u \in \Lambda$ such that

$$(m \circ \Delta)(a) = \ell_u(a)$$

for every element $a \in \Lambda$. □

References

[1] R. Farnsteiner.*Self-injective algebras: Comparison with Frobenius algebras.* Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`

[2] ———.*The Theorem of Wedderburn-Malcev: Conjugacy of maximal separable suablgebras.* Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`