INDUCED MODULES: GRADED ALGEBRAS AND GREEN'S INDECOMPOSABILITY THEOREM

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Throughout, k is assumed to be an algebraically closed field of characteristic $\mathrm{char}(k) = p > 0$. Let $N \trianglelefteq G$ be a normal subgroup of a finite group G. As usual, kG denotes the group algebra of G.

Theorem (Green's Indecomposability Theorem, [3])**.** *Suppose that G/N is a p-group. If M is an indecomposable kN-module, then the kG-module $kG \otimes_{kN} M$ is indecomposable.*

We shall prove this result by establishing a general statement on group-graded algebras, that has also been useful in the representation theory of infinitesimal group schemes (cf. [2]). In the sequel, all algebras and modules are finite-dimensional.

Definition. Let G be a finite group. A k-algebra

$$R = \bigoplus_{g \in G} R_g$$

is said to be *G-graded* if $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We call R *strongly G-graded* if $R_g R_h = R_{gh}$ for all $g, h \in G$.

For instance, if $N \trianglelefteq G$ is a normal subgroup, then kG is a strongly G/N-graded k-algebra. In fact, if $R = \bigoplus_{g \in G} R_g$ is G-graded and $\pi : G \longrightarrow H$ is a surjective homomorphism of finite groups, then R obtains the structure of an H-graded algebra via

$$R_h := \bigoplus_{g \in \pi^{-1}(h)} R_g.$$

Indecomposability corresponds to endomorphism rings being local, so we are interested in the question when a graded algebra is local. By way of motivation we record a few necessary conditions.

Lemma 1. *Suppose that $R = \bigoplus_{g \in G} R_g$ is a local algebra. Then the following statements hold:*
(1) *The algebra R_1 is local.*
(2) *If $R_g \not\subseteq \mathrm{Rad}(R)$ for every $g \in G$, then G is a p-group.*

Proof. By assumption, there exists an algebra homomorphism $\varepsilon : R \longrightarrow k$ such that $\ker \varepsilon = \mathrm{Rad}(R)$.
(1) Since $\ker \varepsilon|_{R_1}$ is a nilpotent ideal of codimension 1, it follows that R_1 is local.
(2) Consider $N := \bigoplus_{g \in G} (\ker \varepsilon) \cap R_g$. Then N is a nilpotent ideal of R. Thus, ε induces an algebra homomorphism $\gamma : S \longrightarrow k$ of the local, G-graded algebra $S := R/N$. By virtue of our current assumption, we have $\dim_k S_g = 1$ for every $g \in G$, and for every $g \in G$ there exists a unique element $s_g \in S_g$ such that $\gamma(s_g) = 1$. Consequently, we have

$$s_g s_h = s_{gh} \qquad \forall\ g, h \in G,$$

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so that the map $G \longrightarrow S \;\; ; \;\; g \mapsto s_g$ induces a surjective algebra homomorphism $\zeta : kG \longrightarrow S$. By equality of dimensions, this map is bijective. As a result, the group algebra kG is local, forcing G to be a p-group. $\square$

We turn to algebras that are graded by some p-group G, beginning with the case where G is abelian.

Lemma 2. *Let $R = \bigoplus_{g \in G} R_g$ be a group-graded k-algebra. Suppose that*
(a) *G is an abelian p-group, and*
(b) *$\dim_k R_g \leq 1$ for every $g \in G$, and*
(c) *the elements of $R_g \setminus \{0\}$ are invertible for every $g \in G$.*
Then there exists a subgroup $H \subseteq G$ with $R \cong kH$.

Proof. In view of (c), $H := \{h \in G \;;\; R_h \neq (0)\}$ is a subgroup of G, and $R = \bigoplus_{h \in H} R_h$. By general theory, the group H is a direct sum of cyclic groups with generators $h_1, \ldots, h_\ell$ of orders $p^{n_1}, \ldots, p^{n_\ell}$, say. Pick $r_i \in R_{h_i}$ with $r_i^{p^{n_i}} = 1$. Given $i, j \in \{1, \ldots, \ell\}$, there exists $\alpha_{ij} \in k$ such that

$$r_i r_j r_i^{-1} = \alpha_{ij} r_j$$

Thus,

$$r_j = r_i^{p^{n_i}} r_j r_i^{-p^{n_i}} = \alpha_{ij}^{p^{n_i}} r_j,$$

so that $\alpha_{ij} = 1$. Consequently, the elements $r_1, \ldots, r_\ell$ commute with each other. Since the subalgebra generated by these elements contains all homogeneous parts of R, we see that R is commutative. By the same token, the map $T_i \mapsto r_i$ defines an isomorphism

$$k[T_1, \ldots, T_r]/(T_1^{p^{n_1}} - 1, \ldots, T_\ell^{p^{n_\ell}} - 1) \xrightarrow{\sim} R,$$

with the truncated polynomial ring being isomorphic to kH. $\square$

The proof of our main result necessitates information on nilpotent elements. A subset $W \subseteq R$ of a k-algebra R is *nil* if every element $w \in W$ is nilpotent. We say that W is *nilpotent* if $W^n = (0)$ for some $n \in \mathbb{N}$. The set W is referred to as *weakly closed* if there exists a function $\gamma : W \times W \longrightarrow k$ such that $vw + \gamma(v, w)wv \in W$ for all $v, w \in W$. Here is the relevant result, which we shall take for granted (see [4, (II.2)]).

Theorem 3 (Jacobson's Theorem on nil weakly closed sets)**.** *Let $W \subseteq R$ be a nil, weakly closed subset of an associative k-algebra R. Then the associative subalgebra $\mathrm{alg}_k(W) \subseteq R$ without identity that is generated by W is nilpotent.* $\square$

Theorem 4 ([2])**.** *Let G be a p-group, $R = \bigoplus_{g \in G} R_g$ be a G-graded algebra. If R_1 is local, then R is local.*

Proof. We first assume that G is abelian and write G additively. Since R_0 is local, there exists a linear map $\alpha : R_0 \longrightarrow k$ such that

$$\ker \alpha = \{r \in R_0 \;;\; r \text{ is nilpotent}\}.$$

Given $g \in G$, we set

$$N_g := \{r \in R_g \;;\; r \text{ is nilpotent}\}.$$

Suppose that $\mathrm{ord}(G) = p^m$. For $g \in G$ and $r \in R_g$, we have $r^{p^m} \in R_{p^m g} = R_0$. By the above, we can write

$$(*) \qquad r^{p^m} = \alpha(r^{p^m})1 + x$$

for some nilpotent element $x \in N_0$. It follows that

$$\psi_g : R_g \longrightarrow k \;\; ; \;\; r \mapsto \alpha(r^{p^m})$$

is a homogeneous polynomial function of degree p^m, whose zero locus $\mathcal{Z}(\psi_g)$ is N_g. Since R_g and k are irreducible varieties and ψ_g is a morphism, it follows from standard results on morphisms that $\dim N_g \geq \dim_k R_g - 1$.

By $(*)$, a homogeneous element $r \in R$ is either nilpotent or invertible. Given $r \in N_g$ and $s \in R_h$, we have $rs \in R_{g+h}$. If rs is invertible, then left multiplication by r is surjective, which contradicts the nilpotence of r. Hence $rs \in N_{g+h}$, and a similar argument shows that $sr \in N_{g+h}$. Consequently, $N := \bigcup_{g\in G} N_g$ is a nil weakly closed subset of R. Theorem 3 now implies that the associative algebra $\mathrm{alg}_k(N)$ without identity generated by N is nilpotent. In particular, N_g is a subspace of R_g, which, by our earlier observation, has codimension ≤ 1. By the above, $J = \bigoplus_{g\in G} N_g$ is a nilpotent ideal of R, such that the factor algebra $S := R/J$ is G-graded with the following properties:

(a) $\dim_k S_g \leq 1$ for every $g \in G$, and

(b) every element of $S_g \setminus \{0\}$ is invertible.

Consequently, Lemma 2 provides a subgroup $H \subseteq G$ such that $S \cong kH$. In particular, S is local and the algebra R thus enjoys the same property.

In the general case, that is, when G is not necessarily abelian, we proceed by induction on the order of G. The p-group G has a non-trivial center $C(G)$. We set $G' := G/C(G)$ and denote by $\pi : G \longrightarrow G'$ the canonical projection. By our introductory remarks, this map endows R with a G' grading such that $R_1 = \bigoplus_{g\in C(G)} R_g$ is graded with respect to the abelian p-group $C(G)$. By the first part of the proof R_1 is local, so that induction ensures that the algebra R is also local. □

Corollary 5. *Let $R = \bigoplus_{g\in G} R_g$ be a group-graded algebra. If $N \trianglelefteq G$ is a normal subgroup of index a p-power such that the subalgebra $\bigoplus_{h\in N} R_h$ is local, then R is local.* □

If G is a finite group that acts on a k-algebra Λ via automorphisms

$$(g, \lambda) \mapsto g.\lambda,$$

then $\Lambda{*}G$ denotes the *skew group algebra* of G with coefficients in Λ. By definition, $\Lambda{*}G$ is the free Λ-module with basis G, whose multiplication is given by

$$(\lambda_g g)(\lambda_h h) := \lambda_g(g.\lambda_h)gh \qquad \forall\ g, h \in G,\ \lambda_g, \lambda_h \in \Lambda.$$

We now obtain the following generalization of Green's theorem:

Corollary 6. *Let G be a finite group that operates on an algebra Λ via automorphisms, and suppose that $N \trianglelefteq G$ is a normal subgroup of index a power of p. If M is an indecomposable $\Lambda{*}N$-module, then the induced module $\Lambda{*}G \otimes_{\Lambda{*}N} M$ is indecomposable.*

Proof. The skew group algebra $\Lambda{*}G$ is strongly graded relative to the p-group G/N, with one-component $(\Lambda * G)_1 = \Lambda{*}N$. Moreover, the induced module $\Lambda{*}G \otimes_{\Lambda{*}N} M$ and its endomorphism ring are also G/N-graded, and [1, (4.8)] provides an isomorphism

$$\mathrm{End}_{\Lambda{*}G}(\Lambda{*}G \otimes_{\Lambda{*}N} M)_1 \cong \mathrm{End}_{\Lambda{*}N}(M)$$

of rings. Our assumption on M in conjunction with Theorem 4 now guarantees that the k-algebra $\mathrm{End}_{\Lambda{*}G}(\Lambda{*}G \otimes_{\Lambda{*}N} M)$ is local. Consequently, the module $\Lambda{*}G \otimes_{\Lambda{*}N} M$ is indecomposable. □

References

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