Selected topics in representation theory 9
The volume of a tilting module for $\mathbb{A}_n$

WS 2005/06

The aim of this lecture is to compute the Δ–volume of a tilting module for a directed quiver of type $\mathbb{A}_n$. The classifiaction of tilting modules is very well-known in this case (Section 1). In particular, there is a bijection between tilting modules and rooted (planar and binary) trees with $n+1$ leaves (Section 2). Moreover, there is an easy way to obtain the Δ–volume of a tilting modules directly from its rooted tree. We also note, that everything is already well-known, we only obtain a new interpretation and we present new proofs using the notion of the volume of a tilting module.

1 Tilting modules

In this note we consider modules over the path algebra Λ of directed quiver of type $\mathbb{A}_n$ (with arrows $\alpha_a : a \longrightarrow a+1$) over a field k. We denote by $M[a,b]$ the unique indecomposable module with dimension vector $\underline{\dim}\, M[a,b]_c = 1$ precisely when $1 \leq a \leq c \leq b \leq n$.

Lemma. *a)* $\mathrm{Hom}(M[a,b], M[c,d]) = k$ *if and only if* $c \leq a \leq d \leq b$.
b) $\mathit{Ext}^1(M[a,b], M[c,d]) = \mathit{Ext}^1(M[c,d], M[a,b]) = 0$ *precisely when* $[a,b] \subseteq [c,d]$, $[c,d] \subseteq [a,b]$, $b < c-1$, *or* $d < a-1$.

Proof. The first claim is obvious from the definition and the second claim can easily deduced from the first one by using the projective resolution

$$0 \longrightarrow P(b+1) = M[b+1,n] \longrightarrow P(a) = M[a,n] \longrightarrow M[a,b].$$

□

It is now obvious how one can construct tilting modules recursively: one starts with the projective and injective module $M[1,n]$ (it is always a direct summand of a titling module) and then one proceeds either with step 1 or step 2 to obtain new indecomposable direct summands:
STEP 1. choose a number $1 < a < n$ and take the modules $M[1,a-1]$ and $M[a+1,n]$ (the weight of this step is by definition $\binom{n-1}{a-1} = (n-1)!/((a-1)!(n-a)!)$), or
STEP 2. take either the module $M[1,n-1]$ or the module $M[2,n]$ as a direct summand (the weight of this step is one).
Then replace $M[1,n]$ by one of the chosen ones ($M[1,a-1]$ first and after finishing with this proceed with $M[a+1,n]$ in step 1; $M[1,n-1]$, respectively $M[2,n]$ in step 2) and proceed in the same way. The *total weight* of a tilting module constructed recursively in this way is the product of its weights (if only step 2 occures the weight is one).

Remember that $\mathrm{vol}_\Delta(M) = \prod_{i=1}^n 1/\dim M_i$ for a basic tilting module M with indecomposable direct summands M_i. Then

$$\sum \mathrm{vol}_\Delta(M) = 1, \text{ where the sum runs over all basic tilting modules.}$$

EXAMPLE. If $n = 3$ we obtain the following tilting modules by using the indicated steps (the same result can be obtained by using the mutations introduced in the first lectures):
1) step 1, step 1: $M[1,3], M[1,2], M[1,1]$; $M[1,3], M[1,2], M[2,2]$; $M[1,3], M[2,3], M[2,2]$; $M[1,3], M[2,3], M[3,3]$ (all have weight one and volume 1/6)
2) step 2: $M[1,3], M[1,1], M[3,3]$ (it has weight 2 and volume 1/3).
We also draw a figure indicating the intersection of the cones of the tilting module with the simplex Δ in the Grothendieck group (note that the indicated points represent lines and they are, in general, not in the simplex Δ).

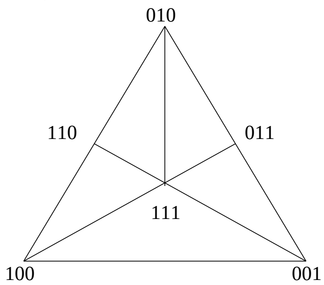

Figure 1. the cones of the tilting modules intersected with Δ for $n = 3$

In this way we get a list of tilting modules. Using the volume introduced in Lecture 6 we can compute the Δ–volume of each titling module a priori using the weight (just knowing in which order we use step 1 or step 2) and show, this list is complete. For we use another interpretation of the recursion in terms of rooted trees.

Theorem. *The list of tilting modules constructed above is complete. Moreover, the Δ–volume* $\mathrm{vol}_\Delta(M)$ *of a tilting module* $M = \oplus_{i=1}^n M_i$ *(with indecomposable direct summands M_i) is* $1/n!$ *times the weight of M. There is a map form the symmetric group S_n to the set of tilting modules, surjective onto the isomorphism classes of tilting modules, so that the weight is precisely the crdinality of the preimage.*

For the proof we introduce rooted trees (they are always planar and binary in this note) and construct a map from S_n onto the set of rooted trees with $n+1$ leaves (following a construction of Loday and Ronco [3]). Then it is obvious that a rooted tree defines a tilting module via the recursion above. Finally, since the cardinality of the preimage is the weight, we obtain

$$\sum_M \mathrm{vol}_\Delta(M) = \sum_T \mathrm{wt}(T) = \sum_{\sigma \in S_n} 1/n! = 1,$$

where the first sum runs over the set of recursively constructed tilting modules M, the second sum runs over all rooted trees (and $\mathrm{wt}(T)$ is its weight), and the third sum runs over the elements in the symmetric group. The first equality follows from the definition of the weight in the next section, the second equality follows from the construction of the surjective map from S_n to the set of rooted trees with n leaves and the last equality is obvious. Consequently, the list of tilting modules constructed above is complete (otherwise we got $\sum_M \mathrm{vol}_\Delta(M) < 1$).

Note that the Δ–volume of a tilting module is always a multiple of $1/n!$. This is no longer true for other orientations of the quiver $\mathbb{A}_n$, as one can already see for $n = 3$.

EXAMPLE. We consider the quiver $\mathbb{A}_3$ with the orientation $1 \longrightarrow 2 \longleftarrow 3$. Then the dimension vector of an indecomposable module is given by an intervall $[a, b]$ with $1 \leq a \leq b \leq 3$ and there are five tilting modules with dimension vectors and volume as follows:
1) $010, 110, 011,\ \mathrm{vol}_\Delta = 1/4$,
2) $110, 011, 111,\ \mathrm{vol}_\Delta = 1/12$,
3) $110, 111, 001,\ \mathrm{vol}_\Delta = 1/6$,
4) $011, 111, 100,\ \mathrm{vol}_\Delta = 1/6$, and
5) $111, 001, 100,\ \mathrm{vol}_\Delta = 1/3$.

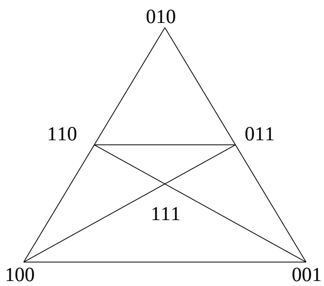

Figure 2. the cones of the tilting modules intersected with Δ for $\mathbb{A}_3$ with two sources

2 Rooted trees

DEFINITION. We introduce the set $\mathcal{T}(n)$ of rooted trees with n leaves: a *rooted tree* is a graph with one distinguished end (we always draw it as the root) and $n+1$ other ends (drawn in the plane and numbered from left to right by $1, 2, \ldots, n, n+1$) that is a tree and each vertex that is not an end has precisely three edges connected to it. Note that such a tree has $n+2$ ends (one root and $n+1$ leaves), so it must have n inner vertices (ordered from left to right and numbered by $1, \ldots, n$, this order is unique if we always assume that above of each vertex there is no edge, see the following figures), $n+2$ edges connected to one end and $n-1$ inner edges. Note that a tree T defines a total order, called the left–right order, and a partial order, the bottom–up order, on the set of inner vertices. We define the weight $\mathrm{wt}(T)$ of a tree to be the number of total orders refining the bottom–up partial order.

We first associate to each T in $\mathcal{T}(n)$ a tilting module $M(T)$ over Λ. For we define for each inner vertex i an indecomposable module $M(T)_i := M[a(i), b(i)]$, where $a(i)$ is the most left place (the number of the vertex with respect to the left–right order) of a vertex above of i and $b(i)$ is the most right place of a vertex above i. There occure two cases, either there is a unique inner vertex j above i, then we construct $M(T)_j$ as in step 2 or there are two inner vertices j_1 and j_2 above i, then we construct $M(T)_{j_1}$ and $M(T)_{j_2}$ as in step 1.

EXAMPLE. For $n = 3$ we obtain the following tilting modules for the five rooted trees with 4 leaves.

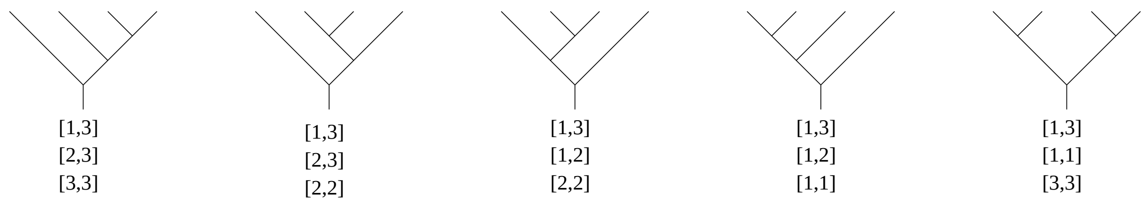

Figure 3. tilting modules for the trees with 4 leaves

3 The symmetric group and trees

Remember that we have two orders (one total order and one partial order) on the set of inner vertices for a rooted tree T. Conversely, let σ be an element of the symmetric group S_n, then we can define an order on the set $\{1, \ldots, n\}$ via $i \leq j$ iff $\sigma(i) \leq \sigma(j)$. Note that for each σ there is precisely one rooted tree $T(\sigma)$ with $n+1$ leaves satisfying the following two conditions:
1) the inner vertices are numbered from left to right and
2) the bottom–up order of $T(\sigma)$ coincides with the order defined above.

The tree $T(\sigma)$ can also be constructed as follows. Let $X(\sigma)$ be the permutation matrix of σ ($X(\sigma)_{\sigma(i),i} = 1$ are the only non-zero entries). Then replace each entry 1 by the rooted tree with 2 leaves and connect them as follows: each leave is connected with the first root above of it on the same side of the matrix with respect to all lower inner vertices (the left leaf with the first left root, the right leaf with the first right root above of it so that above of each vertex there is no edge). We illustrate the construction in an example:

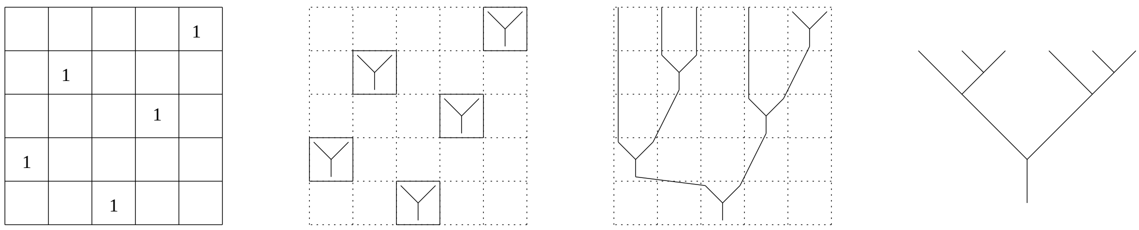

Figure 4. the construction of $T(\sigma)$ for $\sigma = (1435)(2)$, $\mathrm{wt}(T(\sigma)) = \binom{4}{2} = 6$

Lemma. *The map* $S_n \longrightarrow \mathcal{T}(n), \sigma \mapsto T(\sigma)$ *is surjective and the cardinality of the preimage of* T *is the weight* $\mathrm{wt}(T)$.

Proof. Starting with a tree T we have two orders, the left–right order and a refinement of the bottom–up order. We enumerate the inner vertices with $1, \ldots, n$ respecting the left–right order. The second order on the set $\{1, \ldots, n\}$ defines an element σ in S_n, so that $T(\sigma) = T$. Then $\sharp\{\sigma \mid T(\sigma) = T\} = \mathrm{wt}(T)$. $\square$

References

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