Selected topics in representation theory 1
Mutations of Exceptional Sequences

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1 Introduction

Let mod$-A$ be the category of finitely generated left modules over a finite dimensional k–algebra A. There exists the notion of an exceptional module (or more generally an exceptional chain complex of modules). For short, this is a module with $\sum_{i=0}^{\infty} \dim \mathrm{Ext}^i(M,M) = 1$ (the dimension is over the skew field $\mathrm{End}(M)$, for chain complexes of modules we take $\sum_{i=-\infty}^{\infty} \dim \mathrm{Hom}(M, M[i]) = 1$, were $[1]$ denotes the shift and Hom denotes the homomorphism space in the corresponding derived category). Thus an exceptional object is a non-trivial object M with $\sum_{i=-\infty}^{\infty} \dim \mathrm{Hom}(M, M[i])$ minimal.

We are mainly interested in a classification of exceptional A–modules. It turns out that it might be sometimes more convenient and natural to consider this classification in the corresponding derived category (see e. g. [2]). However, under certain additional assumptions, we can also work in the module category itself.

The principal idea for such a classification is to consider exceptional sequences instead of exceptional objects. Then there is a construction, called mutation, that produces from one exceptional sequence further ones. The definition of exceptional sequences and mutations is rather general, we do not need to work with module categories. We only need a few assumptions on our category $\mathcal{C}$ we work with: it should be an abelian or triangulated k–category and for two objects M and N the group $\mathrm{Ext}^l(M,N)$ (or in the triangulated case the group $\mathrm{Hom}(M, N[i])$) is defined. Moreover, for two objects M and N the group $\oplus_l \mathrm{Ext}^l(M,N)$ (respectively $\oplus_l \mathrm{Hom}(M, N[l])$) must be a finite dimensional k-vector space.

Example. 1. Let A be a finite dimensional k–algebra, then $\mod - A$ and $\mathcal{D}^b(\mod - A)$ satisfy this condition, provided the algebra A is of finite global dimension.
2. Let X be a smooth projective algebraic variety, then the category of coherent sheaves $\mathrm{Coh}(X)$ and its bounded derived category satisfy the condition above.
3. Let X be a weighted projective space in the sense of Geigle and Lenzing, then the category of coherent sheaves and its bounded derived category satisfy this condition.

In fact there is an action of the braid group on the set of all exceptional sequences (see lecture 2), in some good cases this group action is transitive. In any of the above examples there appear naturally several questions:

Problem 1. Does there exist at least one full exceptional sequence.

Problem 2. Does each exceptional object occure in at least one full exceptional sequence.

Problem 3. When is the mutation of an exceptional sequence of modules (or coherent sheaves, or objects in an abelian category) again a sequence of modules?

Problem 4. When is the mutation of a strongly exceptional sequence again strongly exceptional.

Problem 5. When does the braid group act transitive on the set of all full exceptional sequences?

We adress Problem 5 to the next lecture. Problem 4 has some partial answer (for the left and the right dual sequence, see below for a definition) in terms of Koszul algebras (lecture 3) and Kokoszul algebras (not dicussed further). Problem 3 is a rather technical and in general an open problem. However, for hereditary algebras it can be solved very elegantly by a modification of the mutations (see also lecture 2). It works, since each indecomposable bounded complex of modules over a hereditary algebra is isomorphic to a shift of some module (using perpendicular categories mutations are defined in [1], [3]). The other case where it is known to work well are coherent sheaves on $\mathbb{P}^2$ ([5]). There are at least some partial answers related to Koszul algebras (we define Koszul algebras in lecture 3) and the Serre functor (discussed later in connection with the Calabi-Yau property).
A positive solution to problem 2 is also known in many cases, in particular it is true for hereditary alegbras. For triangular algebras there is a standard full and strongly exceptional sequence, the sequence of indecomposable projective modules (ordered in an adapted way), provided $\mathrm{End}(P)$ is a skew field for each indecomposable projective module P.

2 Exceptional Sequences

Let $\mathcal{C}$ be an abelian or a triangulated k–category satisfying the following conditions:
1.) For any two objects the group $\mathrm{Ext}^l(M,N)$ is defined and a finite dimensional k–vector space (we denote Hom also by Ext^0, in the derived category $\mathrm{Ext}^l(M,N) = \mathrm{Hom}(M,N[l])$ should be defined for all integers l).
2.) The direct sum $\oplus_l \mathrm{Ext}^l(M,N)$ is a finite dimensional k–vector space (together with condition 1. we assume it is bounded below and above).

Definition. An object E in $\mathcal{C}$ is called *exceptional* if it satisfies
1.) $\mathrm{Hom}(E,E) = D(E)$ is a skew field, and
2.) $\mathrm{Ext}^l(E,E) = 0$ for all $l \neq 0$.
A sequence $\varepsilon = (E_1,\ldots,E_t)$ of objects in $\mathcal{C}$ is called *exceptional sequence* if
3.) each object E_i is exceptional for $i = 1,\ldots,t$, and
4.) $\mathrm{Ext}^l(E_j,E_i) = 0$ for each pair $1 \leq i < j \leq t$ and all l.
An exceptional sequence is called *strongly exceptional* it it is exceptional and
5.) $\mathrm{Ext}(E_i,E_j) = 0$ for all $1 \leq i,j \leq t$ and all $l \neq 0$.
Finally we say an exceptional sequence is *full* (or *complete*) if it generates the derived category.

Example. Let A be a triangular algebra and assume we have ordered the isoclasses of indecomposable projective modules $P(i)$ (for $i = 1,\ldots,r$) so that $\mathrm{Hom}(P(j),P(i)) = 0$ for all $j > i$. Moreover, we assume $\mathrm{Hom}(P(i),P(i))$ is a skew field for each i. Then $\varepsilon = (P(1),\ldots,P(r))$ is full and strongly exceptional. Similar, the sequence of indecomposable injective modules forms a full and strongly exceptional sequence.
Let A be as above, and $S(i) := P(i)/\operatorname{rad} P(i)$. Then the sequence $\eta = (S(r),\ldots,S(1))$ is a full and exceptional sequence. It is not strongly exceptional, if A is not a direct product of skew fields.
Let $T = \oplus_{i=1}^r T(i)$ be a partial tilting module with triangular endomorphism algebra (we can

and do assume that the tilting module is basic and we have chosen the index set so that $\mathrm{Hom}(T(j), T(i)) = 0$ for all $j > i$) and $\mathrm{End}(T(i))$ a skew field. Then $(T(1), \ldots, T(r))$ is a strongly exceptional sequence. If T is a tilting module, then the sequence is even full.

3 Mutations of Pairs

We first consider exceptional pairs and define mutations for them. Let (E, F) be an exceptional pair. There exists a natural map

$$\mathrm{Hom}(E, F) \otimes_{D(E)} E \longrightarrow F, \quad \phi \otimes e \mapsto \phi(e)$$

and similar for each integer l a natural map

$$\mathrm{Hom}(E[-l], F) \otimes_{D(E)} E[-l] = \mathrm{Hom}(E, F[l]) \otimes_{D(E)} E[-l] \longrightarrow F.$$

If we take direct sums we get a map, called the *canonical map*, and a triangle in the derived category

$$L_E F \longrightarrow \oplus_l \mathrm{Hom}(E, F[l]) \otimes_{D(E)} E[-l] \xrightarrow{\text{can}} F \xrightarrow{[1]} L_F E[1].$$

In this way we defined a new object $L_E F$, the *left mutation* of F by E. If we consider the dual map we get a *dual canonical map*

$$E \xrightarrow{\text{can}^*} \oplus_l \mathrm{Hom}\,(E, F[l])^*[l] \otimes_{D(F)} F$$

and we can complete it to a triangle

$$E \xrightarrow{\text{can}^*} \oplus_l \mathrm{Hom}\,(E, F[l])^*[l] \otimes_{D(F)} F \longrightarrow R_F E \xrightarrow{[1]} E[1].$$

We define $R_F E$ to be the *right mutation* of E with respect to F.

If we wish $L_E F$, respectively $R_F E$, to be an object in the abelian category up to some shift, then we need some additional properties. We consider these cases explicitly.

Example. We assume E and F are modules (or objects in an abelian category). As an example we consider Λ to be a hereditary algebra wih two non-isomorphic projective modules (e. g. a generalized Kronecker algebra).
1.) Assume $\mathrm{Ext}^l(E, F) = 0$ for $l \neq 0$ and assume further the canonical map $\mathrm{Hom}(E, F) \otimes E \longrightarrow F$ is surjective. Then $L_E F$ is also a module and defined as the kernel of the canonical map. To give a concrete example we consider Λ–modules. We take two preprojective (and not projective) modules M and N, that are neighbours in the AR-quiver (the space $\mathrm{Hom}(M, N)$ consists of irreducible maps). They form an exceptional pair (M, N) and the canonical map is surjective. For the algebra Λ this is the generic situation, however also the following two examples may occure.
2.) Assume again $\mathrm{Ext}^l(E, F) = 0$ for $l \neq 0$ and assume further the canonical map $\mathrm{Hom}(E, F) \otimes E \longrightarrow F$ is injective. Then $L_E F[1]$ is a module. If we take $(P(1), P(2))$ the two indecomposable projective modules (where $P(1)$ is simple), then the canonical map is injective.

3.) Assume now $\mathrm{Ext}^l(E,F)=0$ for all $l\neq 1$. Then the left mutation is defined as an universal extension

$$0 \longrightarrow F \longrightarrow L_E F[1] \longrightarrow \mathrm{Ext}^1(E,F)\otimes_{D(E)} E \longrightarrow 0,$$

and the object $L_E F[1]$ is a module. If we consider Λ–modules this occures for the sequence of simple modules $(S(2),S(1))$.
4.) Finally, if $\mathrm{Ext}^l(E,F)=0$ for all $l\neq 2$ and the first map in the following sequence (the "canonical map" for $\mathrm{Hom}(E,F)$) is injective, then there is a mutation of the form

$$0 \longrightarrow \mathrm{Hom}(E,F)\otimes E \longrightarrow F \longrightarrow L_F E[1] \longrightarrow \mathrm{Ext}^1(E,F)\otimes E \longrightarrow 0$$

and $L_F E[1]$ is a module.
5.) We will see in the next lecture, that left and right mutations are inverse to each other. Using this property one also gets examples for right mutations.

Theorem. *Let (E,F) be exceptional, then $(L_E F, E)$ and $(F, R_F E)$ are both exceptional pairs.*

Proof. To simplify notation we write $(E,F)^l$ for $\mathrm{Ext}^l(E,F)$, (E,F) for $\mathrm{Hom}(E,F)$ and $(E,F)^\bullet$ for $\oplus_l \mathrm{Ext}^l(E,F)[-l]$. The main property we need is the following:

$$\mathrm{Hom}(E,\mathrm{can}) : (E,F)^\bullet \otimes_{D(E)} (E,E) \simeq (E,F)^l \longrightarrow (E,F)^l$$

is the identity map for all l. We prove only the first part, the second is dual. For objects X and Y we get exact sequences

$$\begin{array}{c} 0 \longrightarrow (X, L_E F) \longrightarrow (X,E)\otimes (E,F)^\bullet \longrightarrow (X,F) \longrightarrow \dots \\ \dots \longrightarrow (X, L_E F)^l \longrightarrow (X,E)^l \otimes (E,F)^\bullet \longrightarrow (X,F) \longrightarrow \dots \end{array}$$

$$\begin{array}{c} 0 \longrightarrow (F,Y) \longrightarrow (E,Y)\otimes (E,F)^{\bullet,*} \longrightarrow (L_E F, Y) \longrightarrow \dots \\ \dots \longrightarrow (F,Y)^l \longrightarrow (E,Y)^l \otimes (E,F)^{\bullet,*} \longrightarrow (L_E F, Y) \longrightarrow \dots . \end{array}$$

To prove the properties of an exceptional pair we use the sequences above as follows:
1. $\mathrm{Hom}(E, L_E F)=0$: take $X=E$.
2. $\mathrm{Hom}(L_E F, E)\simeq \mathrm{Hom}(E,F)^*$: take $Y=E$.
3. $\mathrm{Hom}(L_E F, L_E F)\simeq D(F)$: take $X=L_E F$ and use 2. □

4 Mutations of Exceptional Sequences

Mutations for exceptional pairs can be generalized to mutations of exceptional sequences as follows. We chose in an exceptional sequences $\varepsilon=(E_1,\dots,E_i,E_{i+1},\dots,E_r)$ a pair of neighbours and apply the mutation to this pair. In this way we define mutations

$$L_i\varepsilon := (E_1,\dots,L_{E_i}E_{i+1},E_i,\dots,E_r) \quad \text{and} \quad R_i\varepsilon := (E_1,\dots,E_{i+1},R_{E_{i+1}}E_i,\dots,E_r).$$

Theorem. *Let ε be an exceptional sequence, then also $L_i\varepsilon$ and $R_i\varepsilon$ are both exceptional.*

Proof. We have already shown several vanishing properties in the theorem above. To prove the remaining ones we only need to show $\mathrm{Ext}^l(E_j, L_{E_i}E_{i+1}) = \mathrm{Ext}^l(E_j, R_{E_{i+1}}E_i)=0$

for all $j > i+1$ and $\mathrm{Ext}^l(L_{E_i}E_{i+1}, E_j) = \mathrm{Ext}^l(R_{E_{i+1}}E_i, E_j) = 0$ for all $j < i$. Using the exact sequences for $X = E_j$ and $Y = E_j$ in the proof of the theorem above, the result follows. □

Finally we define the left and the right dual sequence of an exceptional sequence by iterating the mutations, so that the "objects are just in the reversed order":

$$\begin{aligned} {}^{\perp}\varepsilon &:= L_{r-1}L_{r-2}L_{r-1}L_{r-3}L_{r-2}L_{r-1}\dots L_1L_2\dots L_{r-2}L_{r-1} \\ &= (L_{E_1}L_{E_2}\dots L_{E_{r-1}}E_r, L_{E_1}L_{E_2}\dots L_{E_{r-2}}E_{r-1}, \dots, L_{E_1}E_2, E_1), \end{aligned}$$

$$\begin{aligned} \varepsilon^{\perp} &:= R_1R_2R_1R_3R_2R_1\dots R_{r-2}\dots R_2R_1R_{r-1}\dots R_2R_1\varepsilon \\ &= (E_r, R_{E_r}E_{r-1}, \dots, R_{E_r}\dots R_{E_3}E_2, R_{E_r}\dots R_{E_2}E_1). \end{aligned}$$

Example. If we start with the strongly exceptional sequence ε of indecomposable, pairwise non-isomorphic projective modules then the left dual is the sequence of simple modules up to some shift (there appears a certain shift for each simple module). Similar, if we start with the strongly exceptional sequence of indecomposable, pairwise non-isomorphic injective modules then the right dual is the sequence of simple modules up to some shift. A partial answer to problem 3 (for the left and the right dual sequence) can be obtained if the endomorphism algebra of $\oplus E_i$ is a Koszul algebra, since then the endomorphism algebra of the left dual sequence is the quadratic dual of this algebra.

References

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