A Model for Nonlocal Advection

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MFO Mini-Workshop on the Mathematics of Peridynamics
16–22 January 2011

SAND2011-0093C

We consider only the 1-D case in this presentation.

Local advection is a well-known subject.

- The general case is the scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

  where f is the flux function.

- The simplest case is the one-way linear wave equation:

$$f(u) = c\,u \;\Rightarrow\; \frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0$$

- Burgers equation is the simplest nonlinear example:

$$f(u) = \frac{u^2}{2} \;\Rightarrow\; \frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0$$

Such equations possess a rich structure.

Investigation of these equations incorporates several important concepts of physics, mathematics, and numerics:

- *Physics*: wave interactions, entropy, EOS
- *Mathematics*: wave structure of HCLs, weak solutions
- *Numerics*: solution algorithms, conservation

There are numerous references on these subjects, including the superb monographs by Dafermos [9], Evans [14], Lax [21], LeVeque [24, 25], Smoler [34], Trangenstein [35], Whitham [39].

The concepts underpinning peridynamics are well established.

Peridynamics provides a nonlocal framework for elasticity [33].

- Nonlocal interactions are *intrinsic* to the theory.
  - These interactions are mediated through the micromodulus.
  - For elasticity, the nonlocal nature admits discontinuous displacements, e.g., fracture.
- Consideration of nonlocality leads to fundamental questions related to continuum mechanics.
- Mathematical and computational investigations have, likewise, revealed a rich and varied structure.

What is the relation between nonlinear advection and peridynamics?

Can we develop a *unified* approach to peridynamics and nonlinear advection that captures, e.g., “shock-like” behavior?

- We would like to expand peridynamics-based simulation capabilities to include impact, energetic materials, etc.
  - This necessarily includes *coupled* mass, momentum, and energy balance equations. . .
  - . . . together with a description of more complex material response, i.e., a functional relationship between the stress (pressure) and the state (density, internal energy, strain).
- What is the simplest model equation we can examine to understand the relevant issues? *Burgers equation.*

Others have considered nonlocal advection (1/4).

- Logan [27]: nonlocal wavespeed related to a specified function $G(u)$ over a fixed domain $\Omega \Rightarrow$

$$u_t + \left(\int_\Omega G(u)\, dy \right) u_x = 0 \,. \tag{1}$$

- Baker et al. [4]: nonlocality introduced through Hilbert transform for vortex sheet modeling $\Rightarrow$

$$u_t + (\mathbb{H}(u))_x = \epsilon\, u_{xx} \,, \tag{2}$$

$$u_t - \mathbb{H}(u)\, u_x = \epsilon\, u_{xx} \,, \tag{3}$$

$$\text{where} \quad \mathbb{H}(u) := \int_{-\infty}^{\infty} dy\, u(y)/(x-y) \,. \tag{4}$$

Castro and Córdoba [7], Parker [31], Deslippe et al. [10], Biello and Hunter [6] consider related forms.

Others have considered nonlocal advection (2/4).

- Veksler and Zarmi [36, 37] consider a nonlocal form of the Burgers equation that is "discretely nonlocal" in that it involves function values at discrete points.
- Droniou [11], Alibaud and co-workers [2, 3] consider the usual 1D Burgers flux and fractional derivative regularization.
- Woyczyński [40] considers fractional derivative operator in the advective term with no regularization.
- Miškinis [28] considers a fractional derivative advective term and local diffusive regularization.
- Benzoni-Gavage [5] and Alì et al. [1] consider a generalized Burgers equation $u_t + \mathcal{F}_x[u] = 0$, where the F.T. of $\mathcal{F}[u]$ is $\hat{\mathcal{F}}[u](k) = \int_{-\infty}^{\infty} \Lambda(k-l)\hat{u}(k-l)\hat{u}(l)\, dl$.

Others have considered nonlocal advection (3/4).

- Fellner and Schmeiser [15] rewrite the system $u_t + u\,u_x = \phi_x,\ \phi_{xx} - \phi = u$ as the single equation $u_t + u\,u_x = \phi_x[u]$, where $\phi[u] = \int_{\mathbb{R}} G(x-y)\,u(y)\,dy$.
- Liu [26] considers nonlocal Burgers equations of the form $u_t + u\,u_x + (G * B[u, u_x])_x = 0$, where G is the same kernel.
- Chmaj [8] considers traveling wave solutions to a generalized nonlocal Burgers equation of the form $u_t + (u^2/2)_x + u - K * u = 0$, for symmetric K.
- Duan et al. [13] examine existence and stability of solutions to equations that are multi-dimensional generalizations of those studied by Chmaj [8] .

Others have considered nonlocal advection (4/4).

- Rohde [32] considers existence and uniqueness of $u_t + \mathrm{div} f(u) = \mathcal{R}[\epsilon, u]$, $\mathcal{R}$ a nonlocal regularization.
- Kissling and Rohde [18] generalize this analysis to $u_t^{\epsilon,\lambda} + f_x(u^{\epsilon,\lambda}) = \mathcal{R}^{\epsilon}[\lambda; u^{\epsilon,\lambda}]$, where ϵ is a scale parameter and λ is an auxiliary parameter.
- Kissling et al. [19] focus on the multidimensional case for a particular form of nonlocal regularization in [18].
- Ignat and Rossi [17] analyze the equation

$$u_t(x,t) = \int_{\mathbb{R}} \big(u(y,t) - u(x,t)\big)\, J(y-x)\, dy + \int_{\mathbb{R}} \big(h(u)(y,t) - h(u)(x,t)\big)\, K(y-x)\, dy\,.$$

We posit the following integro-differential equation:

For $(x, t) \in \mathbb{R} \times (0, \infty)$:

$$u_t(x,t) + \int_{\mathbb{R}} dy\, \psi\left(\frac{u(y,t) + u(x,t)}{2}\right) \phi_a(y,x) = 0\,, \tag{5a}$$

$$u(x,0) = g(x)\,. \tag{5b}$$

- The kernel (i.e., *micromodulus*) is antisymmetric:
  $$\phi_a(y,x) = -\phi_a(x,y)$$
- ϕ_a is typically a translation-invariant function:
  $$\phi_a(y,x) = -\phi_a(y-x)$$

(5a) is a nonlocal, nonlinear advection equation.

Why does this equation represent advection?

Let $\phi_a(y,x) \equiv -\partial\delta(x-y)/\partial y$ and evaluate:

$$\int_{\mathbb{R}} dy\, \psi\left(\frac{u(y,t)+u(x,t)}{2}\right)\phi_a(y,x) \tag{6a}$$

$$= -\left[\psi\left(\frac{u(y,t)+u(x,t)}{2}\right)\delta(y-x)\right]\Bigg|_{y=-\infty}^{y=\infty} \tag{6b}$$

$$+ \int_{\mathbb{R}} dy\, \psi_y\left(\frac{u(y,t)+u(x,t)}{2}\right)\delta(y-x) \tag{6c}$$

$$= \psi_x\big(u(x,t)\big) \tag{6d}$$

$$\Longrightarrow \boxed{u_t + f_x(u) = 0 \quad \text{where} \quad f \leftrightarrow \psi}$$

Why does this equation represent conservation?

From asymmetry of the integrand,

$$\int_a^b \int_a^b \psi\left(\frac{u(y,t)+u(x,t)}{2}\right) \phi_a(y,x)\, dy\, dx = 0\,. \tag{7}$$

Therefore, integrating (5a) equation over (a, b) implies

$$\frac{d}{dt}\int_a^b u(x,t)\, dx + \int_a^b \int_{\mathbb{R}\setminus(a,b)} \psi\left(\frac{u(y,t)+u(x,t)}{2}\right) \phi_a(y,x)\, dy\, dx = 0\,. \tag{8}$$

Extending the interval (a, b) to the entire line and using the asymmetry of this integrand gives the result that

$$\boxed{\frac{d}{dt}\int_{\mathbb{R}} u(x,t)\, dx = 0, \quad i.e., \int_{\mathbb{R}} u(x,t)\, dx \text{ is conserved.}}$$

We develop a more general notion of a flux...

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be open intervals such that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$. Define

$$\Psi\big(\mathcal{I}_1, \mathcal{I}_2, t\big) := \int_{\mathcal{I}_1} \int_{\mathcal{I}_2} \psi\left(\frac{u(y,t) + u(x,t)}{2}\right) \phi_a(y,x)\, dy\, dx\,, \tag{9}$$

The antisymmetry of the integrand leads to the following result.

Lemma 1

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be open intervals such that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$. Then

$$\begin{aligned} \Psi\big(\mathcal{I}_1, \mathcal{I}_2, t\big) + \Psi\big(\mathcal{I}_2, \mathcal{I}_1, t\big) &= 0\,, \\ \Psi\big(\mathcal{I}_1, \mathcal{I}_1, t\big) &= 0\,. \end{aligned} \tag{10}$$

With these ideas, we generalize the concept of flux.

$$\Psi(\mathcal{I}_1, \mathcal{I}_2, t) + \Psi(\mathcal{I}_2, \mathcal{I}_1, t) = 0\,, \quad \Psi(\mathcal{I}_1, \mathcal{I}_1, t) = 0\,. \tag{11}$$

We identify $\Psi(\mathcal{I}_1, \mathcal{I}_2, t)$ with the flux of u from $\mathcal{I}_1$ into $\mathcal{I}_2$.

(11) shows that the flux is equal and opposite between disjoint intervals, and there is no flux from an interval into itself.

This contrasts with the usual flux concept with a unit normal on a surface separating $\mathcal{I}_1$ and $\mathcal{I}_2$ carrying the direction for the flux.

We conclude that the relation below is an *abstract balance law*:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + \Psi\big((a,b), \mathbb{R} \setminus (a,b), t\big) = 0\,. \tag{12}$$

The production of a quantity inside an interval is balanced by the flux of the same quantity out of the same interval.

These properties are central to the concept of the flux.

Both the production and flux are additive and biadditive, respectively, over disjoint intervals; e.g.,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{I}_1} u(x,t)\,dx + \frac{d}{dt}\int_{\mathcal{I}_2} u(x,t)\,dx &= \frac{d}{dt}\int_{\mathcal{I}_1\cup\mathcal{I}_2} u(x,t)\,dx \\ &= -\Psi\big(\mathcal{I}_1\cup\mathcal{I}_2, \mathbb{R}\setminus(\mathcal{I}_1\cup\mathcal{I}_2), t\big) \\ &= \Psi\big(\mathbb{R}\setminus(\mathcal{I}_1\cup\mathcal{I}_2), \mathcal{I}_1\cup\mathcal{I}_2, t\big)\,.\end{aligned}$$

These additive and biadditive properties for the production and flux of a quantity can be shown to be a necessary and sufficient condition for the antisymmetry of the integrand of Ψ given in (9), as discussed by Du et al. [12, Section 6].

Noll's Lemma I gives an alternative flux expression.

For general antisymmetric ϕ_a, and with certain boundedness and smoothness assumptions, Noll's Lemma I [23, 30] gives an explicit expression for the flux function:

$$f(u; x, t) = -\frac{1}{2} \int_{\mathbb{R}} dz \int_0^1 d\lambda \, \psi\left(\frac{u(x - (1-\lambda)z, t) + u(x + \lambda z, t)}{2} \right) \times z \, \phi_a(x - (1-\lambda)z, x + \lambda z) \tag{13}$$

such that

$$f_x(u; x, t) = \int_{\mathbb{R}} dy \, \psi\left(\frac{u(y, t) + u(x, t)}{2} \right) \phi_a(y, x) \, . \tag{14}$$

There is an another expression for the alternative flux.

The expression for the nonlocal flux function given in (13) can be recast in the following form (c.f. [38, Eq. 9],[22, Def. 2]):

$$f(u; x, t) = \int_0^\infty dz \int_0^\infty dy \; \psi\left(\frac{u(x+y,t)+u(x-z,t)}{2}\right) \times \phi_a(x+y, x-z) \,. \tag{15}$$

The flux function depends on:

values to the *right* of x, labeled by $x + y$, and
values to the *left* of x, labeled by $x - z$.

This differs from the local flux.

We seek a conservative numerical scheme.

Discretize space into cells $[x_{i-1/2}, x_{i+1/2}]$ and time into intervals $[t^n, t^{n+1}]$. On the ith cell, define

$$\Psi(x_{i-1/2}, x_{i+1/2}, t) := \sum_{j \neq i} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} \psi\left(\frac{u(y,t)+u(x,t)}{2}\right) \times \phi_a(y,x)\, dy\, dx\,. \tag{16}$$

The quantity $\Psi(x_{i-1/2}, x_{i+1/2}, t)$ represents the flux of u over the interval $[x_{i-1/2}, x_{i+1/2}]$. The spatially integrated form of the nonlocal conservation law (5a) can now be written as

$$\int_{x_{i-1/2}}^{x_{i+1/2}} u_t(x,t)\, dx + \Psi(x_{i-1/2}, x_{i+1/2}, t) = 0\,. \tag{17}$$

We devise such a scheme as follows...

Integrating both sides of (16) over $[t^n, t^{n+1}]$ implies:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \Big(u(x, t^{n+1}) - u(x, t^n) \Big)\, dx + \int_{t^n}^{t^{n+1}} \Psi(x_{i-1/2}, x_{i+1/2}, t)\, dt = 0\,. \tag{18}$$

This is a nonlocal statement that the change in the u over the cell $[x_{i-1/2}, x_{i+1/2}]$ in the time interval $[t^n, t^{n+1}]$ is balanced by the flux over the cell $[x_{i-1/2}, x_{i+1/2}]$ in the time interval $[t^n, t^{n+1}]$.

. . .and obtain a familiar form:

$$\bar{u}_i^n := \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n)\, dx \quad \text{and} \tag{19a}$$

$$\bar{\Psi}_i^{n-1/2} := \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \Psi(x_{i-1/2}, x_{i+1/2}, t)\, dt\,, \tag{19b}$$

we write the nonlocal equation on $[x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$ as

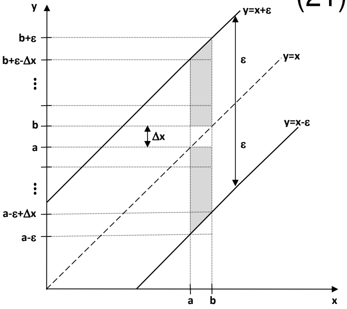
$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\, \bar{\Psi}_i^{n+1/2}\,. \tag{20}$$

A conservative numerical scheme results by application of a quadrature rule in the expression for Ψ in (19b) (i.e., (16)).

Using a simple quadrature rule. . .

$$\Psi\left(x_{i-1/2}, x_{i+1/2}, t\right) = \sum_{j=-r}^{r} \omega_j\, \psi\left(\frac{u(x_{i+j}, t) + u(x_i, t)}{2}\right) \phi_a(x_{i+j}, x_i)(\Delta x)^2\,, \tag{21}$$

$$\omega_j = \begin{cases} 0\,, & j = 0\,, \\ 1\,, & j = \pm 1, \ldots, \pm(r-1)\,, \\ 1/2\,, & j = -r, r\,. \end{cases}$$



. . .the scheme has familiar stability properties.

The kernel: $\phi_a^P(x,y) = \frac{1}{\varepsilon^2}\begin{cases} 1\,, & y > x\,,\\ 0\,, & y = x\,,\\ -1\,, & y < x\,,\end{cases}$ gives the scheme:

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{1}{\varepsilon^2}\frac{\Delta t}{\Delta x}\left(\sum_{j=1}^{r}\left(\frac{u_{i+j}^n + u_i^n}{2}\right)(\Delta x)^2 - \sum_{j=1}^{r}\left(\frac{u_{i-j}^n + u_i^n}{2}\right)(\Delta x)^2\right). \tag{22}$$

The linear stability limit is: $\Delta t < \frac{\beta(\Delta x)\varepsilon^2}{r\Delta x} = \beta(\Delta x)\varepsilon\,,$ where

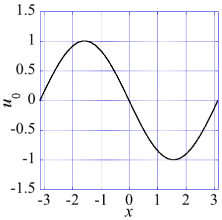
$$\beta^2(\Delta x) := 1 - \max_{k\in\mathbb{K}\setminus\mathbb{K}_1}\left\{\cos^2(k\Delta x)\right\} \quad \text{and} \quad r = \varepsilon/\Delta x \in \mathbb{Z}^+\,,$$

for $\mathbb{K} := \{m\pi/L,\ m = 1, \ldots, L/\Delta x\}$, $\mathbb{K}_1 := \{k : k\Delta x = 0 \mod \pi\}$

We perform computations for two initial conditions.

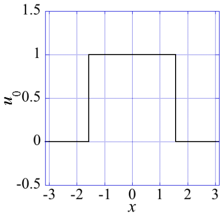
- *Nonlocal Burgers Flux Function*: $\psi(u) = u^2/2$
- *Domain*: $-\pi \leq x < \pi$, N_x cells with $dx = L/N_x$, $L = \pi$
- *Boundary conditions*: $u(x + kL, t) = u(x),\ k \in \mathbb{Z}$
- *Initial conditions*:

$$u_0(x) = -\sin x$$



"Sinusoid"

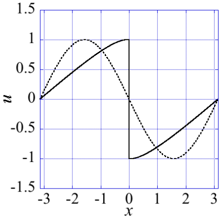
$$u_0(x) = H(x + \pi/2) - H(x - \pi/2)$$



"Tophat"

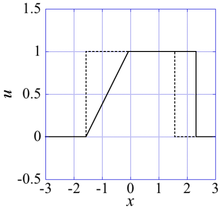
Local Burgers equation results are a reference.

Sinusoid ICs (Muraki [29]) $\Rightarrow$
shock formation at $t = 1$



$t = 1.5$
Shock fixed at $x = 0$;
$t \to \infty \Rightarrow N$-wave.

Tophat ICs $\Rightarrow$
rarefaction (L) + shock (R)



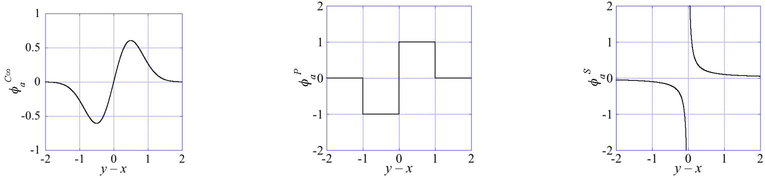
$t = 1.5$
This structure persists
up to $t = 2\pi$.

For nonlocal cases, we consider different micromoduli.

$$C^\infty: \quad \phi_a^{C^\infty}(y,x) \propto (y-x)\exp\left(-|y-x|^2/B(\varepsilon)\right)$$

$$\text{"Parks"}: \phi_a^{P}(y,x) \propto H(y-x+\varepsilon) - 2H(y-x) + H(y-x-\varepsilon)$$

$$\textit{Singular}: \phi_a^{S}(y,x) \propto \operatorname{sgn}(y-x)\,|y-x|^{-\alpha}$$



$\phi_a^{C^\infty}(y,x)/A^C(\varepsilon)$ $\quad$ $\phi_a^{P}(y,x)/A^P(\varepsilon)$ $\quad$ $\phi_a^{S}(y,x)/A^S(\varepsilon)$

$$\textit{Note: } \varepsilon \to 0^+ \;\Rightarrow\; \phi_a(y,x) \to \delta'(y-x)$$

There are two primary nondimensional length scales.

- $\boxed{\varepsilon/L \in (0,1)}$: ratio of PD horizon to problem length scale

  ε/L measures the degree of nonlocality

  $\varepsilon/L \to 0^+$ is the local limit

  $\varepsilon/L \to 1^-$ is the extreme nonlocal limit

- $\boxed{\varepsilon/\Delta x \in (1,\infty)}$: ratio of PD horizon to cell size

  $\varepsilon/\Delta x$ measures how well the nonlocality is resolved

  $\varepsilon/\Delta x \to 1^+$ is an under-resolved micromodulus

  $\varepsilon/\Delta x \gg 1$ is a well-resolved micromodulus

We have hypotheses about these parameters' effects.

- For $\varepsilon/L \ll 1$, the effect of nonlocality should be decreased.
  - $\rightarrow$ Different $\phi_a \Rightarrow$ results should be similar.
  - $\rightarrow$ Nonlocal results should approach local results.
- For $\varepsilon/L \rightarrow 1^-$, differences between the various ϕ_a should be highlighted.
- For $\varepsilon/\Delta x \gg 1$, the computed solution may be more faithful to the continuum solution.
- For $\varepsilon/\Delta x \rightarrow 1^+$, the computed result may not reflect the continuum solution.

These tables summarize the computational study.

The domain with characteristic length $L = \pi$ and $N - 1$ cells each of width Δx.

ε-refinement: effect of nonlocality				
N	10000	10000	10000	10000
Δx	6.28e-4	6.28e-4	6.28e-4	6.28e-4
ε	1.26e-2	6.28e-2	1.57e-1	3.14e-1
ε/L	4.00e-3	2.00e-2	5.00e-2	1.00e-1
$\varepsilon/\Delta x$	20	100	250	500

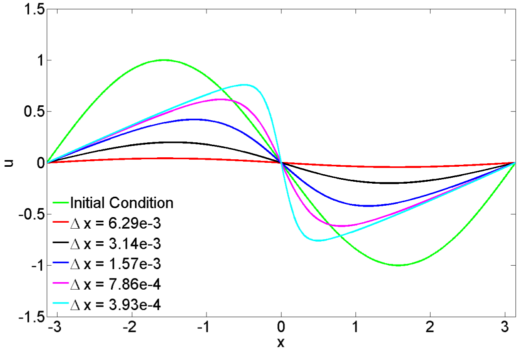
Δx-refinement: effect of mesh resolution					
N	1000	2000	4000	8000	16000
Δx	6.29e-3	3.14e-3	1.57e-3	7.86e-4	3.93e-4
ε	5.00e-2	5.00e-2	5.00e-2	5.00e-2	5.00e-2
ε/L	1.59e-2	1.59e-2	1.59e-2	1.59e-2	1.59e-2
$\varepsilon/\Delta x$	8	16	32	64	128

The smallest and largest values of ε are equal to 0.004 L and 0.1 L, respectively.

Sine IC: mesh refinement effects are significant.

Results for Parks micromodulus, fixed $\varepsilon/L \approx 1.59 \times 10^{-2}$, varying Δx.

Init. Cond.
$\varepsilon/\Delta x = 8$
$\varepsilon/\Delta x = 16$
$\varepsilon/\Delta x = 32$
$\varepsilon/\Delta x = 64$
$\varepsilon/\Delta x = 128$



Larger $\Delta x \Rightarrow$ the scheme has greater numerical dissipation.

Sine IC: horizon refinement effects are less obvious.

Results for Parks micromodulus, fixed $\Delta x/L = 2 \times 10^{-4}$, varying ε.

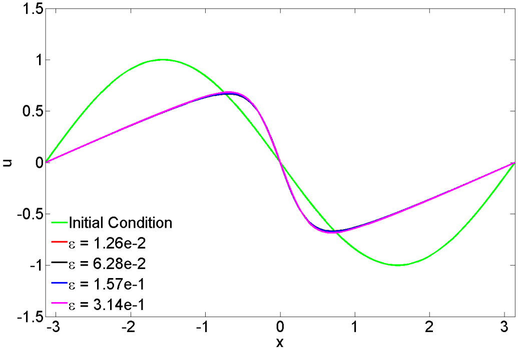
Init. Cond.

$\varepsilon/L = 4 \times 10^{-3}$

$\varepsilon/L = 2 \times 10^{-2}$

$\varepsilon/L = 5 \times 10^{-2}$

$\varepsilon/L = 1 \times 10^{-1}$

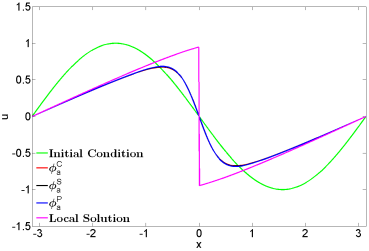


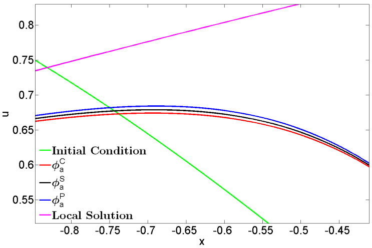
Larger horizon does not have much effect on the solution.

Sine IC: kernel function effects are also small.

Fix $\Delta x \approx 6.28 \times 10^{-4}$ with $\varepsilon \approx 3.14 \times 10^{-1}$ and vary ϕ_a.

Full domain *Close-up*





Init. Cond. $\phi_a^{C^\infty}$ ϕ_a^S ϕ_a^P *Local Burgers Sol'n*

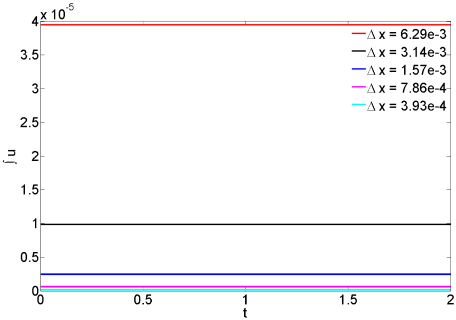
$\Delta x/L = 2\times 10^{-4}$, $\varepsilon/L = 0.1 \Rightarrow$ varying ϕ_a has negligible effect.

Sine IC: conservation under mesh refinement.

Fix $\varepsilon \approx 5.0 \times 10^{-1}$ and vary Δx for $0 < t < 2$.

$$\int_{-\pi}^{\pi} u(x,t)\,dx$$

$\varepsilon/\Delta x = 8$
$\varepsilon/\Delta x = 16$
$\varepsilon/\Delta x = 32$
$\varepsilon/\Delta x = 64$
$\varepsilon/\Delta x = 128$

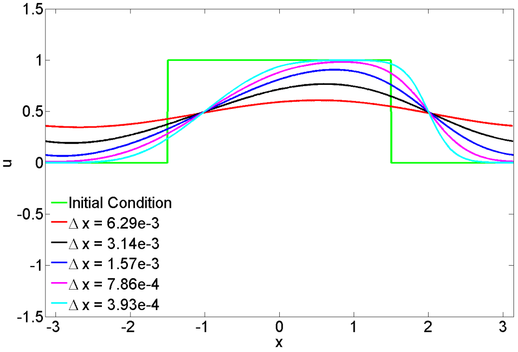


The integral of u is conserved.

Tophat IC: mesh refinement effects are significant.

Results for Parks micromodulus, fixed $\varepsilon/L \approx 1.59 \times 10^{-2}$, varying Δx.

Init. Cond.
$\varepsilon/\Delta x = 8$
$\varepsilon/\Delta x = 16$
$\varepsilon/\Delta x = 32$
$\varepsilon/\Delta x = 64$
$\varepsilon/\Delta x = 128$

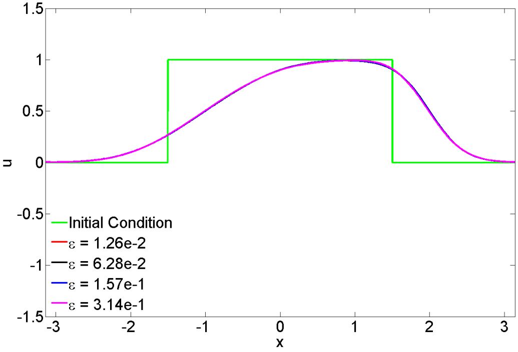


Larger $\Delta x \Rightarrow$ the scheme as greater numerical dissipation.

Tophat IC: horizon refinement effects are less obvious.

Results for Parks micromodulus, fixed $\Delta x/L = 2 \times 10^{-4}$, varying ε.

Init. Cond.
$\varepsilon/L = 4 \times 10^{-3}$
$\varepsilon/L = 2 \times 10^{-2}$
$\varepsilon/L = 5 \times 10^{-2}$
$\varepsilon/L = 1 \times 10^{-1}$

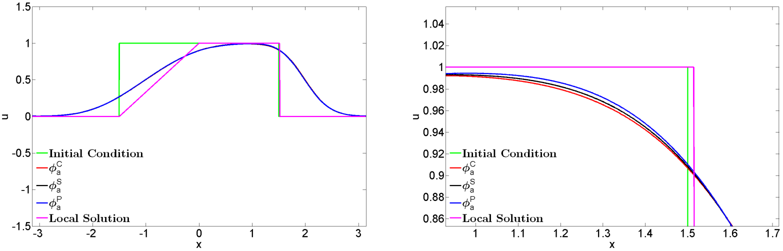


Larger horizon does not have much effect on the solution.

Tophat IC: kernel functions effects are also small.

Fix $\Delta x \approx 6.28 \times 10^{-4}$ with $\varepsilon \approx 3.14 \times 10^{-1}$ and vary ϕ_a.

Full domain *Close-up*



Init. Cond. $\phi_a^{C^\infty}$ ϕ_a^S ϕ_a^P *Local Burgers Sol'n*

$\Delta x/L = 2\times 10^{-4}$, $\varepsilon/L = 0.1 \Rightarrow$ varying ϕ_a has negligible effect.

Our hypotheses were not all substantiated.

- For $\varepsilon/L \ll 1$, the effect of nonlocality should be decreased.

  $\rightarrow$ *Decreasing $\varepsilon/L \Rightarrow$ little difference in solutions.*

- For $\varepsilon/L \rightarrow 1^-$, differences between the various ϕ_a should be highlighted.

  $\rightarrow$ *$\varepsilon/L = 0.1 \Rightarrow$ different ϕ_a had little effect.*

- For $\varepsilon/\Delta x \gg 1$, the computed solution may be more faithful to the continuum solution.

  $\rightarrow$ *Small $\Delta x \Rightarrow$ less dissipation.*

- For $\varepsilon/\Delta x \rightarrow 1^+$, the computed result may not reflect the continuum solution.

  $\rightarrow$ *Large $\Delta x \Rightarrow$ more dissipation.*

A summary of this presentation:

- *Why Nonlocal Advection?*

  This is the first step toward the marriage of nonlinear advection with peridynamics.

- *Are There Other Approaches to Nonlocal Advection?*

  Yes — but not (to our knowledge) from the perspective of peridynamics.

- *A New Approach to Nonlocal Advection*

  The preliminary results presented for our peridynamics-inspired approach appear plausible, both analytically and computationally.

There remain many open questions. . .

- Can we employ a more sophisticated numerical scheme?
- Can we extend this to nonlocal *viscous* Burgers?
- How does this *nonlocally regularized* equation relate to its local analogue?
- How does one *verify* computed results? *Exact solutions.*
- Can one conduct a modified equation analysis?
- What is the nonlocal analogue of entropy solutions? Should we concern ourselves with this issue?
- How does one extend these concepts to systems?
- How does one extend these concepts to more general material response (i.e., more general flux function)?
- What is the nonlocal analogue of singularity structure in the complex plane?

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