Minimal infinite cogeneration-closed subcategories.

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Let Λ be an artin algebra, and $\operatorname{mod}\Lambda$ the category of Λ-modules of finite length. The subcategories to be considered will be full subcategories closed under isomorphisms, direct sums and dirct summands, we call such subcategories *additive* subcategories. Let $\mathcal{C}$ be an additive subcategory. We say that $\mathcal{C}$ is *finite* provided it contains only finitely many isomorphism classes of indecomposable modules, otherwise $\mathcal{C}$ is said to be *infinite.* We say that $\mathcal{C}$ is *minimal infinite* provided $\mathcal{C}$ is infinite, but any proper additive subcategory $\mathcal{D} \subset \mathcal{C}$ is finite. Finally, $\mathcal{C}$ is *cogeneration-closed,* provided it is also closed under submodules. Given a class $\mathcal{X}$ of modules (or of isomorphism classes of modules), we denote by $\operatorname{add}\mathcal{X}$ the smallest additive subcategory containing $\mathcal{X}$.

Theorem. *Let $\mathcal{C}$ be an infinite cogeneration-closed subcategory of* $\operatorname{mod}\Lambda$*. Then $\mathcal{C}$ contains a minimal infinite cogeneration-closed subcategory $\mathcal{C}'$.*

Proof. We denote by $\mathbb{N} = \mathbb{N}_1$ the natural numbers starting with 1. Given a Gabriel-Roiter measure I, let $\mathcal{C}(I)$ be the set of isomorphism classes of indecomposable objects in $\mathcal{C}$ with Gabriel-Roiter measure I. An obvious adaption of one of the main results of [R1] asserts:

There is an infinite sequence of Gabriel-Roiter measures $I_1 < I_2 < \cdots$ such that $\mathcal{C}(I_t)$ is non-empty for any $t \in \mathbb{N}$ and such that for any J with $\mathcal{C}(J) \neq \emptyset$, either $J = I_t$ for some t or else $J > I_t$ for all t. Moreover, all the sets $\mathcal{C}(I_t)$ are finite. (Note that the sequence of measures I_t depends on $\mathcal{C}$, thus one should write $I_t^{\mathcal{C}} = I_t$; the papers [R1,R2] were dealing only with the case $\mathcal{C} = \operatorname{mod}\Lambda$, but the proofs carry over to the more general case of dealing with a cogeneration-closed subcategory $\mathcal{C}$).

Since $\operatorname{add}\bigcup_{t\in\mathbb{N}}\mathcal{C}(I_t)$ is cogeneration-closed, we can assume that $\mathcal{C} = \operatorname{add}\bigcup_{t\in\mathbb{N}}\mathcal{C}(I_t)$. In order to construct $\mathcal{C}'$, we will construct a sequence of subcategories

$$\mathcal{C} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots$$

with the following properties:
(a) Any subcategory $\mathcal{C}_i$ is infinite and cogeneration-closed,

(b) $$\mathcal{C}_i(I_t) = \mathcal{C}_t(I_t) \qquad \text{for} \qquad t \le i.$$

(c) If $\mathcal{D} \subseteq \mathcal{C}_i$ is infinite and cogeneration-closed, then

$$\mathcal{D}(I_t) = \mathcal{C}_t(I_t) \qquad \text{for} \qquad t \le i.$$

We start with $\mathcal{C}_0 = \mathcal{C}$ (the t in conditions (b) and (c) satisfies $t \geq 1$, thus nothing has to be verified). Assume, we have constructed $\mathcal{C}_i$ for some $i \geq 0$, satisfying the conditions (a), and the conditions (b), (c) for all pairs (i,t) with $t \leq i$. We are going to construct $\mathcal{C}_{i+1}$.

Call a subset $\mathcal{X}$ of $\mathcal{C}_i(I_{i+1})$ *good*, provided there is a subcategory $\mathcal{D}_{\mathcal{X}}$ of $\mathcal{C}_i$ which is infinite and cogeneration-closed and such that $\mathcal{D}_{\mathcal{X}}(I_{i+1}) = \mathcal{X}$. For example $\mathcal{C}_i(I_{i+1})$ itself is good (with $\mathcal{D}_{\mathcal{X}} = \mathcal{C}_i$). Since $\mathcal{C}_i(I_{i+1})$ is a finite set, we can choose a minimal good subset $\mathcal{X}' \subseteq \mathcal{X}$. For $\mathcal{X}'$, there is an infinite and cogeneration-closed subcategory $\mathcal{D}_{\mathcal{X}'}$ of $\mathcal{C}_i$ such that $\mathcal{D}_{\mathcal{X}'}(I_{i+1}) = \mathcal{X}'$. (Note that in general neither $\mathcal{X}'$ nor $\mathcal{D}_{\mathcal{X}'}$ will be uniquely determined: usually, there may be several possible choices. Also note that $\mathcal{X}'$ may be empty.) Let $\mathcal{C}_{i+1} = \mathcal{D}_{\mathcal{X}'}$. By assumption, $\mathcal{C}_{i+1}$ is infinite and cogeneration-closed, thus (a) is satisfied. In order to show (b) for all pairs $(i+1,t)$ with $t \leq i+1$, we first consider some $t \leq i$. We can apply (c) for $\mathcal{D} = \mathcal{C}_{i+1} \subseteq \mathcal{C}_i$ and see that $\mathcal{D}(I_t) = \mathcal{C}_t(I_t)$, as required. But for $t = i+1$, nothing has to be shown. Finally, let us show (c). Thus let $\mathcal{D} \subseteq \mathcal{C}_{i+1}$ be an infinite cogeneration-closed subcategory. Since $\mathcal{D} \subseteq \mathcal{C}_i$, we know by induction that $\mathcal{D}(I_t) = \mathcal{C}_t(I_t)$ for $t \leq i$. It remains to show that $\mathcal{D}(I_{i+1}) = \mathcal{C}_{i+1}(I_{i+1})$. Since $\mathcal{D} \subseteq \mathcal{C}_{i+1}$, we have $\mathcal{D}(I_{i+1}) \subseteq \mathcal{C}_{i+1}(I_{i+1})$. But if this would be a proper inclusion, then $\mathcal{X} = \mathcal{D}(I_{i+1})$ would be a good subset of $\mathcal{C}_i(I_{i+1})$ which is properly contained in $\mathcal{C}_{i+1}(I_{i+1}) = \mathcal{D}_{\mathcal{X}'}(I_{i+1})$, a contradiction to the minimality of $\mathcal{X}'$. This completes the inductive construction of the various $\mathcal{C}_i$.

Now let

$$\mathcal{C}' = \bigcap\nolimits_{i \in \mathbb{N}} \mathcal{C}_i.$$

Of course, $\mathcal{C}'$ is cogeneration-closed. Also, we see immediately

(b′) $\qquad \mathcal{C}'(I_t) = \mathcal{C}_t(I_t) \qquad \text{for all} \qquad t,$

since $\mathcal{C}'(I_t) = \bigcap_{i \geq t} \mathcal{C}_i(I_t) = \mathcal{C}_t(I_t)$, according to (b).

First, we show that $\mathcal{C}'$ is infinite. Of course, $\mathcal{C}'(I_1) \neq \emptyset$, since $I_1 = \{1\}$ and a good subset of $\mathcal{C}_0(I_1)$ has to contain at least one simple module. Assume that $\mathcal{C}'(I_s) \neq \emptyset$ for some s, we want to see that there is $t > s$ with $\mathcal{C}'(I_t) \neq \emptyset$. For every Gabriel-Roiter measure I, let $n(I)$ be the minimal number n with $I \subseteq [1,n]$, thus $n(I)$ is the length of the modules in $\mathcal{C}(I)$. Let $n(s)$ be the maximum of $n(I_j)$ with $j \leq s$, thus $n(s)$ is the maximal length of the modules in $\bigcup_{j \leq s} \mathcal{C}(I_j)$. Let s' be a natural number such that $n(I_j) > n(s)pq$ for all $j > s'$ (such a number exists, since the modules in I_j with j large, have large length); here p is the maximal length of an indecomposable projective module, q that of an indecomposable injective module.

We claim that $\mathcal{C}'(I_j) \neq \emptyset$ for some j with $s < j \leq s'$. Assume for the contrary that $\mathcal{C}'(I_j) = \emptyset$ for all $s < j \leq s'$. We consider $\mathcal{C}_{s'}$. Since $\mathcal{C}_{s'}$ is infinite, there is some $t > s$ with $\mathcal{C}_{s'}(I_t) \neq \emptyset$, and we choose t minimal. Now for $s < j \leq s'$, we know that $\mathcal{C}_{s'}(I_j) = \mathcal{C}_j(I_j) = \mathcal{C}'(I_j) = \emptyset$, according to (b) and (b′). This shows that $t > s'$. Let Y be an indecomposable module with isomorphism class in $\mathcal{C}_{s'}(I_t)$. Let X be a Gabriel-Roiter submodule of Y. Then X belongs to $\mathcal{C}_{s'}(I_j)$ with $j < t$. If $j \leq s$, then the length of X is bounded by $n(s)$, and therefore Y is bounded by $n(s)pq$ (see [R2], 3.1 Corollary), in contrast to the fact that $n(I_t) > n(s)pq$. Thus $j > s$. Buth then $s < j < t$ and $\mathcal{C}_{s'}(I_j) \neq \emptyset$ — this contradicts the minimality of t. This final contradiction shows that $\mathcal{C}'$ is infinite.

Now, let $\mathcal{D}$ be an infinite cogeneration-closed subcategory of $\mathcal{C}'$. We show that $\mathcal{D}[I_t] = \mathcal{C}'[I_t]$ for all t. Consider some fixed t and choose an i with $i \geq t$. Since $\mathcal{C}' \subseteq \mathcal{C}_i$, we see that $\mathcal{D}[t] = \mathcal{C}_t[t]$ the given t, according to (b) for $\mathcal{C}_i$. But according to (b$'$), we also know that $\mathcal{C}'[t] = \mathcal{C}_t[t]$. This completes the proof.

Example 1. *Any tame concealed algebras has a unique minimal infinite cogeneration-closed subcategory $\mathcal{C}$, namely the subcategory of all preprojective modules.*

Example 2. Let I be a twosided ideal in Λ. The category of Λ-modules annihilated by I is obviously cogeneration-closed and of course equivalent (or even equal) to the category of all Λ/I-modules. If Λ/I is representation-finite, then $\operatorname{mod}\Lambda/I$ will contain a minimal infinite cogeneration-closed subcategory. Consider for example the generalized Kronecker-algebra $K(3)$ with three arrows α, β, γ. The one-dimensional ideals of $K(3)$ correspond bijectively to the elements of the projective plane $\mathbb{P}^2$, say $a = (a_0 : a_1 : a_2) \in \mathbb{P}^2$ yields the ideal $I_a = \langle a_0\alpha + a_1\beta + a_2\gamma \rangle$. Let $\mathcal{C}_a$ be additive subcategory of $\operatorname{mod} K(3)$ of all preprojective $K(3)/I_a$-modules. Then these are pairwise different minimal infinite cogeneration-closed subcategories (the intersection of any two of these subcategories is the subcategory of semisimple projective modules). In particular, *if the base field is finite, there are infinitely many subcategories in* $\operatorname{mod} K(3)$ *which are minimal infinite and cogeneration-closed.* (Note that the preprojective $K(3)$-modules provide a further subcategory which is minimal infinite and cogeneration-closed.)

Example 3. There can be several different take-off categories containing all the indecomposable projective modules: Take the take-off part, as well as the preprojective component of the algebra with 3 vertices a, b, c, two arrows $b \to a$, and two arrows $c \to b$.

References.

[R1] Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005), 726-748.

[R2] Ringel: Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure. To appear in the Proceedings of the Queretaro Workshop 2004. Contemporary Mathematics. Amer.Math.Soc.