TWO RESULTS ON PROJECTIVE MODULES

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We wish to present two results.

The first is from the paper

Pavel Příhoda, Projective modules are determined by their radical factors, J. Pure Applied Algebra 210 (2007) 827–835.

Theorem A. *Let Λ be a ring, with Jacobson radical J, and let P and Q be two projective (right) Λ-modules. Then we can lift any isomorphism $\bar{f}\colon P/PJ \xrightarrow{\sim} Q/QJ$ to an isomorphism $f\colon P \xrightarrow{\sim} Q$.*

In particular, projective modules are determined by their tops.

Special cases have been known for a long time. For example, we say that Λ is right perfect if every right Λ-module admits a projective cover. Then $P \twoheadrightarrow P/PJ$ is a projective cover, so the result follows (Bass 1960). In general, though, $P \to P/PJ$ will not be a projective cover.

Beck (1972) showed that if P/PJ is a free Λ/J-module, then P is free.

Note that if $J = 0$ then the theorem tells us nothing!

The second result is from the appendix to the paper

Michael Butler and Alastair King, Minimal resolutions of algebras, J. Algebra 212 (1999) 323–362.

Theorem B. *Let $\Lambda = T_A(M)$ be an hereditary tensor algebra, where A is semisimple and M is an A-bimodule. Then every projective module is isomorphic to an induced module, so one of the form $X \otimes_A \Lambda$; equivalently, every projective module is gradable.*

This is trivial if there are no oriented cycles; that is, if the graded radical Λ_+ is nilpotent, so Λ is semiprimary and $J(\Lambda) = \Lambda_+$. In general, if there are enough arrows, then $J(\Lambda) = 0$.

Green (2000) has an alternative (and more well-known) proof for path algebras of quivers using Gröbner bases, but this is actually more involved than the Butler-King proof.

Bergman's theory of modules over coproducts of rings (1974) can be used to reduce the problem to when A is a division ring, in which case Cohn's work on firs (free ideal rings) can be used to finish the result. This approach is extremely complicated, with Bergman's paper being technically demanding.

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1. Projective modules are determined by their tops

We fix a ring Λ with Jacobson radical J, and write $\bar{X} := X/XJ$ for a Λ-module X.

The proof of Theorem A is based around generalising two classical results.

Lemma 1. *Let P be a projective module, $X \subset P$ a finite subset, and $\theta \in \operatorname{End}(P)$ such that $\theta(p) - p \in PJ$ for all $p \in P$. Then there exists $\phi \in \operatorname{End}(P)$ satisfying*

(1) *$\phi(p) - p \in PJ$ for all $p \in P$, and*
(2) *$\phi\theta(x) = x$ for all $x \in X$.*

By taking $\theta = 0$ and any element $x \in P$, this recovers a result of Bass (1960) showing that $PJ = P$ implies $P = 0$.

Proof. Assume first that P is free, and take a finitely generated free summand F, say of rank n, containing both X and $\theta(X)$. Note that $\operatorname{End}(F) \cong \mathbb{M}_n(\Lambda)$ and $J(\operatorname{End}(F) \cong \mathbb{M}_n(J)$, so that $\alpha \in J(\operatorname{End}(F))$ if and only if $\operatorname{Im}(\alpha) \subseteq FJ$.

Let $P \underset{\iota}{\overset{\pi}{\rightleftarrows}} F$ be the natural maps, and set $\theta' := \pi\theta\iota \in \mathrm{id}_F + J(\operatorname{End}(F))$. This has an inverse of the form $\mathrm{id}_F + \phi'$ for some $\phi' \in J(\operatorname{End}(F))$, and we can take $\phi := \mathrm{id}_P + \iota\phi'\pi$.

In general, suppose $P \oplus Q$ is free and consider $\hat{\theta} := \iota_P\theta\pi_P + \iota_Q\pi_Q$. This satisfies $\hat{\theta}(a) - a \in PJ$ for all $a \in P \oplus Q$, so we can find $\hat{\phi}$ as above. We then set $\phi = \pi_P\hat{\phi}\iota_P$. $\square$

This is enough to prove Theorem A for countably generated projectives.

Proposition 2. *Let P and Q be countably generated projective modules. Then we can lift any isomorphism $\bar{f}\colon \bar{P} \xrightarrow{\sim} \bar{Q}$ to an isomorphism $f\colon P \xrightarrow{\sim} Q$.*

Proof. Let $\bar{g}$ be the inverse to $\bar{f}$, and choose lifts $P \underset{\tilde{g}}{\overset{\tilde{f}}{\rightleftarrows}} Q$. Let $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$ be generating sets for P and Q respectively.

We will construct recursively homomorphisms $P \underset{g_n}{\overset{f_n}{\rightleftarrows}} Q$ and finite sets $X_n \subset P$ and $Y_n \subset Q$ as follows. We start by setting $X_1 := \{x_1\}$ and $f_1 := \tilde{f}$, and then apply the constructions $\beta_1, \alpha_2, \beta_2, \dots$, where

(α_n) Note that $\tilde{f}g_{n-1}(q) - q \in QJ$ for all $q \in Q$, so by the lemma there exists $\phi_n \in \operatorname{End}(Q)$ such that
  (a) $\phi_n(q) - q \in QJ$ for all $q \in Q$, and
  (b) $\phi_n\tilde{f}g_{n-1}(y) = y$ for all $y \in Y_{n-1}$.
  Set $f_n := \phi_n\tilde{f}$, which is a lift of $\bar{f}$, and $X_n := g_{n-1}(Y_{n-1}) \cup \{x_n\}$.

(β_n) Note that $\tilde{g}f_n(p) - p \in PJ$ for all $p \in P$, so by the lemma there exists $\theta_n \in \operatorname{End}(P)$ such that
  (a) $\theta_n(p) - p \in PJ$ for all $p \in P$, and
  (b) $\theta_n\tilde{g}f_n(x) = x$ for all $x \in X_n$.
  Set $g_n := \theta_n\tilde{g}$, which is a lift of $\bar{g}$, and $Y_n := f_n(X_n) \cup \{y_n\}$.

Now, $X_{n+1} = g_n(Y_n) \cup \{x_n\} = X_n \cup \{g_n(y_n), x_n\}$. Also, if $x \in X_n$, then $x = g_nf_n(x)$ and $f_n(x) \in Y_n$, so $f_{n+1}(x) = f_n(x)$. Since $\bigcup_n X_n$ generates P, we can define $f\colon P \to Q$ such that $f(x) = f_n(x)$ for all $x \in X_n$. Note that f is indeed a lift of $\bar{f}$.

Similarly, we have a lift $g\colon Q\to P$ of $\bar g$ such that $g(y)=g_n(y)$ for all $y\in Y_n$. Then $gf(x_n)=x_n$ and $fg(y_n)=y_n$ for all n, so that f and g are mutually inverse. $\square$

Lemma 3. *Let $M=\bigoplus_{i\in I}M_i$ be a direct sum of countably generated modules, and let $\Theta:=\{\mathrm{id}_M,\dots\}$ be a countable set of endomorphisms of M. Then there exists a well-ordered increasing sequence of subsets $I_\alpha\subseteq I$ such that*

1. *$I=\bigcup_\alpha I_\alpha$,*
2. *$I_\beta=\bigcup_{\alpha<\beta}I_\alpha$ for a limit ordinal β,*
3. *$I_{\alpha+1}-I_\alpha$ is countable, and*
4. *$M_\alpha:=\bigoplus_{i\in I_\alpha}M_i$ is Θ-stable.*

Proof. We begin by observing that if $U\subseteq M$ is a countable subset, then there exists a countable subset $J\subseteq I$ such that $\theta(u)\in\bigoplus_{j\in J}M_j$ for all $\theta\in\Theta$ and $u\in U$.

We set $I_0:=\emptyset$. Assume that we have constructed I_α, and suppose $i\in I-I_\alpha$. Let U_1 be a countable generating set for M_i. By the observation there exists a countable subset $J_1\subseteq I$ such that $\theta(u)\in\bigoplus_{j\in J_1}M_j$ for all $\theta\in\Theta$ and $u\in I_1$. Let U_2 be a countable generating set for this direct sum, and repeat. This yields an increasing sequence of countable subsets $J_1\subseteq J_2\subseteq\cdots$ of I, and we set $I_{\alpha+1}$ to be their union together with I_α.

Now, M_α is Θ-stable by induction. If $x\in M_{\alpha+1}$, then $x\in M_\alpha\oplus_{j\in J_r}M_j$ for some r, so that $\theta(x)\in M_\alpha\oplus j\in J_{r+1}M_j$ for all $\theta\in\Theta$, and hence $M_{\alpha+1}$ is Θ-stable.

For a limit ordinal β we set $I_\beta:=\bigcup_{\alpha<\beta}I_\beta$, so M_β is clearly Θ-stable. $\square$

By taking $\Theta=\{\mathrm{id}_m,\varepsilon\}$ for an idempotent ε, this recovers a construction of Kaplansky (1958), which he used to prove the following result.

Proposition 4. *Every projective module is isomorphic to a direct sum of countably generated modules.*

Proof. Let P be a direct summand of a free module F, and let $\varepsilon\in\mathrm{End}(F)$ be the corresponding idempotent. Since F is a direct sum of cyclic modules, we can apply the lemma to F and $\Theta=\{\mathrm{id}_F,e\}$ to obtain a sequence of summands F_α. Then $P_\alpha:=\varepsilon(F_\alpha)$ is a summand of F_α, yielding the required decomposition $P\cong\bigoplus_\alpha(P_{\alpha+1}/P_\alpha)$. $\square$

Theorem 5. *Let P and Q be projective modules. Then we can lift any isomorphism $\bar f\colon\bar P\xrightarrow{\sim}\bar Q$ to an isomorphism $f\colon P\xrightarrow{\sim}Q$.*

Proof. Let $\bar g$ be the inverse to $\bar f$, and choose lifts $P\underset{\tilde g}{\overset{\tilde f}{\rightleftarrows}}Q$. We apply the lemma to the module $P\oplus Q$ and $\Theta=\{\mathrm{id},\varepsilon,\phi\}$, where ε is the idempotent corresponding to P, and $\phi:=\left(\begin{smallmatrix}0&\tilde g\\ \tilde f&0\end{smallmatrix}\right)$, yielding the increasing sequence of submodules $P_\alpha\oplus Q_\alpha$.

Note that the isomorphism $\bar f$ restricts to give isomorphism $\bar f_\alpha\colon\bar P_\alpha\xrightarrow{\sim}\bar Q_\alpha$ for all α. We construct recursively isomorphisms $f_\alpha\colon P_\alpha\xrightarrow{\sim}Q_\alpha$ lifting $\bar f_\alpha$.

Choose complements $P_{\alpha+1}=P_\alpha\oplus P'_\alpha$ and $Q_{\alpha+1}=Q_\alpha\oplus Q'_\alpha$, yielding the decomposition $\bar f_{\alpha+1}=\left(\begin{smallmatrix}\bar f_\alpha&\bar\xi_\alpha\\ 0&\bar\eta_\alpha\end{smallmatrix}\right)$. Since P'_α and Q'_α are countably generated, we can apply the earlier proposition to lift the isomorphism $\bar\eta_\alpha$ to an isomorphism $\eta_\alpha\colon P'_\alpha\xrightarrow{\sim}Q'_\alpha$. Finally, take any lift ξ_α of $\bar\xi_\alpha$, and set $f_{\alpha+1}:=\left(\begin{smallmatrix}f_\alpha&\xi_\alpha\\ 0&\eta_\alpha\end{smallmatrix}\right)$.

If β is a limit ordinal, then $P_\beta = \bigcup_{\alpha<\beta} P_\alpha$ and the maps f_α are all compatible, so we have an isomorphism f_β lifting $\bar{f}_\beta$. By induction we thus obtain an isomorphism $f\colon P \to Q$ lifting $\bar{f}$. □

Corollary 6. *If Λ is semiperfect, then every projective module is isomorphic to a direct sum of indecomposable projectives of the form $e\Lambda$ for primitive idempotents e, each having local endomorphism ring.*

Proof. Let P be projective. We know that $\bar{\Lambda}$ is semisimple, so $\bar{P}$ is a direct sum of simple modules, each of the form $\bar{e}\bar{\Lambda}$ for some primitive idempotent $\bar{e} \in \bar{\Lambda}$. We can lift $\bar{e}$ to a primitive idempotent $e \in \Lambda$, and set Q to be the corresponding direct sum of the indecomposable projectives $e\Lambda$. Then $\bar{P} \cong \bar{Q}$, so $P \cong Q$. □

2. Projective modules for hereditary tensor algebras

First, some history.

Dedekind proved that every subgroup of a free abelian group is again free. This generalises to the case of free modules over a principal ideal domain.

Schreier (1927) showed that subgroups of free groups are again free. This extended Nielsen's result (1921) which covered finitely generated subgroups of free groups.

Cohn (1964) showed that a free algebra is a free ideal ring (fir), and hence that all projective modules are free. Lewin (1969) gave a direct proof of this following techniques developed by Schreier.

Butler and King's proof can be seen as a further extension of Schreier's techniques to hereditary tensor algebras.

Let A be a semisimple algebra, M an A-bimodule, and set $\Lambda := T_A(M)$ to be the tensor algebra. Thus

$$\Lambda = \bigoplus_{n\geq 0} M^{\otimes n} = A \oplus M \oplus (M \otimes_A M) \oplus (M \otimes_A M \otimes_A M) \oplus \cdots$$

with multiplication given by concatenation of tensors.

Example. All path algebras of quivers are tensor algebras, where $A = K^n$ is a product of fields, one for each vertex, and M has K-basis given by the arrows. Thus $M^{\otimes n}$ has basis the paths of length n.

Set $\Lambda_+ := \bigoplus_{n\geq 1} M^{\otimes n}$. Then $\Lambda_+ \cong M \otimes_A \Lambda$ as right Λ-modules, and every right Λ-module X has a projective presentation of the form

$$0 \to X \otimes_A \Lambda_+ \to X \otimes_A \Lambda \to X \to 0.$$

It follows that Λ is an hereditary algebra.

Theorem 7. *Every projective Λ-module is isomorphic to a module of the form $X \otimes_A \Lambda$ for some A-module X.*

Proof. We know that every free module is isomorphic to an induced module, so it is enough to show that every submodule of an induced module is again induced. In particular, this argument will give an alternative proof that Λ is hereditary.

Let $F = F_0 \otimes_A \Lambda$ be an induced module. Note that this has a natural grading $F = \bigoplus_n F_n$, where $F_n := F_0 \otimes_A M^n$. The corresponding filtration is given by $F_{\leq n} := F_0 \oplus \cdots \oplus F_n$.

Let $L \leq F$ be a submodule, with induced filtration $L_{\leq n} := L \cap F_{\leq n}$. For each n we choose an A-module complement X_n to the A-submodule $L_{<n} + L_{<n}M$ of $L_{\leq n}$. We set $X := \bigoplus_n X_n$, and claim that the multipliction map $\mu\colon X \otimes_A \Lambda \to L$ is an isomorphism.

By induction we see that $L_{\leq n} = \sum_{p+q\leq n} X_p M^q$. Since $L = \bigcup_n L_{\leq n}$, we deduce that μ is surjective. It remains to show that μ is injective. Equivalently, that each $X \otimes_A M^n \to XM^n$ is an isomorphism, and that the sum $\sum_n XM^n$ is direct.

Observe that $-\otimes_A M$ is an exact functor. If now U is an A-submodule of F, then the multiplication map $U \otimes_A M \to UM$ is an isomorphism. For, it is clearly surjective, and is injective since it equals the composite $U \otimes_A M \rightarrowtail F \otimes_A M \xrightarrow{\sim} FM$. By induction, $X \otimes_A M^n \to XM^n$ is an isomorphism for all n.

It remains to prove that the sum $\sum_n XM^n$ is direct. We do this in three steps.

(a) We have $L_{\leq n} \cap LM = L_{<n}M$.

For, denote by $p_n \colon F \twoheadrightarrow F_{\geq n}$ the natural projection of A-modules. Since F is an induced module we have $F_{\geq n}M = F_{>n}$, yielding the commutative diagram

$$\begin{tikzcd}
L \otimes_A M \arrow[r, tail] \arrow[d, "\wr"] & F \otimes_A M \arrow[r, "p_n \otimes \mathrm{id}_M"] \arrow[d, "\wr"] & F_{\geq n} \otimes M \arrow[d, "\wr"] \\
LM \arrow[r, tail] & FM \arrow[r, "p_{n+1}"] & F_{>n}.
\end{tikzcd}$$

The composite of the maps in the top row has kernel $L_{<n} \otimes_A M$, whereas for the bottom row it is $L_{\leq n} \cap LM$. This shows that $L_{<n}M = L_{\leq n} \cap LM$ as required.

(b) We have $XM^n \cap LM^{n+1} = 0$ for all n.

Observe that, if $U, V \leq F$ are A-submodules, then $UM \cap VM = (U \cap V)M$. For, the exact functor $- \otimes_A M$ preserves pullbacks, so $(U \otimes_A M) \cap (V \otimes_A M) = (U \cap V) \otimes_A M$. Thus $XM^n \cap LM^{n+1} = \big(X \cap LM\big)M^n$, and it is enough to prove that $X \cap LM = 0$.

Suppose therefore that $x = x_0 + \cdots + x_n \in X \cap LM$, where $x_i \in X_i$. Then $x \in L_{\leq n} \cap LM = L_{<n}M$ and $x_0 + \cdots + x_{n-1} \in L_{<n}$, so that $x_n \in X_n \cap \big(L_{<n} + L_{<n}M\big) = 0$. By induction on n we deduce that $x_i = 0$ for all i.

(c) The sum $\sum_n XM^n$ is direct.

Suppose we have $y_i \in XM^i$ such that $y_r + \cdots + y_s = 0$. Then $-y_r = y_{r+1} + \cdots + y_s \in XM^r \cap LM^{r+1} = 0$, so by induction $y_i = 0$ for all i. $\square$

Corollary 8. *Assume A is basic and write $1 = \sum_i e_i$ as a sum of primitive orthogonal idempotents. Then the $P_i := e_i\Lambda$ are pairwise non-isomorphic indecomposable projective modules, and every projective module is isomorphic to a direct sum of the P_i.* $\square$