SELF-INJECTIVE ALGEBRAS: EXAMPLES AND MORITA EQUIVALENCE

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Let Λ be a finite dimensional algebra, defined over a field k. The category of finite dimensional left Λ-modules and the set of isoclasses of simple Λ modules will be denoted by $\operatorname{mod}\Lambda$ and $\mathcal{S}(\Lambda)$, respectively. Given a simple Λ-module S, we let $P(S)$ be its projective cover.

With our previous lectures in hand, it is fairly easy to construct examples of classes of self-injective algebras:

Examples. (1) The initial example of a Frobenius algebras was the group algebra of a finite group. Writing $\Lambda := kG$, we consider the linear form $\pi : kG \longrightarrow k$ given by

$$\pi(g) := \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases} .$$

If $x = \sum_{g \in G} \alpha_g g$ belongs to $\ker \pi \setminus \{0\}$, then we have $\alpha_{g_0} \neq 0$ for some $g_0 \in G$, whence $\pi(g_0^{-1}x) \neq 0$. Accordingly, $\ker \pi$ does not contain any non-zero left ideals. Moreover, we have

$$\pi(gh) = \pi(hg) \quad \forall\ g, h \in G,$$

so that the corresponding non-degenerate associative form $(\, , \,)_\pi$ is symmetric.

(2) In contrast to self-injective algebras, the class of Frobenius algebras is not stable under Morita equivalence. Let

$$\Lambda := k[\tilde{A}_1]/k[\tilde{A}_1]_{(2)}$$

be the bound quiver algebra of the quiver $\tilde{A}_1$ with cyclic orientation, and with paths of length ≥ 2 being zero. Then Λ is a Frobenius algebra with Nakayama permutation $\nu(i) = i + 1 \mod(2)$ (cf. [3, Theorem 3]).

Being Morita equivalent to Λ, the algebra $\Gamma := \operatorname{End}_\Lambda(2P_1 \oplus P_2)^{\mathrm{op}}$ is self-injective. Since the dimensions of the simple Γ-modules S_1 and S_2 are 2 and 1, respectively, we have $1 = \dim_k \nu(S_1) \neq \dim_k S_1$. Consequently, [3, Theorem 3] implies that Γ is not a Frobenius algebra.

(3) If Λ is a basic Frobenius algebra, then each simple Λ-module occurs in $\operatorname{Soc}(\Lambda)$ with multiplicity one (cf. [2, Thm.]), so that

$$\operatorname{Soc}(\Lambda) = \bigoplus_{S \in \mathcal{S}(\Lambda)} S.$$

Let $\pi : \Lambda \longrightarrow k$ be a linear form such that $\pi|_S \neq 0$ for each constituent of $\operatorname{Soc}(\Lambda)$. If $J \subset \Lambda$ is a non-zero left ideal, then $\operatorname{Soc}(J) \neq (0)$, so that J contains at least one constituent of the above isotypic decomposition. As a result, $J \not\subset \ker \pi$, and π endows Λ with the structure of a Frobenius algebra (cf. [3, Lemma 1]).

In practice, it is often difficult to determine $\operatorname{Soc}(\Lambda)$. However, if Λ is given as a bound quiver algebra $k[Q]/I$, then we obtain an explicit definition of π. Let Q be the double of the quiver $\tilde{A}_{n-1}$, with arrows $\alpha_i : i \mapsto i + 1$ and $\beta_i : i \mapsto i - 1$ for $i \in \mathbb{Z}/(n)$. The relations are

$$I = (\{\alpha_{i-1}\beta_i - \beta_{i+1}\alpha_i,\ \alpha_{i+1}\alpha_i,\ \beta_{i-1}\beta_i\ ;\ i \in \mathbb{Z}/(n)\}).$$

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Letting $k[Q]_{(3)}$ be the vector space generated by all paths of length ≥ 3, we define a linear map $\omega : k[Q] \longrightarrow k$ via

$$\omega(e_i) = 0 = \omega(\alpha_i) = \omega(\beta_i), \ \ \omega(\alpha_{i-1}\beta_i) = 1 = \omega(\beta_{i+1}\alpha_i),$$

and

$$\omega(\alpha_{i+1}\alpha_i) = 0 = \omega(\beta_{i-1}\beta_i), \ \ \omega(k[Q]_{(3)}) = (0).$$

Then ω induces a linear map $\pi : k[Q]/I \longrightarrow k$ that endows $k[Q]/I$ with the structure of a Frobenius algebra.

Definition. The k-algebra Λ is *symmetric* if it possesses a non-degenerate, symmetric, associative form. We say that Λ is *weakly symmetric* if $\mathrm{Soc}(P(S)) \cong S$ for every simple Λ-module S.

Remarks. (i) If Λ is symmetric, then its Nakyama functor is the identity, so that Λ is weakly symmetric (cf. [3]). Example (1) shows that group algebras of finite groups are symmetric. This fact plays an important rôle in the classification of tame blocks of group algebras [1].

(ii) The class of weakly symmetric algebras is stable under Morita equivalence.

Example. (4) Let $q \in k \setminus \{0\}$ and consider

$$\Lambda_q := k\langle x, y\rangle/(\{x^2, y^2, yx - qxy\}).$$

The algebra is local with $\mathrm{Soc}(\Lambda_q) = kxy$. In view of Example (3), the linear forms $\pi : \Lambda \longrightarrow k$ giving Λ_q the structure of a Frobenius algebra satisfy $\pi(xy) \neq 0$. If the corresponding bilinear form $(\, , \,)_\pi$ is symmetric, then

$$q\,\pi(xy) = \pi(qxy) = \pi(yx) = \pi(xy),$$

so that $q = 1$. Thus, Λ_q is weakly symmetric, but usually not symmetric. A Nakayama automorphism $\mu_q : \Lambda_q \longrightarrow \Lambda_q$ is given by

$$\mu_q(x) = qx \ \text{ and } \ \mu_q(y) = q^{-1}y,$$

so that μ_q usually has infinite order.

Proposition. *Let Λ be a symmetric algebra.*

(1) *For any idempotent $e \in \Lambda$, the algebra $e\Lambda e$ is symmetric.*

(2) *If Γ is Morita equivalent to Λ, then Γ is symmetric.*

Proof. (1) Let $(\, , \,) : \Lambda \times \Lambda \longrightarrow k$ be a non-degenerate, symmetric, associative form. We will show that the restriction $(\, , \,)_e$ of $(\, , \,)$ to $(e\Lambda e) \times (e\Lambda e)$ is non-degenerate. Given $x \in \mathrm{Rad}(\, , \,)_e$, we obtain for every $y \in \Lambda$:

$$(x, y) = (exe, y) = (ex, ey) = (ey, ex) = (eye, x) = (eye, x)_e = 0.$$

Thus, $x \in \mathrm{Rad}(\, , \,) = (0)$, as desired.

(2) By general theory, there exists a projective Λ-module P such that

$$\Gamma^{\mathrm{op}} \cong \mathrm{End}_\Lambda(P).$$

Since Γ is symmetric if and only if Γ^{op} enjoys this property, it thus suffices to show that $\mathrm{End}_\Lambda(P)$ is symmetric. We may write

$$\Lambda^n \cong P \oplus P'$$

for a suitable $n \in \mathbb{N}$ and some $P' \in \mathrm{mod}\,\Lambda$. Consequently, the map $e : \Lambda^n \xrightarrow{\mathrm{pr}} P \xrightarrow{\iota} \Lambda^n$ is an idempotent of $\mathrm{Mat}_n(\Lambda)$ such that $\mathrm{End}_\Lambda(P) \cong e\mathrm{Mat}_n(\Lambda)e$.

Let $\pi : \Lambda \longrightarrow k$ be a linear map such that $(\, , \,)_\pi$ is a non-degenerate symmetric form on Λ. Then

$$\varrho : \mathrm{Mat}_n(\Lambda) \longrightarrow k \quad ; \quad A \mapsto \pi(\mathrm{tr}(A))$$

satisfies $\varrho(AB) = \varrho(BA)$ and defines a non-degenerate, symmetric, associative form on $\mathrm{Mat}_n(\Lambda)$. Owing to (1), the algebra $\mathrm{End}_\Lambda(P)$ is also symmetric. □

Examples. (5) We consider the trivial extension $T(\Lambda) := \Lambda \ltimes \Lambda^*$ of Λ by its bimodule Λ^*. By definition, the multiplication of $T(\Lambda)$ is given by

$$(a, \varphi) \cdot (b, \psi) := (ab, a.\psi + \varphi.b) \qquad \forall \; a, b \in \Lambda, \; \varphi, \psi \in \Lambda^*.$$

The linear form

$$\pi : T(\Lambda) \longrightarrow k \quad ; \quad (a, \varphi) \mapsto \varphi(1)$$

endows $T(\Lambda)$ with the structure of a symmetric algebra. If $n \in \mathbb{N}$ is even, then the trivial extension of the radical square zero algebra $k[\tilde{A}_{n-1}]$ (i.e., the orientation is chosen such that there are no paths of length 2) is just the algebra discussed in Example (3).

(6) Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over a field k of characteristic $p > 0$. As usual, we denote the left multiplication effected by $x \in \mathfrak{g}$ by

$$\mathrm{ad}\, x : \mathfrak{g} \longrightarrow \mathfrak{g} \quad ; \quad y \mapsto [x, y].$$

Given $\chi \in \mathfrak{g}^*$, the *reduced enveloping algebra*

$$U_\chi(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} - \chi(x)^p 1 \; ; \; x \in \mathfrak{g}\})$$

is a Frobenius algebra, whose Nakayama automorphism is determined by the formula

$$\mu(x) = x + \mathrm{tr}(\mathrm{ad}\, x)1 \qquad \forall \; x \in \mathfrak{g}.$$

The verification of these assertions is more technical and can be found in [6, (V.4)].

(7) According to a theorem by Larson and Sweedler [5], every finite dimensional Hopf algebra H is a Frobenius algebra. A Nakayama automorphism is given by the formula

$$\mu = \eta^{-2} \circ (\zeta * \mathrm{id}_H) = (\zeta * \mathrm{id}_H) \circ \eta^{-2}.$$

Here η denotes the antipode of H, $\zeta : H \longrightarrow k$ is the *right modular function* of H, and the convolution is defined via

$$(\zeta * \mathrm{id}_H)(h) = \sum_{(h)} \zeta(h_{(1)}) h_{(2)} \qquad \forall \; h \in H,$$

where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ (see [4] for more details).

References

[1] K. Erdmann. *Blocks of Tame Representation Type and Related Algebras.* Lecture Notes in Mathematics **1428**. Springer Verlag, 1990

[2] R. Farnsteiner. *Self-injective algebras: The Nakayama permutation.* Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`

[3] ———. *Self-injective algebras: Comparison with Frobenius algebras.* Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`

[4] D. Fischman, S. Montgomery and H. Schneider. *Frobenius extensions of subalgebras of Hopf algebras.* Trans. Amer. Math. Soc. **349** (1996), 4857-4895

[5] R. Larson and M. Sweedler. *An associative orthogonal bilinear form for Hopf algebras.* Amer. J. Math. **91** (1969), 75-93

[6] H. Strade and R. Farnsteiner. *Modular Lie Algebras and their Representations.* Pure and Applied Mathematics **116**. Marcel Dekker 1988