The Harder-Narasimhan filtration

(based on talks by Markus Reineke at the ICRA 12 conference in Torun, August 2007)

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1 Definitions

Let Q be a finite quiver with set of vertices I, and let $\theta : \mathbb{Z}I \to \mathbb{Z}$ be a linear function, called *stability*. We also define dim on $\mathbb{Z}I$ by $\dim d = \sum_{i \in I} d_i$.

Let us recall those definitions which are needed to define the Harder-Narasimhan filtration:

Definition 1.1. .

1. *For a non-zero dimension vector $d \in \mathbb{N}I$, we define its* slope *by*

$$\mu(d) = \frac{\theta(d)}{\dim d} \in \mathbb{Q}.$$

We define the slope of a non-zero representation X of Q (over some field) as the slope of its dimension vector, thus $\mu(X) = \mu(\dim X) \in \mathbb{Q}$.

2. *We call the representation X* semistable *if $\mu(U) \leq \mu(X)$ for all non-zero subrepresentations U of X, and we call X* stable *if $\mu(U) < \mu(X)$ for all non-zero proper subrepresentations U of X.*

2 Harder-Narasimhan filtration

Definition 2.1. *A filtration $0 = X_0 \subset X_1 \subset \ldots \subset X_s = X$ of a representation X is called* Harder-Narasimhan *(abbreviated by* HN*) if the subquotients X_i/X_{i-1} are semistable for $i = 1, \ldots, s$ and $\mu(X_1/X_0) > \mu(X_2/X_1) > \ldots > \mu(X_s/X_{s-1})$.*

Theorem 2.1. *Any representation X possesses a unique Harder-Narasimhan filtration.*

For the proof of this theorem, we need the concept of strongly contradicting semistability:

Definition 2.2. *A subrepresentation U of a representation X is called* strongly contradicting semistable *(or just* scss*) if its slope is maximal among the slopes of subrepresentations of X, that is, $\mu(U) = max\{\mu(V) \,|\, V \subset X\}$, and it is of maximal dimension with this property.*

In a previous lecture it was shown that any representation X admits a unique scss subrepresentation. This is crucial for the proof of the theorem:

Proof of Theorem 2.1. We will first prove existence, then uniqueness.

Existence is proved by induction over the dimension of X. Let X_1 be the scss of X. By induction, we have a HN filtration

$$0 = Y_0 \subset Y_1 \subset \ldots \subset Y_{s-1} = X/X_1.$$

Via the projection $\pi : X \to X/X_1$, we pull this back to a filtration of X defined by $X_i = \pi^{-1}(Y_{i-1})$ for $i = 1, \ldots, s$.

Now X_1/X_0 is semistable since X_1 is the scss of X, and $X_{i+1}/X_i \cong Y_i/Y_{i-1}$ is semistable by the choice of the Y_i for $i = 1, \ldots, s-1$.

We also get $\mu(X_2/X_1) > \ldots > \mu(X_s/X_{s-1})$ from the corresponding property of the slopes of the subquotients in the HN filtration of X/X_1. Since X_2 is a subrepresentation of X of strictly larger dimension than X_1, we have $\mu(X_1) > \mu(X_2)$ since X_1 is scss in X, and thus $\mu(X_1/X_0) = \mu(X_1) > (X_2/X_1)$.

Existence is also proved by induction on the dimension. Assume that

$$0 = X'_0 \subset \ldots \subset X'_s = X$$

is a HN filtration of X. Let t be minimal such that X_1 is contained in X'_t, thus the inclusion induces a non-zero map from X_1 to X'_t/X'_{t-1}. Both representations being semistable and X_1 being scss, we have

$$\mu(X'_1) \leq \mu(X_1) \leq \mu(X'_t/X'_{t-1}) \leq \mu(X'_1),$$

thus $\mu(X_1) = \mu(X'_1)$ and $t = 1$, which means $X_1 \subset X'_1$. Again since X_1 is scss, we conclude that $X_1 = X'_1$.

By induction, we know that the induced filtrations on the factor X/X_1 coincide, thus the filtrations of X coincide, which proves uniqueness.

□

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