Homological structures on the category $\mathcal{P}_d$ of strict polynomial functors

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The category $\mathcal{P}_d$

An object of $\mathcal{P}_d$ is determined by:

1. $V \mapsto F(V)$,
2. $F_{V,W} : \Gamma^d(\mathrm{Hom}_{\mathbf{k}}(V, W)) \longrightarrow \mathrm{Hom}_{\mathbf{k}}(F(V), F(W))$.

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Evaluation $F \mapsto F(\mathbf{k}^n)$ gives an equivalence of categories

$$\mathcal{P}_d \simeq \Gamma^d(\mathrm{End}_{\mathbf{k}}(\mathbf{k}^n))\text{-mod} =: S_{n,d}(\mathbf{k})\text{-mod}$$

provided that $n \geq d$.

Examples of strict polynomial functors

$$
\begin{aligned}
&V \rightsquigarrow V^{\otimes d} && (I^d),\\
&V \rightsquigarrow (V^{\otimes d})_{\Sigma_d} && (S^d),\\
&V \rightsquigarrow (V^{\otimes d})^{\Sigma_d} && (\Gamma^d),\\
&V \rightsquigarrow ((V^{\otimes d})^{alt})^{\Sigma_d} \simeq ((V^{\otimes d})^{alt})_{\Sigma_d} && (\Lambda^d),
\end{aligned}
$$

If char($\mathbf{k}$)=p, $p > 0$,

$$
V \rightsquigarrow V^{(1)} \qquad (I^{(1)}),
$$

$$
F^{(i)} := F \circ I^{(1)} \circ \ldots \circ I^{(1)}.
$$

Young diagram of weight d: $\lambda = (\lambda_1, \dots, \lambda_k)$, $\sum \lambda_j = d$.

$\{F_\lambda\}$ – complete set of simple objects in $\mathcal{P}_d$

Problem Compute $\mathrm{Ext}^*_{\mathcal{P}_d}(F_\mu, F_\lambda)$.

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Problem Compute $\mathrm{Ext}^*_{\mathcal{P}_d}(F_\mu, F_\lambda)$.

$$S_\lambda := \mathrm{im}(\Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_k} \longrightarrow I^d \longrightarrow S^{\widetilde{\lambda}_1} \otimes \ldots \otimes S^{\widetilde{\lambda}_s}),$$
$$W_\lambda := \mathrm{im}(\Gamma^{\widetilde{\lambda}_1} \otimes \ldots \otimes \Gamma^{\widetilde{\lambda}_s} \longrightarrow I^d \longrightarrow \Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_k}),$$

$F_\lambda \hookrightarrow S_\lambda$, $W_\lambda \twoheadrightarrow F_\lambda$.

The Lusztig Conjecture

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Program Compute $\mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(W_\mu^{(i)}, S_\lambda)$ for $i > 0$.

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$$\begin{array}{rcl} F & \mapsto & F^{(1)} \\ \mathbf{C} : \mathcal{DP}_d & \longrightarrow & \mathcal{DP}_{dp} \end{array}$$

$$\mathbf{K}^r(F)(V) := \mathrm{RHom}_{\mathcal{DP}_{dp}}(\Gamma^d(V^* \otimes I^{(1)}), F),$$
$$\mathbf{K}^l(F) := \mathbf{K}^r(F^\#)^\#.$$

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Theorem 1 There is a two-sided adjunction

$$\mathbf{C} : \mathcal{DP}_d \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \mathcal{DP}_{dp} : \mathbf{K}^r, \mathbf{K}^l.$$

Hence $\mathrm{Ext}^*_{\mathcal{P}_{dp}}(W_\mu^{(1)}, S_\lambda) \simeq \mathrm{HExt}^*_{\mathcal{P}_d}(W_\mu, \mathbf{K}^r(S_\lambda))$.

$\mathrm{H}^*(\mathbf{K}^r(F))(V) := \mathrm{Ext}^*_{\mathcal{P}_{dp}}(\Gamma^d(V^* \otimes I^{(1)}), F)$
is a graded module over
$\mathrm{Ext}^*_{\mathcal{P}_{dp}}(\Gamma^d(V^* \otimes I^{(1)}), \Gamma^d(V^* \otimes I^{(1)}))$

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$\mathcal{P}_d^{af} := \{F : \text{A–mod}^{fr} \longrightarrow \mathbf{k}\text{–mod}^{gr}$,
strict polynomial of degree d over $\mathbf{k}\}$.

Evaluation $F \mapsto F(\mathbf{k}^n \otimes A)$
gives an equivalence of categories

$$\mathcal{P}_d^{af} \simeq \Gamma^d(\mathrm{End}_{\mathbf{k}}(\mathbf{k}^n) \otimes A)\text{-mod}^{gr}$$

provided that $n \geq d$.

Examples of affine strict polynomial functors:

For $0 \leq j \leq p-1$, $\chi_j(W) := x^j W / x^{j+1} W[-j]$,
$S^{af}_{(\emptyset;\ldots;\lambda;\ldots;\emptyset)}(W) := S_\lambda(\chi_j(W))$,
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$$A \simeq \mathbf{k}[\mathbf{Z}/p] \simeq \mathbf{k}[x]/x^p \quad \longleftrightarrow \quad \mathbf{k}[\mathbf{Z}] \simeq \mathbf{k}[x, x^{-1}]$$

$$\Gamma^d(M_n(\mathbf{k}[x]/x^p))\text{-mod}^{gr} \quad \longleftrightarrow \quad \Gamma^d(M_n(\mathbf{k}[x, x^{-1}]))\text{-mod}$$

$\mathrm{Kom}(\mathcal{P}_d^{af}) := \{F : \mathrm{A\text{–}mod}^{\mathrm{fr}} \longrightarrow \mathrm{Kom}_{\mathbf{k}}$, strict polynomial of degree d over $\mathbf{k}\}$

$\mathcal{DP}_d^{af} := \mathrm{Kom}(\mathcal{P}_d^{af})/\mathrm{qis}$

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Theorem 3
There is a commutative diagram of adjunctions where the bottom row is a left part of recollement of triangulated categories.

$$
\begin{array}{ccccc}
 & \mathcal{DP}_d & & & \\
 & f\downarrow\uparrow t & \nwarrow\searrow\nwarrow & & \\
\mathbf{C}^{af}: & \mathcal{DP}_d^{af} & \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} & \mathcal{DP}_{pd} & : \mathbf{K}^{\mathrm{r,af}}, \mathbf{K}^{\mathrm{l,af}}
\end{array}
$$

Moreover, we have

$$\mathbf{K}^{r,af}(S_\lambda) = S_{q(\lambda)}^{af}[-h(\lambda)].$$

The idea of proof

1. $P_* \longrightarrow \Gamma^d(V^* \otimes I^{(1)})$ a projective resolution,
   $B := \mathrm{Hom}_{\mathcal{P}_{pd}}(P_*, P_*)$.
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$$\mathcal{D}(B\text{-mod}) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{DP}_{pd}.$$

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   $B := \mathrm{Hom}_{\mathcal{P}_{pd}}(P_*, P_*)$.
   By dg–Morita theory we have a recollement
   $$\mathcal{D}(B\text{-mod}) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{DP}_{pd}.$$

2. $\mathrm{H}^*(B) \simeq \Gamma^d(\mathrm{End}_{\mathbf{k}}(V) \otimes A)$,
   B is formal as a $\Gamma^d(\mathrm{End}_{\mathbf{k}}(V))$–algebra.
   Hence
   $$\mathcal{D}(B\text{-mod}) \simeq \mathcal{DP}_d^{af}.$$

Application: The Collapsing Conjecture

Corollary
For any $F, G \in \mathcal{P}_d$

$$\mathrm{Ext}^*_{\mathcal{P}_{pd}}(F^{(1)}, G^{(1)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(F, G_A)$$

where $G_A(V) := G(V \otimes A)$.

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Proof:

$$\mathrm{Ext}^*_{\mathcal{P}_{pd}}(F^{(1)}, G^{(1)}) = \mathrm{Ext}^*_{\mathcal{P}_{pd}}(\mathbf{C}(F), \mathbf{C}(G)) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(F, \mathbf{K}^r(\mathbf{C}(G))) \simeq$$

$$\mathrm{Ext}^*_{\mathcal{P}_d}(F, t(f(G))) = \mathrm{Ext}^*_{\mathcal{P}_d}(F, G_A).$$

$$\mathcal{DP}_d^{af} \;\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}\; \mathcal{DP}_{pd} \;\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}\; ?$$

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The Kuhn stratification

There is a stratification of $\mathcal{P}_d$ by abelian categories with a set of strata indexed by a poset
$\mathcal{T} := \{\alpha = (\alpha_0, \dots, \alpha_s) : \alpha_j \geq 0, \sum \alpha_j p^j = d\}$,
with ordering generated by
$(\alpha_0, \dots, \alpha_s) < (\alpha_0, \dots, \alpha_j + p, \alpha_{j+1} - 1, \dots, \alpha_s)$.

$$\mathcal{P}_{\mathcal{J}} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{P}_{\mathcal{K}} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{P}_{\mathcal{K}/\mathcal{J}}$$

$$\mathcal{DP}_d^{af} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{DP}_{pd} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} ?$$

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$$\mathcal{P}_{\mathcal{J}} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{P}_{\mathcal{K}} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{P}_{\mathcal{K}/\mathcal{J}}$$

$$\mathcal{P}_{\mathcal{J}_{max}} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{P}_d \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{k}[\Sigma_d]\text{-mod}$$

Hopefully, we have

$$\mathcal{DP}^{af}_{\mathcal{J}} \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{DP}^{af}_{\mathcal{K}} \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{DP}^{af}_{\mathcal{K}/\mathcal{J}},$$

$$\mathcal{DP}^{af}_{\mathcal{J}_{max}} \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{DP}_{d} \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(\mathbf{k}[\Sigma_d]\text{-mod}).$$