QUASI-HEREDITARY ALGEBRAS: BGG RECIPROCITY

ROLF FARNSTEINER

Let A be a finite-dimensional associative algebra over an algebraically closed field k. We let $S(1), \dots, S(n)$ denote a full set of representatives of the simple A-modules and define $P(i)$ to be the projective cover of $S(i)$. Reciprocity laws, such as Brauer reciprocity [2], Humphreys reciprocity [4, §4], or Bernstein-Gel'fand-Gel'fand reciprocity [1] involve A-modules $\Delta(1), \dots, \Delta(n)$ such that each $P(i)$ admits a Δ-filtration, that is, a filtration

$$(0) = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = P(i)$$

such that $M_i/M_{i-1} \in \{\Delta(1), \dots, \Delta(n)\}$ for $i \in \{1, \dots, r\}$. The actual law then reads as

$$(P(i) : \Delta(j)) = [\Delta(j) : S(i)],$$

where the brackets denote the relevant filtration multiplicities.

Let us record one immediate consequence. Recall that $C_A = (c_{ij})_{1 \le i,j \le n}$ with $c_{ij} := [P(j) : S(i)] = \dim_k \operatorname{Hom}_A(P(i), P(j))$ is a *Cartan matrix* of A. In view of the exactness of $\operatorname{Hom}_A(P(i), -)$, we obtain

$$\begin{aligned} c_{ij} &= \dim_k \operatorname{Hom}_A(P(i), P(j)) = \sum_{\ell=1}^{n} (P(j) : \Delta(\ell)) \dim_k \operatorname{Hom}_A(P(i), \Delta(\ell)) \\ &= \sum_{\ell=1}^{n} (P(j) : \Delta(\ell))[\Delta(\ell) : S(i)] = \sum_{\ell=1}^{n} (P(j) : \Delta(\ell))(P(i) : \Delta(\ell)). \end{aligned}$$

Thus, setting $D := ((P(j) : \Delta(i))_{1 \le i,j \le n}$, we arrive at $C_A = D^{\mathrm{tr}} D$, so that C_A is symmetric with $\det(C_A) = \det(D)^2$ being a square.

The main tools for the proof of reciprocity laws are certain subcategories of the module categories to be considered. For complex semi-simple Lie algebras or Kac-Moody algebras one considers the BGG-categories $\mathcal{O}$ (cf. [1, 9, 10]); for Frobenius kernels of reductive groups Jantzen [6] introduced the category of G_rT-modules. In addition, certain categories of perverse sheaves also satisfy BGG-reciprocity, see [7]. With so many examples in hand, people started thinking about a proper axiomatic set-up for reciprocity theorems. The initial ideas in this direction appear to be due to Irving [5]; the more general framework of a *highest weight category* was introduced by Cline-Parshall-Scott, see [3].

Let A be a finite-dimensional k algebra, $(I, \le)$ be a finite partially ordered set with $(S(i))_{i \in I}$ being a complete set of representatives for the isomorphism classes of the simple A-modules. For each $i \in I$, let $P(i)$ and $I(i)$ be the projective cover and the injective hull of $S(i)$, respectively. Throughout, we will be working in the category $\operatorname{mod} A$ of finite-dimensional A-modules. The Jordan-Hölder multiplicity of $S(i)$ in M will be denoted $[M : S(i)]$.

The following definition is motivated by properties of the so-called *Verma modules* occurring in Lie theory.

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Definition. A family $(\Delta(i))_{i\in I}$ of A-modules is referred to as a *collection of standard modules* (*for A relative to* $(I,\leq)$) if
1. $\mathrm{Top}(\Delta(i)) \cong S(i)$ and $[\Delta(i):S(i)] = 1$ for all $i \in I$, and
2. $[\Delta(i):S(j)] = 0$ for $j \not\leq i$.

An A-module M is called Δ-*good* if it affords a filtration, whose filtration factors are standard modules. In that case, the element $[M]$ in the Grothendieck group $K_0(A)$ of $\mathrm{mod}\, A$ corresponding to M can be written as

$$[M] = \sum_{j\in I} n_j[\Delta(j)] = \sum_{i\in I}(\sum_{j\in I}[\Delta(j):S(i)]n_j)[S(i)].$$

Since the matrix $([\Delta(j):S(i)])_{i,j\in I}$ is unipotent upper triangular, the coefficients n_j are uniquely determined. In other words, the multiplicities

$$(M:\Delta(j)) = n_j$$

do not depend on the choice of the Δ-filtration of M.

Definition. Let A be a k-algebra, $\{\Delta(i);\ i \in I\}$ be a collection of standard modules. Then A is called *quasi-hereditary* if
1. each $P(i)$ is Δ-good, and
2. for each $i \in I$, we have $(P(i):\Delta(i)) = 1$ and $(P(i):\Delta(j)) = 0$ for $i \not\leq j$.

From now on we fix a quasi-hereditary algebra A. We consider truncated subcategories of $\mathrm{mod}\, A$. For each $i \in I$, we let $\mathrm{mod}_{\leq i} A$ be the full subcategory of $\mathrm{mod}\, A$, whose objects have all their composition factors lying in $\{S(j)\ ;\ j \leq i\}$. We record the following properties:

- $\mathrm{mod}_{\leq i} A$ is closed under submodules, factor modules and extensions.
- If $M_1, M_1 \subseteq M$ are submodules of M, then
$$M_1/(M_1 \cap M_2) \cong (M_1 + M_2)/M_2.$$
Thus, if $M_1, M_2 \in \mathrm{mod}_{\leq i} A$, then $M_1 + M_2 \in \mathrm{mod}_{\leq i} A$.
- Similarly, if M/M_1 and M/M_2 belong to $\mathrm{mod}_{\leq i} A$, then $M/(M_1 \cap M_2)$ belongs to $\mathrm{mod}_{\leq i} A$.

As a result, M possesses a unique largest factor module $\mathrm{Tr}_{\leq i}(M) \in \mathrm{mod}_{\leq i}(M)$ and a unique largest submodule belonging to $\mathrm{mod}_{\leq i} A$. We define $\nabla(i)$ to be the largest submodule of $I(i)$ such that $\nabla(i) \in \mathrm{mod}_{\leq i} A$. Note that $\Delta(i) \in \mathrm{mod}_{\leq i} A$.

Given an A-module M, we let $\Omega_A(M)$ be the kernel of a projective cover $P \longrightarrow M$. In the following, $\mathcal{F}(\Delta)$ denotes the full subcategory of $\mathrm{mod}\, A$, whose objects are the Δ-good modules.

Lemma 1. *Let A be quasi-hereditary. Then the following statements hold:*
1. *The module $\Omega_A(\Delta(i))$ is Δ-good, with filtration factors belonging to $\{\Delta(\ell)\ ;\ \ell > i\}$.*
2. $\dim_k \mathrm{Hom}_A(\Delta(i), \nabla(j)) = \delta_{ij}$ *for all* $i, j \in I$.
3. *If $N \subseteq I(j)$ is a submodule such that* $\dim_k \mathrm{Hom}_A(\Delta(i), N) = \delta_{ij}$ *for all $i \in I$, then $N \subseteq \nabla(j)$.*

Proof. (1) Since A is quasi-hereditary, there exists a surjection $P(i) \xrightarrow{\pi} \Delta(j)$ for some $j \geq i$ such that $\ker \pi$ is Δ-good. Thus, $S(i) \cong \mathrm{Top}(\Delta(j)) \cong S(j)$, whence $i = j$. Consequently, $\Omega_A(\Delta(i))$ is a Δ-good module with filtration factors belonging to $\{\Delta(\ell)\ ;\ \ell > i\}$.

(2) Suppose that $i \not\leq j$. Then $S(i)$ is not a composition factor of $\Delta(j)$ or $\nabla(j)$, whence

$$\mathrm{Hom}_A(P(i), \nabla(j)) = (0) = \mathrm{Hom}_A(\Delta(j), I(i)).$$

As Hom_A is left exact, we obtain $\mathrm{Hom}_A(\Delta(i), \nabla(j)) = (0) = \mathrm{Hom}_A(\Delta(j), \nabla(i))$. Consequently, $\mathrm{Hom}_A(\Delta(i), \nabla(j)) = (0)$ for $i \neq j$. Since $\Delta(i) \in \mathrm{mod}_{\leq i} A$, we also have

$$\dim_k \mathrm{Hom}_A(\Delta(i), \nabla(i)) = \dim_k \mathrm{Hom}_A(\Delta(i), I(i)) = [\Delta(i) : S(i)] = 1,$$

as desired.

(3) Suppose that $\mathrm{Hom}_A(P(i), N) \neq (0)$. Since $P(i)$ is filtered with filtration factors $(\Delta(\ell))_{\ell \geq i}$, there exists $\ell \geq i$ such that $\mathrm{Hom}_A(\Delta(\ell), N) \neq (0)$. By assumption, we obtain $\ell = j$, so that $i \leq j$. Consequently, $N \in \mathrm{mod}_{\leq i} A$, whence $N \subseteq \nabla(j)$. □

Lemma 2. *The following statements hold:*

(1) $\mathrm{Ext}^1_A(\Delta(i), \nabla(j)) = (0)$ *for all* $i, j \in I$.

(2) *If* M *is a* Δ*-good module, then* $(M : \Delta(j)) = \dim_k \mathrm{Hom}_A(M, \nabla(j))$.

Proof. Let $\leq_1$ be a total ordering on I containing $\leq$. If $(A, \leq)$ is quasi-hereditary with standard modules $\{\Delta(i), i \in I\}$, then $(A, \leq_1)$ is quasi-hereditary with the same standard modules. Letting $\nabla_1(j) \subseteq I(j)$ be the co-standard module relative to $\leq_1$, we have $\nabla(j) \subseteq \nabla_1(j)$. Moreover, Lemma 1(1) yields $\dim_k \mathrm{Hom}_A(\Delta(i), \nabla_1(j)) = \delta_{ij}$ for all $i \in I$, so that (3) of Lemma 1 gives $\nabla_1(j) = \nabla(j)$.

We may therefore assume without loss of generality that $\leq$ is a total ordering on I.

(1) If $i \geq j$, then (1) and (2) of Lemma 1 yield $\mathrm{Hom}_A(\Omega_A(\Delta(i)), \nabla(j)) = (0)$, whence $\mathrm{Ext}^1_A(\Delta(i), \nabla(j)) = (0)$. Alternatively, $i < j$ and we consider the canonical projection $\pi : I(j) \longrightarrow I(j)/\nabla(j)$. If $f \in \mathrm{Hom}_A(\Delta(i), I(j)/\nabla(j))$, then $\pi^{-1}(f(\Delta(i))) \in \mathrm{mod}_{\leq j} A$, so that $\pi^{-1}(f(\Delta(i))) = \nabla(j)$. Consequently, $f(\Delta(i)) = \pi(\pi^{-1}(f(\Delta(i)))) = (0)$. Since the connecting homomorphism

$$\mathrm{Hom}_A(\Delta(i), I(j)/\nabla(j)) \longrightarrow \mathrm{Ext}^1_A(\Delta(i), \nabla(j))$$

is surjective, we obtain $\mathrm{Ext}^1_A(\Delta(i), \nabla(j)) = (0)$.

(2) Let M be a Δ-good module. Thanks to (1), we have $\mathrm{Ext}^1_A(M, \nabla(j)) = (0)$ for all $j \in I$. This implies that the functor $\mathrm{Hom}_A(-, \nabla(j))|_{\mathcal{F}(\Delta)}$ is exact. Our assertion is now a consequence of Lemma 1(2). □

The weak BGG reciprocity principle reads as follows:

Theorem 3. *Let* A *be a quasi-hereditary algebra. Then we have*

$$(P(i) : \Delta(j)) = [\nabla(j) : S(i)]$$

for all $i, j \in I$.

Proof. According to (2) of Lemma 2, we have

$$(P(i) : \Delta(j)) = \dim_k \mathrm{Hom}_A(P(i) : \nabla(j)).$$

Since k is algebraically closed, the latter number coincides with $[\nabla(j) : S(i)]$. □

The above reciprocity law has a blemish residing in the seemingly contrived definition of the *co-standard modules* $\nabla(i)$. Our next lemma addresses this issue by showing that the $\Delta(i)$ may be constructed in a similar fashion. Many papers on quasi-hereditary algebras use this as the defining property of standard modules, see for instance [8, p.213f].

Lemma 4. *We have*

$$\mathrm{Tr}_{\leq i}(P(i)) \cong \Delta(i)$$

for every $i \in I$.

Proof. Let $\pi : P(i) \longrightarrow \Delta(i)$ and $\tilde{\pi} : P(i) \longrightarrow \mathrm{Tr}_{\leq i}(P(i))$ be the canonical projections. By definition of $\mathrm{Tr}_{\leq i}(P(i))$, there exists a linear map $\omega : \mathrm{Tr}_{\leq i}(P(i)) \longrightarrow \Delta(i)$ such that $\omega \circ \tilde{\pi} = \pi$. Since $\Omega_A(\Delta(i)) = \ker \pi$ is Δ-good with filtration factors $(\Delta(\ell))_{\ell > i}$ and

$$\mathrm{Hom}_A(\Delta(\ell), \mathrm{Tr}_{\leq i}(P(i))) = (0)$$

for $\ell > i$, it follows that $\mathrm{Hom}_A(\Omega_A(\Delta(i)), \mathrm{Tr}_{\leq i}(P(i))) = (0)$. Consequently, the map

$$\pi^* : \mathrm{Hom}_A(\Delta(i), \mathrm{Tr}_{\leq i}(P(i))) \longrightarrow \mathrm{Hom}_A(P(i), \mathrm{Tr}_{\leq i}(P(i)))$$

is surjective, and there exists $\gamma : \Delta(i) \longrightarrow \mathrm{Tr}_{\leq i}(P(i))$ with

$$\tilde{\pi} = \gamma \circ \pi.$$

Thus, $(\omega \circ \gamma) \circ \pi = \omega \circ \tilde{\pi} = \pi$, so that $\omega \circ \gamma = \mathrm{id}_{\Delta(i)}$. This shows that $\Delta(i)$ is a direct summand of $\mathrm{Tr}_{\leq i}(P(i))$. Being a factor module of $P(i)$, $\mathrm{Tr}_{\leq i}(P(i))$ is indecomposable, whence $\Delta(i) \cong \mathrm{Tr}_{\leq i}(P(i))$. □

The classical BGG reciprocity principle necessitates an additional ingredient, a duality functor which interchanges standard modules and costandard modules. By definition, a *duality* $D : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A$ is a contravariant functor which is an equivalence $(\mathrm{mod}\, A)^{\mathrm{op}} \longrightarrow \mathrm{mod}\, A$. Such functors are available in the aforementioned contexts.

Definition. A quasi-hereditary algebra A is called a *BGG-algebra* if there exists a duality $D : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A$ such that $D(S(i)) \cong S(i)$ for every $i \in I$.

Theorem 5 (BGG Reciprocity)**.** *Let A be a BGG-algebra. Then we have*

$$(P(i) : \Delta(j)) = [\Delta(j) : S(i)]$$

for all $i, j \in I$.

Proof. By assumption, there exists a duality $D : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A$ such that $D(S(i)) \cong S(i)$ for all $i \in I$. Consequently, $D(I(i)) \cong P(i)$ and $D(\mathrm{mod}_{\leq i} A) = \mathrm{mod}_{\leq i} A$. In view of Lemma 4, we thus have $D(\nabla(i)) \cong \Delta(i)$, so that

$$[\nabla(j) : S(i)] = [D(\nabla(j)) : D(S(i))] = [\Delta(j) : S(i)]$$

for all $i, j \in I$. The assertion now follows from Theorem 3. □

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