BURNSIDE'S THEOREM FOR HOPF ALGEBRAS

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In our lecture [1] we have introduced Burnside's Theorem for complex representations of finite groups, whose original proof employed characters of finite groups. It took about 50 years until the subject was taken up again by Steinberg [5], who produced a precursor to the modern account using Hopf algebras [4, 3]. This broader point of view is suggested by results of a similar flavor, such as the realization of the finite dimensional simple $\mathfrak{sl}(2,\mathbb{C})$-modules via polynomials in two variables.

Throughout, we will be working over an arbitrary field k. Let us informally recall the notion of a Hopf algebra. Given an algebra H, we consider the algebra $H \otimes_k H$, whose product is defined on simple tensors via

$$(a\otimes b)(c\otimes d) := ab\otimes cd \qquad \forall\ a,b,c,d \in H.$$

If V and W are H-modules, then

$$(a\otimes b).(v\otimes w) := a.v\otimes c.w \qquad \forall\ a,b\in H,\ v\in V,\ w\in W$$

endows $V\otimes_k W$ with the structure of an $H\otimes_k H$-module. Letting $\mathrm{ann}_H(M)$ denote the annihilator of the H-module M, we obtain

$$(*) \qquad \mathrm{ann}_{H\otimes_k H}(V\otimes_k W) = \mathrm{ann}_H(V)\otimes_k H + H\otimes_k \mathrm{ann}_H(W).$$

One key ingredient of a Hopf algebra is the presence of an algebra homomorphism $\Delta : H \longrightarrow H\otimes_k H$, the so-called *comultiplication*. Here are three examples to bear in mind:

- If $H = kG$ is the group algebra of a group G, then Δ is given by
$$\Delta(g) = g\otimes g \quad \forall\ g\in G.$$
- Let $H = U(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then Δ is the unique algebra homomorphism extending
$$\Delta(x) = x\otimes 1 + 1\otimes x \qquad \forall\ x\in\mathfrak{g}.$$
- Let $(\mathfrak{g},[p])$ be a restricted Lie algebra over a field of characteristic p. Then
$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]}\ ;\ x\in\mathfrak{g}\})$$
is a Hopf algebra, whose comultiplication is inherited from that of $U(\mathfrak{g})$.

Returning to H, we note that pull-back along Δ endows $V\otimes_k W$ with the structure of an H-module. Equation $(*)$ now implies

$$(**) \qquad \Delta(\mathrm{ann}_H(V\otimes_k W)) \subset \mathrm{ann}_H(V)\otimes_k H + H\otimes_k \mathrm{ann}_H(W).$$

Definition. Let H be a Hopf algebra. An ideal $I \lhd H$ is called a *bi-ideal* if

$$\Delta(I) \subset I\otimes_k H + H\otimes_k I.$$

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If M is an H-module such that

$$\mathrm{ann}_H(M) \subset \mathrm{ann}_H(M \otimes_k M),$$

then $(**)$ shows that $\mathrm{ann}_H(M)$ is a bi-ideal.

The coassociativity of the comultiplication gives rise to isomorphisms

$$(U \otimes_k V) \otimes_k W \cong U \otimes_k (V \otimes_k W)$$

of H-modules. This fact is used in the proof of the following basic result:

Lemma 1. *Let V be an H-module. Then $\mathrm{ann}_H(\bigoplus_{m\geq 1} V^{\otimes m})$ is a bi-ideal of H.*

Proof. Setting $M := \bigoplus_{m\geq 1} V^{\otimes m}$, we obtain

$$M \otimes_k M \cong \bigoplus_{m,n\geq 1} V^{\otimes m} \otimes_k V^{\otimes n} \cong \bigoplus_{m\geq 2} (m-1)V^{\otimes m},$$

so that the inclusion

$$\mathrm{ann}_H(M) = \bigcap_{m\geq 1} \mathrm{ann}_H(V^{\otimes m}) \subset \mathrm{ann}_H(M \otimes_k M)$$

implies our assertion. □

Theorem 2. *Let V be a module for a finite dimensional Hopf algebra H. If $\mathrm{ann}_H(V)$ does not contain any non-zero bi-ideals, then every simple H-module S is a submodule of a tensor power $V^{\otimes m}$ for some $m \geq 1$.*

Proof. Thanks to Lemma 1, the H-module $M := \bigoplus_{m\geq 1} V^{\otimes m}$ is faithful. Let $I \subset H$ be a minimal left ideal. Then $I.M \neq (0)$, and there exist $m \in \mathbb{N}$ and $v \in V^{\otimes m}$ such that $I.v \neq (0)$. Since I is a simple H-module, the map $x \mapsto x.v$ thus defines an embedding $I \hookrightarrow V^{\otimes m}$.

As a result, every simple module belonging to $\mathrm{Soc}(H)$ is a submodule of a suitable tensor power of V. By the Theorem of Larson-Sweedler [2], the algebra H is self-injective. Consequently, every simple H-module occurs in $\mathrm{Soc}(H)$, so that our assertion follows. □

Remark. Let $e_S \in S$ be an idempotent corresponding to the simple H-module S. Since M is faithful, we have

$$(0) \neq e_S M \cong \mathrm{Hom}_H(He_S, M).$$

Consequently, the occurrence of S as a composition factor of some $V^{\otimes m}$ can be shown without reference to [2].

To retrieve Burnside's Theorem from the above result, we require further structural ingredients of Hopf algebras. By definition, there exists an algebra homomorphism $\varepsilon : H \longrightarrow k$ such that $V \otimes_k k_\varepsilon \cong V$ for every H-module V. The map ε, which turns out to be the unit element of the dual algebra H^*, is called the *counit* of H. Another important map, which corresponds to $g \mapsto g^{-1}$ in case $H = kG$, is the *antipode* $\eta : H \longrightarrow H$ of H. By definition, we have

$$\sum_{(h)} \eta(h_{(1)})h_{(2)} = \varepsilon(h)1 = \sum_{(h)} h_{(1)}\eta(h_{(2)}) \quad \forall\, h \in H,$$

where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$.

Definition. A bi-ideal satisfying $\eta(I) \subset I$ is referred to as a *Hopf ideal*.

Note that every Hopf ideal $I \subsetneq H$ is contained in the *augmentation ideal* $H^\dagger := \ker \varepsilon$. We require the following Lemma.

Lemma 3. *Let H be a finite dimensional Hopf algebra. Then every bi-ideal is a Hopf ideal.* □

Let H be a Hopf algebra. An element is $h \neq 0$ is *group-like* if $\Delta(h) = h \otimes h$. The set $G(H)$ of group-like elements is linearly independent and forms a subgroup of the group of invertible elements of H. Every homomorphism $f : H \longrightarrow H'$ of Hopf algebras induces a homomorphism $f : G(H) \longrightarrow G(H')$ of groups. Burnside's Theorem now follows from Theorem 2, Lemma 3 and the determination of the Hopf ideals of group algebras.

Lemma 4. *Let $H = kG$ be the group algebra of a finite group, $I \subsetneq H$ a Hopf ideal. Then there exists a normal subgroup $N \lhd G$ such that $I = (kG)(kN)^\dagger$.*

Proof. The canonical projection $\pi : H \longrightarrow H/I$ is a homomorphism of Hopf algebras, and H/I is generated by $\pi(G(H)) \subset G(H/I)$. Thus, $G(H) = G$ and $G(H/I) = \pi(G)$, so that $\dim_k H/I = |G/N|$, where $N := \ker \pi|_G$ is a normal subgroup of G. On the other hand, $J_N := (kG)(kN)^\dagger$ is a Hopf ideal of H with $J_N \subset I$. Thus, π induces a surjection

$$kG/J_N \longrightarrow H/I$$

of Hopf algebras. The left-hand side, being isomorphic to the group algebra $k(G/N)$, has dimension $\dim_k H/I$. As a result, $I = (kG)(kN)^\dagger$. □

Remarks. (1) Passing to dual modules, one can show that every simple module occurs as a factor of some tensor power of a "Hopf-faithful" H-module.

(2) Lemma 4 generalizes to finite dimensional cocommutative Hopf algebras. By general theory [6], such an algebra is the "group algebra" $H = k\mathcal{G}$ of a finite group scheme $\mathcal{G}$, and one can prove that every proper Hopf ideal $I \subset k\mathcal{G}$ is of the form $I = (k\mathcal{G})(k\mathcal{N})^\dagger$, for a suitable normal subgroup scheme $\mathcal{N} \subset \mathcal{G}$.

In particular, if V is a finite dimensional faithful restricted module of the finite dimensional restricted Lie algebra $(\mathfrak{g}, [p])$, then every simple restricted $\mathfrak{g}$-module occurs in a suitable tensor power of V.

(3) In view of (2), the comments of [1] pertaining to McKay quivers readily generalize to the context of linearly reductive group schemes (semisimple cocommutative Hopf algebras).

(4) If $\mathfrak{g}$ is a finite dimensional Lie algebra, then its universal enveloping algebra $U(\mathfrak{g})$ is free of zero divisors, so that 0 and 1 are the only idempotents of $U(\mathfrak{g})$. Thus, the arguments of Theorem 2 and its succeeding remark do not carry over to this context. For the classical case $\mathfrak{g} = \mathfrak{sl}(2)$, $V = L(1)$, the analog of Burnside's Theorem for finite dimensional modules follows from the Clebsch-Gordan formula, which implies that the McKay quiver $\Psi_{L(1)}$ is the double of a quiver with underlying graph A_∞.

References

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