Tilting complexes for local left artinian algebras are trivial.

Claus Michael Ringel

Prerequisites. We consider a local left artinian ring R, and the bounded derived category $D^b(\operatorname{mod} R)$ of finitely generated left R-modules.

> We may define $D^b(\operatorname{mod} R)$ as the homotopy category $K^{-b}(\operatorname{proj} R)$ of "(cochain) complexes of finitely generated projective left R-modules bounded above and with homology bounded below", thus we deal with complexes $C^\bullet$ with the following properties:
> (1) Any module C^i is finitely generated projective.
> (2) $C^i = 0$ for $i \gg 0$ (the minus sign in the exponent of $K^{-b}(\operatorname{proj} R)$ refers to this condition).
> (3) $H^i(C^\bullet) = 0$ for $i \ll 0$ (the letter b in the exponent of $K^{-b}(\operatorname{proj} R)$ refers to this condition)
> Here, one has to know what a bounded complex of R-modules is; the maps between complexes are assumed to be of degree 0. And one needs to know what it means that two maps between complexes are homotopic. Finally, we need to know what $H^i(C^\bullet)$ is. Later, we also will need the shift $[t]$ in the derived category and that we may consider $\operatorname{mod} R$ as being embedded into $D^b(\operatorname{mod} R)$ by identifying a module M with the corresponding stalk complex concentrated at 0, or better with a projective resolution of M, since we want to work with $K^{-b}(\operatorname{proj} R)$.

The title refers to "tilting complexes", they are defined as complexes $C^\bullet$ with the following properties

(T1) Any module C^i is finitely generated projective.
(T2) $C^i = 0$ for $|i| \gg 0$.
(T3) For $n \neq 0$, any map $C^\bullet \to C^\bullet[n]$ is homotopic to zero.
(T4) The complex $C^\bullet$ generates $D^b(\operatorname{mod} R)$ as a triangulated category.

> We will not comment on the last condition, since the result to be shown deals with **all** complexes satisfying the conditions T1, T2, T3. The result does not involve the fact that $K^{-b}(\operatorname{proj} R)$ is a triangulated category.

Proposition. *Let R be a local artinian ring and $C^\bullet$ a complex satisfying the conditions* T1, T2, T3. *Then $C^\bullet$ is homotopy equivalent to a complex of the form $P[t]$, with P a finitely generated projective module of finite rank.*

(Note that for a local ring R, any finitely generated projective module is free.)

Proof: Let $C^\bullet$ be a non-zero complex satisfying the conditions T1, T2, T3. Applying a suitable shift, we can assume $C^0 \neq 0$ and $C^i = 0$ for $i > 0$. There is some $s \geq 0$ such that $C^{-s} \neq 0$ and $C^i = 0$ for all $i < -s$. We denote the differential of $C^\bullet$ by $\delta^\bullet$ and we can assume that $\delta^i : C^i \to C^{i+1}$ maps into the radical of C^{i+1} (otherwise we replace $C^\bullet$ by a corresponding complex homotopic to $C^\bullet$). Consider the following diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{-1} & \xrightarrow{\delta^{-1}} & C^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & C^{-s} & \xrightarrow{\delta^{-s}} & C^{-s+1} & \longrightarrow & \cdots \end{array}$$

it provides a chain map $C^\bullet \to C^\bullet[s]$, provided $f\colon C^0 \to C^{-s}$ is a homomorphism with $f\delta^{-1} = 0 = \delta^{-s} f$. Important: such a chain map cannot be homotopic to zero unless $f = 0$.

We claim that there is a non-zero map f with $f\delta^{-1} = 0 = \delta^{-s} f$. If $s > 0$, we obtain in this way a contradiction to T3. We need:
(a) the cokernel X of δ^{-1} is non-zero,
(b) the kernel Y of δ^{-s} is non-zero.
Note that for a local artinian ring R and non-zero modules X, Y, always $\operatorname{Hom}(X, Y) \neq 0$.

The assertion (a) is never a problem. (As we have mentioned, we may assume that any δ^i maps into the radical. Another formulation would be: if the map δ^{-1} is surjective, it is a split epimorphism and thus $C^\bullet$ is homotopic to a complex with smaller width).

For (b) we need that the **right** socle of R is non-zero: the elements of the right socle are those elements x of R which satisfy $xr = 0$ for all radical elements r of R. (Since C^{-s} and C^{-s+1} are free R-modules of finite rank, we can write δ^{-s} as a matrix with entries in R, but even in $\operatorname{rad} R$, since we assume that δ^{-s} maps into the radical of C^{-s+1}.) Note that if $J = \operatorname{rad} R$, and $J^t = 0$ whereas $J^{t-1} \neq 0$, then J^{t-1} is contained in the right socle (as well as the left socle).

The Relevance. A famous theorem of Rickard asserts that any derived equivalence between rings R, R' is given by a tilting complex (say a tilting complex $T^\bullet$ of R-modules, with the endomorphism ring of $T^\bullet$ (in the homotopy category) being R'). Thus we see: *if R' is derived equivalent to a local artinian ring R, then* $R' = M_n(R)$, the ring of all $(n \times n)$-.matrices with coefficients in R.

Generalizations. The same proof works for other classes of rings. For example, *let R be artinian with radical square zero and with the following property: if S, T are simple modules, then* $\operatorname{Ext}^1(T, S) \neq 0$. *Then any complex with properties T1, T2, T3 is homotopy equivalent to a complex of the form $P[t]$, with P a finitely generated projective module.*

(Proof: We can assume that R is not semisimple. Then all the indecomposable projective modules have Loewy length 2. Let P be an indecomposable direct summand von C^{-s}. Since δ^{-s} maps into the radical of C^{-s+1}, the image of δ^{-s} is semisimple. Thus the socle of P is contained in the kernel of δ^{-s}.)

For example, the quiver of R may look as follows: