SELF-INJECTIVE ALGEBRAS: THE NAKAYAMA PERMUTATION

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Let Λ be a finite dimensional algebra, defined over a field k. The category of finite dimensional left Λ-modules and the set of isoclasses of simple Λ modules will be denoted by $\operatorname{mod}\Lambda$ and $\mathcal{S}(\Lambda)$, respectively. We will occasionally identify $\mathcal{S}(\Lambda)$ with a complete set of its representatives. Given a simple Λ-module S, we consider its projective cover $P(S)$ and its injective envelope $E(S)$. Recall that $\operatorname{Top}(P(S)) \cong S$ and $\operatorname{Soc}(E(S)) \cong S$. Moreover, for every projective (injective) indecomposable Λ-module Q there exists exactly one $S \in \mathcal{S}(\Lambda)$ with $Q \cong P(S)$ ($Q \cong E(S)$) (cf. [1, (I.4),(II.4)]).

What can be said about the structure of $\operatorname{Soc}(P(S))$ (or $\operatorname{Top}(E(S))$)? Let us consider simple examples.

Examples. (1) Let $\Lambda = k[1 \to 2]$. Setting $P_i := P(S_i)$, we obtain $\operatorname{Soc}(P_1) = S_2 = \operatorname{Soc}(P_2)$.
(2) If $\Lambda = k[1 \leftarrow 2 \to 3]$, then

$$\operatorname{Soc}(P_1) = S_1 \quad ; \quad \operatorname{Soc}(P_2) = S_1 \oplus S_3 \quad ; \quad \operatorname{Soc}(P_3) = P_3.$$

In general, one can thus not hope for a fixed pattern. We are going to introduce a class of algebras where such a pattern exists and show that this class is in fact determined by the presence of a correspondence between tops and socles of the principal indecomposable modules $P(S)$. These so-called *quasi-Frobenius algebras* were introduced and studied by Nakayama [3, 4], whose work was inspired by results of Brauer and Nesbitt [2]. Nowadays, the following notion is commonly used.

Definition. The algebra Λ is *self-injective* if $\Lambda \in \operatorname{mod}\Lambda$ is injective.

The principal result of this lecture was proved by Nakayama in the context of basic algebras.

Theorem. *The following statements are equivalent:*
(1) *The algebra Λ is self-injective.*
(2) *The rule $[S] \mapsto [\operatorname{Soc}(P(S))]$ defines a permutation $\nu : \mathcal{S}(\Lambda) \longrightarrow \mathcal{S}(\Lambda)$.*

The permutation ν is referred to as the *Nakayama permutation* of the self-injective algebra Λ.

Given $M \in \operatorname{mod}\Lambda$, its dual $M^* := \operatorname{Hom}_k(M, k)$ has the structure of a right Λ-module. Thus, $M \mapsto M^*$ is a duality between the categories $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$, where Λ^{op} denotes the opposite algebra of Λ. In particular, $?^*$ takes projectives to injectives and vice versa.

Lemma 1. *Let Λ be self-injective.*
(1) *A Λ-module M is projective if and only if it is injective.*
(2) $\operatorname{Soc}(P(S))$ *is simple for every* $S \in \mathcal{S}(\Lambda)$.
(3) $P(S) \cong E(\operatorname{Soc}(P(S))$ *for every* $S \in \mathcal{S}(\Lambda)$.
(4) *Let S, T be simple Λ-modules. If* $\operatorname{Soc}(P(S)) \cong \operatorname{Soc}(P(T))$, *then* $S \cong T$.

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Proof. If M is projective, then M is a direct summand of a free module Λ^n. Thus, M is, as a direct summand of an injective module, injective.

It follows that $\{[P(S)]\ ;\ [S] \in \mathcal{S}(\Lambda)\}$ is a set of isoclasses of injective indecomposable Λ modules of cardinality $|\mathcal{S}(\Lambda)|$. It thus coincides with the set of isoclasses of indecomposable injectives, and there exists a permutation $\nu : \mathcal{S}(\Lambda) \longrightarrow \mathcal{S}(\Lambda)$ such that

$$P(S) \cong E(\nu(S)) \quad \forall\ [S] \in \mathcal{S}(\Lambda).$$

In particular, we have:

- Every indecomposable injective module is projective, so that (1) follows.
- $[\mathrm{Soc}(P(S))] = [\mathrm{Soc}(E(\nu(S)))] = [\nu(S)] \quad \forall\ [S] \in \mathcal{S}(\Lambda)$. Thus, we obtain (2)-(4).

□

Remarks. (a) Owing to (1), the class of self-injective algebras is stable under Morita equivalence.

(b) Since the right module $\Lambda \in \mathrm{mod}\,\Lambda^{\mathrm{op}}$ is projective, its dual is injective, hence projective, so that Λ_Λ is injective. In other words, the algebra Λ^{op} is self-injective.

By general theory, the principal indecomposable Λ modules are of the form

$$P = \Lambda e,$$

where $e \in \Lambda$ is a primitive idempotent. If $M \in \mathrm{mod}\,\Lambda$, then

$$\mathrm{Hom}_\Lambda(P, M) \longrightarrow eM \quad ; \quad f \mapsto f(e)$$

is an isomorphism of vector spaces, which is right Λ-linear in case M is a (Λ, Λ)-bimodule.

The Nakayama permutation is a combinatorial tool that does not provide any information concerning the endomorphism rings of S and $\nu(S)$ (these are actually isomorphic). For algebras over algebraically closed fields, this is of course not a problem. However, a better understanding of ν necessitates a module theoretic (functorial) description.

Definition. Let Λ be a k-algebra. The functor

$$\mathcal{N} : \left\{ \begin{array}{ccc} \mathrm{mod}\,\Lambda & \longrightarrow & \mathrm{mod}\,\Lambda \\ M & \mapsto & \mathrm{Hom}_\Lambda(M, \Lambda)^* \end{array} \right.$$

is called the *Nakayama functor* of Λ.

The above observations yield

$$\mathcal{N}(\Lambda e) \cong (e\Lambda)^*,$$

so that $\mathcal{N}$ sends indecomposable projectives to indecomposable injectives. (However, it does in general not send indecomposables to indecomposables.)

Lemma 2. *Let Λ be a k-algebra that affords a Nakayama permutation $\nu : \mathcal{S}(\Lambda) \longrightarrow \mathcal{S}(\Lambda)$. Then the following statements hold:*

(1) $\mathcal{N}(P(S)) \cong E(S) \quad \forall\ S \in \mathcal{S}(\Lambda)$.

(2) $\mathcal{N}(S) \cong \nu^{-1}(S) \quad \forall\ S \in \mathcal{S}(\Lambda)$.

Proof. (1) Let S be a simple Λ-module. Pick a primitive idempotent $e_S \in \Lambda$ with $P(S) = \Lambda e_S$. Since

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(e_S\Lambda, S^*) \cong (S^*e_S) \cong (e_S S)^* \cong \mathrm{Hom}_\Lambda(P(S), S)^* \neq (0),$$

it follows that the principal indecomposable right Λ-module $e_S\Lambda$ has top S^*. As the duality $?^*$ maps tops to socles, we obtain

$$S \cong \mathrm{Top}(e_S\Lambda)^* \cong \mathrm{Soc}((e_S\Lambda)^*) \cong \mathrm{Soc}(\mathcal{N}(\Lambda e_S)) \cong \mathrm{Soc}(\mathcal{N}(P(S))),$$

so that the indecomposable injective Λ-module $\mathcal{N}(P(S))$ is isomorphic to $E(S)$.

(2) Let S, T be simple Λ-modules, e_S, e_T the corresponding primitive idempotents. Given any Λ-module M, we have

$$\dim_k \mathrm{Hom}_\Lambda(P(T), M) = [M : T] \dim_k \mathrm{End}_\Lambda(T),$$

where $[M : T]$ denotes the multiplicity of T in M. From the k- vector space isomorphisms

$$\begin{aligned} \mathrm{Hom}_\Lambda(P(T), \mathcal{N}(S)) &\cong e_T\mathcal{N}(S) \cong e_T \mathrm{Hom}_\Lambda(S, \Lambda)^* \cong (\mathrm{Hom}_\Lambda(S, \Lambda)e_T)^* \\ &\cong \mathrm{Hom}_\Lambda(S, \Lambda e_T)^* \cong \mathrm{Hom}_\Lambda(S, \nu(T))^* \\ &\cong \begin{cases} \mathrm{End}_\Lambda(S) & \text{if } T \cong \nu^{-1}(S) \\ (0) & \text{otherwise} \end{cases}, \end{aligned}$$

we see that $\nu^{-1}(S)$ is the only composition factor of $\mathcal{N}(S)$. By the same token, we have

$$[\mathcal{N}(S) : \nu^{-1}(S)] \dim_k \mathrm{End}_\Lambda(\nu^{-1}(S)) = \dim_k \mathrm{Hom}_\Lambda(P(\nu^{-1}(S)), \mathcal{N}(S)) = \dim_k \mathrm{End}_\Lambda(S).$$

By applying this formula successively to the modules $\nu^{-i}(S)$, we obtain, observing $\nu^{-n}(S) \cong S$ for some $n \in \mathbb{N}$, a natural number $m \in \mathbb{N}$ such that

$$\dim_k \mathrm{End}_\Lambda(S) = m[\mathcal{N}(S) : \nu^{-1}(S)] \dim_k \mathrm{End}_\Lambda(S).$$

Thus, $[\mathcal{N}(S) : \nu^{-1}(S)] = 1$, and $\mathcal{N}(S) \cong \nu^{-1}(S)$. □

Proof of the Theorem. In view of Lemma 1, it suffices to verify (2) ⇒ (1). Let $\ell(M)$ denote the length of the Λ-module M. The Nakayama functor is right exact and Lemma 2 ensures that it takes simples to simples. Induction on $\ell(M)$ then implies

$$\ell(\mathcal{N}(M)) \leq \ell(M) \quad \forall\, M \in \mathrm{mod}\,\Lambda.$$

Lemma 2 now yields

$$\ell(E(S)) \leq \ell(P(S)) \quad \forall S \in \mathcal{S}(\Lambda).$$

Since $\mathrm{Soc}(P(S)) = \nu(S)$, we have an embedding $\iota_S : P(S) \hookrightarrow E(\nu(S))$. Iteration gives rise to a chain

$$\ell(E(S)) \leq \ell(P(S)) \leq \ell(E(\nu(S))) \leq \ell(P(\nu(S)) \leq \ell(E(\nu^2(S)) \leq \cdots \leq \ell(E(S)),$$

so that $\ell(P(S)) = \ell(E(\nu(S)))$. As a result, ι_S is bijective, showing that $P(S)$ is injective. This implies the self-injectivity of Λ. □

References

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