Multiscale Dynamics for Heterogeneous Peridynamic Media

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Joint work with

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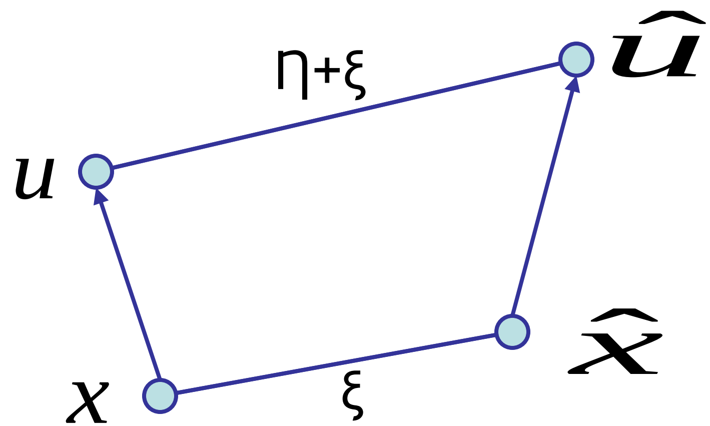
Background & Formulation:

Bond based Peridynamics, S. Silling JMPS 2000



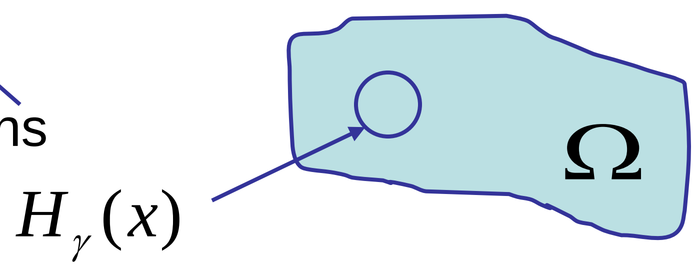
$$\xi = \hat{x} - x$$

$$\eta = \hat{u} - u$$

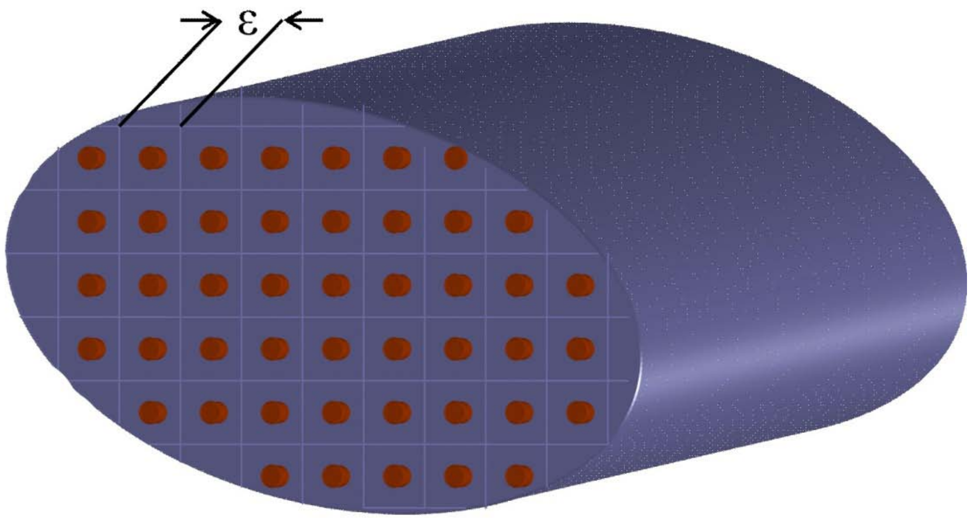
$$\hat{\rho}\, \partial_t^2 u(x,t) = \int_{H_\gamma(x)\cap\Omega} f(u(\hat{x},t) - u(x,t), \hat{x} - x, x)\, d\hat{x} + b(u,x,t),$$

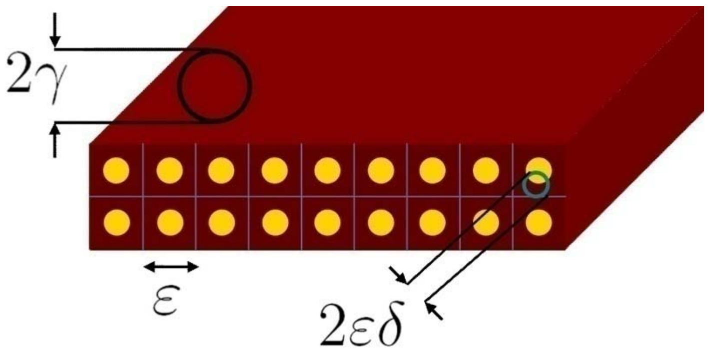
for $(x,t) \in \Omega \times (0,T)$,

Traction free boundary conditions



Heterogeneous media





$$\rho_\varepsilon(x)\partial_t^2 u^\varepsilon(x,t) = \int_{H_\gamma(x)\cap\Omega} f_{\mathrm{long}}(u^\varepsilon(\hat{x},t) - u^\varepsilon(x,t),\xi)\,d\hat{x}$$

$$+\int_{H_{\varepsilon\delta}(x)\cap\Omega} f^\varepsilon_{\mathrm{short}}((u^\varepsilon(\hat{x},t) - u^\varepsilon(x,t)),\xi,x)\,d\hat{x}$$

$$+\,b^\varepsilon(x,t)\,,\ \text{for } x \text{ in } \Omega.$$

Initial Conditions

$$u^\varepsilon(x,0) = u_0^\varepsilon(x)$$
$$\partial_t u^\varepsilon(x,0) = v_0^\varepsilon(x).$$

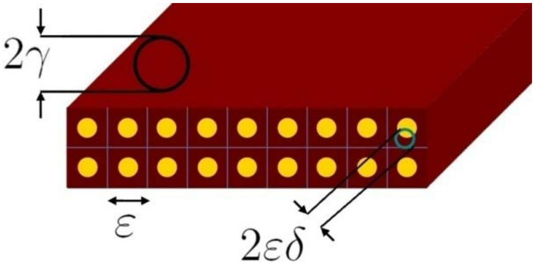
The Bond Force & Density Fluctuations

$$\chi_{\mathrm{f}}(y) = \begin{cases} 1, \text{ y is in the inclusion phase,} \\ 0, \text{ otherwise,} \end{cases}$$

and χ_{m} is given by

$$\chi_{\mathrm{m}}(y) = 1 - \chi_{\mathrm{f}}(y).$$

$$\rho(y) = \chi_{\mathrm{f}}(y)\rho_{\mathrm{f}} + \chi_{\mathrm{m}}(y)\rho_{\mathrm{m}}.$$



Local density fluctuation $\rho_{\varepsilon}(x) = \rho(x/\varepsilon)$

Local bond force strength for short range bond forces

$$\alpha_{\delta}(y,\hat{y}) = \begin{cases} C_{\mathrm{f}}, \text{ if } y \text{ and } \hat{y} \text{ are in the same inclusion and } |y-\hat{y}| < \delta \\ C_{\mathrm{m}}, \text{ if } y \text{ and } \hat{y} \text{ are in the matrix phase and } |y-\hat{y}| < \delta \\ C_{\mathrm{i}}, \text{ if } y \text{ and } \hat{y} \text{ are separated by an interface and } |y-\hat{y}| < \delta \\ 0, \text{ if } |y-\hat{y}| \geq \delta. \end{cases}$$

The Bond Forces

Mathematically we can express the local bond strengths

$$\alpha_\delta(y,\hat{y}) = \chi_\delta(y-\hat{y})\alpha(y,\hat{y}), \tag{1.5}$$

where $\chi_\delta(z) = 1$ for $|z| < \delta$ and $\chi_\delta(z) = 0$ for $|z| \geq \delta$ and α is given by

$$\alpha(y,\hat{y}) = C_{\mathrm{f}}\,\chi_{\mathrm{f}}(y)\chi_{\mathrm{f}}(\hat{y}) + C_{\mathrm{m}}\,\chi_{\mathrm{m}}(y)\chi_{\mathrm{m}}(\hat{y}) + C_{\mathrm{i}}\left(\chi_{\mathrm{f}}(y)\chi_{\mathrm{m}}(\hat{y}) + \chi_{\mathrm{m}}(y)\chi_{\mathrm{f}}(\hat{y})\right). \tag{1.6}$$

The short-range peridynamic force defined on Ω is given by

$$f^{\varepsilon}_{\mathrm{short}}(\eta,\xi,x) = \frac{1}{\varepsilon^2}\,\alpha_{\varepsilon\delta}\left(\frac{x}{\varepsilon},\frac{x+\xi}{\varepsilon}\right)\,\frac{\xi\otimes\xi}{|\xi|^3}\eta.$$

Linearized
Bond stretch model

The long range peridynamic force

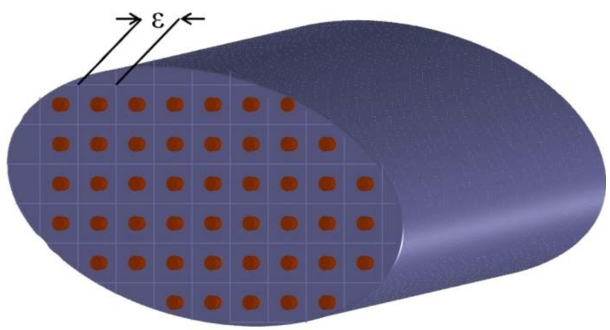
$$f_{\mathrm{long}}(\eta,\xi) = \begin{cases} \lambda\dfrac{\xi\otimes\xi}{|\xi|^3}\eta, & |\xi| \leq \gamma \\ 0, & \text{otherwise.} \end{cases}$$

A well posed initial boundary value problem

$$\rho_\varepsilon(x)\partial_t^2 u^\varepsilon(x,t) = \int_{H_\gamma(x)\cap\Omega} f_{\text{long}}(u^\varepsilon(\hat{x},t) - u^\varepsilon(x,t), \xi)\, d\hat{x}$$

$$+ \int_{H_{\varepsilon\delta}(x)\cap\Omega} f^\varepsilon_{\text{short}}((u^\varepsilon(\hat{x},t) - u^\varepsilon(x,t)), \xi, x)\, d\hat{x}$$

$$+ b^\varepsilon(x,t), \text{ for } x \text{ in } \Omega.$$



Traction free boundary conditions

Initial Conditions

$$u^\varepsilon(x,0) = u_0^\varepsilon(x)$$
$$\partial_t u^\varepsilon(x,0) = v_0^\varepsilon(x).$$

For initial data uniformly bounded in $L^p(\Omega)^3$

and body forces uniformly bounded in $C([0,T];L^p(\Omega)^3)$

An application of the theory of semigroups shows that the solution exists and lies in $C^2([0,T];L^p(\Omega)^3)$ for p greater than or equal to 1.

Strong Approximations

We now introduce strong approximations to
$u^{\varepsilon}(x,t) \in C^2([0,T]; L^p(\Omega)^3), \, p \geq 3/2$
Here the lower bound 3/2 arises from the cubic nature of the bond force.

For body forces and initial conditions are continuous at the coarse length scale but possess discontinuous oscillations over the finer length scales

The strong approximations are of the form $u(x, x/\varepsilon, t)$

Where the function $u(x,y,t)$

is unit periodic in the y variable and

$$\| u^{\varepsilon}(x,t) - u(x, x/\varepsilon, t) \|_{L^p(\Omega)^3} \to 0$$

For 0<t<T as ε→0.

Initial data and body forces associated with this strong approximation

The strong approximations are of the form

$$u(x, x/\varepsilon, t) \in C^2([0,T]; L^p(\Omega)^3)$$

When the initial data is of the form

$$u^\varepsilon(x,0) = u_0(x, x/\varepsilon)$$

$$\partial_t u^\varepsilon(x,0) = v_0(x, x/\varepsilon)$$

$$b^\varepsilon(x,t) = b(x, x/\varepsilon, t)$$

Where the functions $u_0(x, y)$ $v_0(x, y)$
are periodic in the y variable with period Y and are in the space
$L^p_{per}(Y; C(\overline{\Omega})^3)$ given by functions $\psi(x, y)$
L^p integrable with respect to y, with values in $C(\overline{\Omega})^3$

and $b(x, y, t) \in C^1([0,T]; L^p_{per}(Y; C(\overline{\Omega})^3))$

Strong approximation from two-scale limits

2-scale convergence: Weak convergence over a special space of test functions

$$\begin{aligned}
\mathcal{K} &= \{\psi \in C_c^\infty(\mathbb{R}^3 \times Y),\ \psi(x,y) \text{ is } Y\text{-periodic in } y\},\\
\mathcal{J} &= \{\psi \in C_c^\infty(\mathbb{R}^3 \times Y \times \mathbb{R}^+),\ \psi(x,y,t) \text{ is } Y\text{-periodic in } y\},\\
\mathcal{L}_p &= \{w \in C([0,T];\, L^p_{\mathrm{per}}(Y; C(\overline{\Omega})^3)\},\\
\mathcal{Q}_p &= \{w \in C^2([0,T];\, L^p_{\mathrm{per}}(Y; C(\overline{\Omega})^3)\}.
\end{aligned}$$

Definition 3 (Two-scale convergence [21, 1]) *A sequence (v^ε) of functions in $L^p(\Omega)$, is said to two-scale converge to a limit $v \in L^p(\Omega \times Y)$ if, as $\varepsilon \to 0$*

$$\int_\Omega v^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_{\Omega\times Y} v(x,y)\psi(x,y)\, dxdy \tag{3.2}$$

for all $\psi \in L^{p'}(\Omega; C_{per}(Y))$. We will often use $v^\varepsilon \xrightarrow{2} v$ to denote that (v^ε) two-scale converges to v.

Definition 4 *A bounded sequence (v^ε) of functions in $L^p(\Omega \times (0,T))$, is said to two-scale converge to a limit $v \in L^p(\Omega \times Y \times (0,T))$ if, as $\varepsilon \to 0$*

$$\int_{\Omega\times(0,T)} v^\varepsilon(x,t)\psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt \to \int_{\Omega\times Y\times(0,T)} v(x,y,t)\psi(x,y,t)\, dxdydt \tag{3.3}$$

for all $\psi \in \mathcal{J}$.

Two-scale convergence & compactness

Theorem 5 *Let* (v^ε) *be a bounded sequence in* $L^p(\Omega)$. *Then there exists a subsequence and a function* $v \in L^p(\Omega \times Y)$ *such that the subsequence two-scale converges to* v.

A similar two-scale compactness holds for time dependent problems and is stated in the following theorem.

Theorem 6 *Let* (v^ε) *be a bounded sequence in* $L^p(\Omega \times (0,T))^3$. *Then there exists a subsequence and a function* $v \in L^p(\Omega \times Y \times (0,T))^3$ *such that the subsequence two-scale converges to* v.

Properties of two-scale convergence for the class of test functions

Proposition 9 *If $\psi(x,y)$ belongs to $\mathcal{K}$ or $L^p_{per}(Y;C(\overline{\Omega})^3)$ then $\psi(x,\frac{x}{\varepsilon})$ two-scale converges to $\psi(x,y)$ and*

$$\lim_{\varepsilon\to 0}\|\psi(x,\frac{x}{\varepsilon})\|^p_{L^p(\Omega)} = \int_{\Omega\times Y}|\psi(x,y)|^p\,dx\,dy. \tag{3.4}$$

Similarly if $\psi(x,y,t)$ belongs to $\mathcal{J}$ or $\mathcal{L}_p$ then $\psi(x,\frac{x}{\varepsilon},t)$ two-scale converges to $\psi(x,y,t)$ and

$$\lim_{\varepsilon\to 0}\|\psi(x,\frac{x}{\varepsilon},t)\|^p_{L^p(\Omega\times(0,T))^3} = \int_{\Omega\times Y\times(0,T)}|\psi(x,y,t)|^p\,dx\,dy\,dt. \tag{3.6}$$

Two-scale convergence and relation to strong & weak convergence

Proposition 7 *Let (v^ε) be a bounded sequence in $L^p(\Omega \times (0,T))^3$ that two-scale converges to $v \in L^p(\Omega \times Y \times (0,T))^3$. Then as $\varepsilon \to 0$*

$$v^\varepsilon \to \int_Y v(x,y,t)\,dy \quad \textit{weakly in } L^p(\Omega \times (0,T))^3.$$

Proposition 8 *If $v^\varepsilon(x)$ converges to $v(x)$ in $L^p(\Omega)^3$ then its two-scale limit is v.*

Two-scale limit equation for dynamics on product space

Weak form of peridynamics using suitable test functions in J

$$\int_{\Omega\times\mathbb{R}^+} u^\varepsilon(x,t)\cdot\partial_t^2\psi\left(x,\frac{x}{\varepsilon},t\right)\rho(\frac{x}{\varepsilon})\,dxdt - \int_\Omega \partial_t u^\varepsilon(x,0)\cdot\psi\left(x,\frac{x}{\varepsilon},0\right)\rho(\frac{x}{\varepsilon})\,dx$$
$$+\int_\Omega u^\varepsilon(x,0)\cdot\partial_t\psi\left(x,\frac{x}{\varepsilon},0\right)\rho(\frac{x}{\varepsilon})\,dx$$
$$=\int_{\Omega\times\mathbb{R}^+}\left((K_L+K_S^\varepsilon)u^\varepsilon(x,t)+b\left(x,\frac{x}{\varepsilon},t\right)\right)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt$$

$$K_L = K_{L,1} - K_{L,2} \text{ and } K_S^\varepsilon = K_{S,1}^\varepsilon - K_{S,2}^\varepsilon$$

Where

$$K_{L,1}v(x) = \int_{H_\gamma(x)\cap\Omega}\lambda\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,v(\hat{x})\,d\hat{x}, \tag{3.8}$$

$$K_{L,2}v(x) = \int_{H_\gamma(x)\cap\Omega}\lambda\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,d\hat{x}\,v(x), \tag{3.9}$$

$$K_{S,1}^\varepsilon v(x) = \int_{H_{\varepsilon\delta}(x)\cap\Omega}\frac{1}{\varepsilon^2}\,\alpha\left(\frac{x}{\varepsilon},\frac{\hat{x}}{\varepsilon}\right)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,v(\hat{x})\,d\hat{x}, \tag{3.10}$$

$$K_{S,2}^\varepsilon v(x) = \int_{H_{\varepsilon\delta}(x)\cap\Omega}\frac{1}{\varepsilon^2}\,\alpha\left(\frac{x}{\varepsilon},\frac{\hat{x}}{\varepsilon}\right)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,d\hat{x}\,v(x). \tag{3.11}$$

Fine scale limit

The following sequences two scale converge to

$$u^{\varepsilon}(x,t) \to u(x,y,t)$$

Where the initial data and body forces two scale converge to

$$u^{\varepsilon}(x,0) \to u_0(x,y)$$

$$\partial_t u^{\varepsilon}(x,0) \to v_0(x,y)$$

$$b^{\varepsilon}(x,t) \to b(x,y,t)$$

Two-scale dynamics

$$\int_{\Omega\times Y\times\mathbb{R}+} u(x,y,t)\cdot\partial_t^2\psi(x,y,t)\rho(y)\,dxdydt - \int_{\Omega\times Y} v_0(x,y)\cdot\psi(x,y,0)\rho(y)\,dxdy$$
$$+\int_{\Omega\times Y} u_0(x,y)\cdot\partial_t\psi(x,y,0)\rho(y)\,dxdy$$
$$=\int_{\Omega\times Y\times\mathbb{R}+}((B_L+B_S)u(x,y,t)+b(x,y,t))\cdot\psi(x,y,t)\,dxdydt \tag{3.15}$$

Where

$$B_L w(x,y) = \int_{H_\gamma(x)\cap\Omega} \lambda\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(\int_Y w(\hat{x},y')\,dy' - w(x,y)\right)\,d\hat{x},$$
$$B_S w(x,y) = \int_{H_\delta(y)} \alpha(y,\hat{y})\frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|\hat{y}-y|^3}\left(w(x,\hat{y})-w(x,y)\right)\,d\hat{y}.$$

Two-scale dynamics

Theorem 11 *Let (u^ε) be the sequence of solutions of (1.9)-(1.11) with initial data $u_0^\varepsilon = u_0(x, \frac{x}{\varepsilon})$, $v_0 = v_0(x, \frac{x}{\varepsilon})$ and body force $b^\varepsilon(x, \frac{x}{\varepsilon}, t)$ with u_0 and v_0 in $L^p_{per}(Y; C(\overline{\Omega})^3)$ and $b \in \mathcal{L}_p$. Then*
$u^\varepsilon \xrightarrow{2} u$ and the periodic extension of $u(x,y,t)$ in the y variable from Y to $\mathbb{R}^3$ also denoted by u belongs to $\mathcal{Q}_p$, with $\frac{3}{2} < p < \infty$, and is the unique solution of

$$\begin{aligned}\rho(y)\partial_t^2 u(x,y,t) = &\int_{H_\gamma(x)\cap\Omega} \lambda \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(\int_Y u(\hat{x},y',t)\,dy' - u(x,y,t)\right) d\hat{x}\\ &+ \int_{H_\delta(y)} \alpha(y,\hat{y})\frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|\hat{y}-y|^3}\left(u(x,\hat{y},t) - u(x,y,t)\right) d\hat{y}\\ &+ b(x,y,t),\end{aligned} \tag{3.19}$$

supplemented with initial conditions

$$u(x,y,0) = u_0(x,y), \tag{3.20}$$

$$\partial_t u(x,y,0) = v_0(x,y). \tag{3.21}$$

Strong approximation

Theorem 12 *Let $u(x,y,t)$ be the solution of the two-scale problem given in Theorem 11 then*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon(x,t) - u(x, \frac{x}{\varepsilon}, t)\|_{L^p(\Omega)^3} = 0, \tag{3.67}$$

for every t in $[0,T]$ and $\frac{3}{2} < p < \infty$.

Summary

Compute the two-scale dynamics to get strong approximation

$$\rho(y)\partial_t^2 u(x,y,t) = \int_{H_\gamma(x)\cap\Omega} \lambda \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(\int_Y u(\hat{x},y',t)\,dy' - u(x,y,t)\right) d\hat{x}$$

$$+\int_{H_\delta(y)} \alpha(y,\hat{y})\frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|\hat{y}-y|^3}\left(u(x,\hat{y},t)-u(x,y,t)\right) d\hat{y}$$

$$+b(x,y,t), \tag{3.19}$$

supplemented with initial conditions

$$u(x,y,0) = u_0(x,y), \tag{3.20}$$

$$\partial_t u(x,y,0) = v_0(x,y). \tag{3.21}$$

$$\lim_{\varepsilon\to 0}\|u^\varepsilon(x,t) - u(x,\frac{x}{\varepsilon},t)\|_{L^p(\Omega)^3} = 0, \tag{3.67}$$

for every t in $[0,T]$ and $\frac{3}{2} < p < \infty$.

Future work – get convergence rates

Macroscopic Dynamics - Homogenization

The strong approximation $u(x, \frac{x}{\varepsilon}, t)$ admits a natural decomposition into a continuous macroscopic component and a possibly discontinuous fluctuating component. The macroscopic component $u^H(x,t)$ is obtained by projecting out the spatial fluctuations and the corrector $r(x, \frac{x}{\varepsilon}, t)$ containing the possibly discontinuous fluctuations is given by the remainder, i.e.,

$$u(x, \frac{x}{\varepsilon}, t) = u^H(x,t) + r(x, \frac{x}{\varepsilon}, t), \tag{4.1}$$

where

$$u^H(x,t) = \langle u \rangle \equiv \int_Y u(x,y,t)\, dy \tag{4.2}$$

and

$$r(x, \frac{x}{\varepsilon}, t) = u(x, \frac{x}{\varepsilon}, t) - u^H(x,t). \tag{4.3}$$

Macroscopic Dynamics - Homogenization

The weak limit u^H tracks the average dynamics

$$\lim_{\varepsilon\to 0}\frac{1}{|V|}\int_V u^\varepsilon(x,t)\,dx = \lim_{\varepsilon\to 0}\frac{1}{|V|}\int_V u(x,\frac{x}{\varepsilon},t)\,dx = \frac{1}{|V|}\int_V u^H(x,t)\,dx, \quad (4.4)$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{|V|}\int_V r(x,\frac{x}{\varepsilon},t)\,dx = 0. \quad (4.5)$$

We now develop an explicit evolution equation for u^H and (following convention) we call u^H

the Homogenized deformation

Coupled Macroscopic & Microscopic Dynamics

We use the two-scale equation to write the coupled evolution equation for u^H and r

Theorem 14

$$\ddot{u}^H(t) = \langle \rho^{-1} \rangle K_L u^H(t) + \langle \rho^{-1} B_S r \rangle(t) - K \langle \rho^{-1} r \rangle(t) + \langle \rho^{-1} b \rangle(t), \quad (4.7)$$

$$\ddot{r}(t) = \left(\rho^{-1} - \langle \rho^{-1} \rangle\right) K_L u^H(t) + \left(\rho^{-1} B_S r(t) - \langle \rho^{-1} B_S r \rangle(t)\right)$$
$$- K \left(\rho^{-1} r(t) - \langle \rho^1 r \rangle(t)\right) + \left(\rho^{-1} b(t) - \langle \rho^{-1} b \rangle(t)\right), \quad (4.8)$$

with initial conditions $u^H(0) = \langle u_0 \rangle$, $\dot{u}^H(0) = \langle v_0 \rangle$, $r(0) = u_0 - \langle u_0 \rangle$, *and* $\dot{r}(0) = v_0 - \langle v_0 \rangle$.

$$\langle v \rangle(t) \equiv \int_Y v(x, y, t)\, dy$$

$$K = \lambda \int_{H_\gamma(0)} \frac{\xi \otimes \xi}{|\xi|^3}\, d\xi$$

Coupled Macroscopic & Microscopic Dynamics

We obtain the evolution equation for u^H by eliminating r from the coupled dynamical system

Let: $$\mathcal{C}r(t) = \rho^{-1}B_S r(t) - \langle \rho^{-1}B_S r\rangle(t) - K\left(\rho^{-1}r(t) - \langle \rho^{-1}r\rangle(t)\right)$$

and $$\ddot{r}(t) = \mathcal{C}r(t) + \left(\rho^{-1} - \langle\rho^{-1}\rangle\right)K_L u^H(t) + \rho^{-1}b(t) - \langle\rho^{-1}b\rangle(t)$$

Since this equation is linear we write r=w+v where

$$\ddot{v}(t) = \mathcal{C}v(t) + \left(\rho^{-1} - \langle\rho^{-1}\rangle\right)K_L u^H(t),$$ with homogenious IC's

$$\ddot{w}(t) = \mathcal{C}w(t) + \rho^{-1}b(t) - \langle\rho^{-1}b\rangle(t),$$

with $w(0) = \hat{u}_0 = u_0 - \langle u_0\rangle$ and $\dot{w}(0) = \hat{v}_0 = v_0 - \langle v_0\rangle$

Towards a Homogenized Dynamics

We use semi-groups to get the explicit representations for v and w

$$v(t) = \left(\sqrt{\mathcal{C}}\right)^{-1} \int_0^t \sinh\left((t-\tau)\sqrt{\mathcal{C}}\right)\left(\rho^{-1} - \langle\rho^{-1}\rangle\right) K_L u^H(\tau)\, d\tau \quad (4.14)$$

$$w(t) = \cosh t\sqrt{\mathcal{C}}\hat{u}_0 + \left(\sqrt{\mathcal{C}}\right)^{-1} \sinh t\sqrt{\mathcal{C}}\hat{v}_0$$

$$+ \left(\sqrt{\mathcal{C}}\right)^{-1} \int_0^t \sinh\left((t-\tau)\sqrt{\mathcal{C}}\right)\left(\rho^{-1}b(\tau) - \langle\rho^{-1}b\rangle(\tau)\right) d\tau. \quad (4.15)$$

We obtain the evolution equation for u^H by eliminating v and w from the coupled dynamical system

Homogenized Dynamics with Memory

The evolution equation for u^H is given by

Theorem 15 *The homogenized deformation $u^H(t)$ is the solution of the integro-differential equation in space and time given by*

$$\langle\rho^{-1}\rangle^{-1}\ddot{u}^H(t) = K_L u^H(t) + \langle\rho^{-1}\rangle^{-1}\mathcal{K}\left(\sqrt{\mathcal{C}}\right)^{-1}\int_0^t \sinh\left((t-\tau)\sqrt{\mathcal{C}}\right)\left(\rho^{-1} - \langle\rho^{-1}\rangle\right)K_L u^H(\tau)\,d\tau + \langle\rho^{-1}\rangle^{-1}\left(\mathcal{K}w(t) + \langle\rho^{-1}b\rangle(t)\right), \quad (4.17)$$

with the initial conditions $u^H(0) = \langle u_0\rangle$ and $\dot{u}^H(0) = \langle v_0\rangle$. The force generated by the homogenized deformation $f^H(t) = f^H(\cdot,t)$ is given by the history dependent constitutive law

$$f^H(t) = K_L u^H(t) + \langle\rho^{-1}\rangle^{-1}\mathcal{K}\left(\sqrt{\mathcal{C}}\right)^{-1}\int_0^t \sinh\left((t-\tau)\sqrt{\mathcal{C}}\right)\left(\rho^{-1} - \langle\rho^{-1}\rangle\right)K_L u^H(\tau)\,d\tau. \quad (4.18)$$

Here $\quad \mathcal{K} = \langle\rho^{-1}B_S r\rangle(t) - K\langle\rho^{-1}b\rangle(t),$

Summary - Overview

1. The solution sequence two scale converges to

$$u^{\varepsilon}(x,t) \to u(x,y,t)$$

Where u(x,y,t) is the solution of a two scale dynamical system in the spatial variables (x,y)

2. Compute using the two scale dynamics to obtain

$$u(x,y,t)$$

3. The strong approximation to $u^{\varepsilon}(x,t)$ is given by the rescaling u(x,x/ε,t)

Summary – Overview Continued

1. Write

$$u(x,y,t) = u^H(x,t) + r(x,y,t)$$

Where u^H is used to characterize the dynamics of volume averages

$$\lim_{\varepsilon\to 0}\frac{1}{|V|}\int_V u^\varepsilon(x,t)\,dx = \lim_{\varepsilon\to 0}\frac{1}{|V|}\int_V u(x,\frac{x}{\varepsilon},t)\,dx = \frac{1}{|V|}\int_V u^H(x,t)\,dx, \quad (4.4)$$

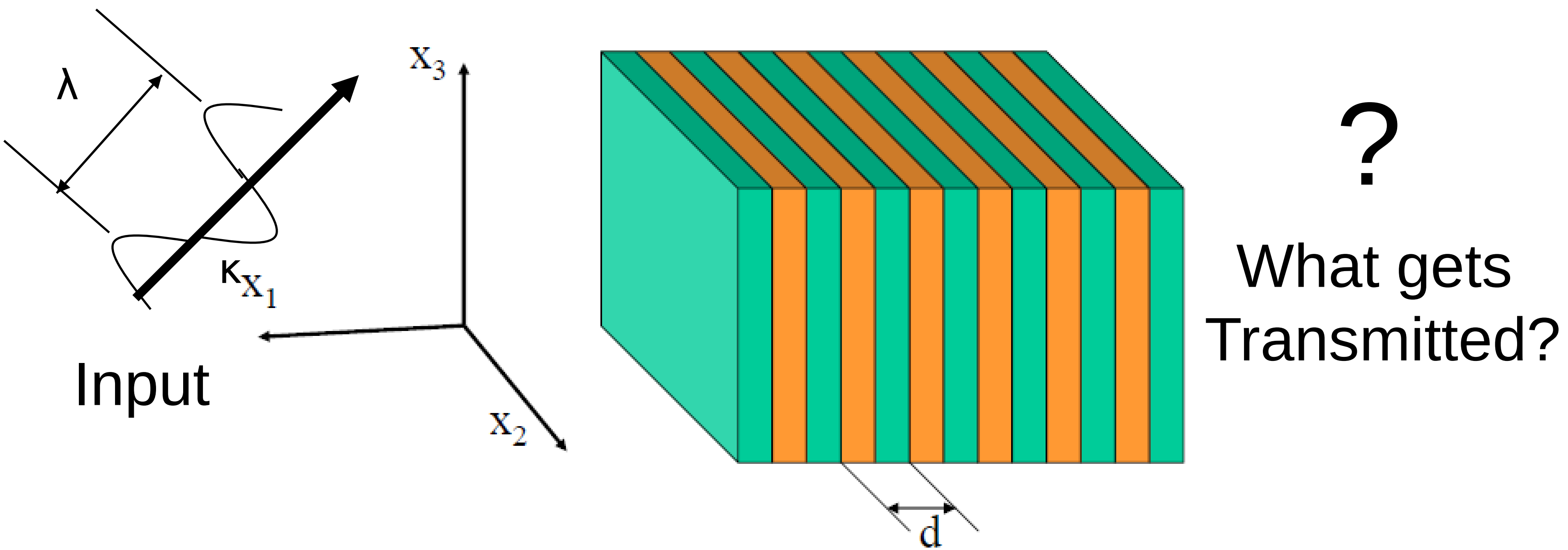
and

$$\lim_{\varepsilon\to 0}\frac{1}{|V|}\int_V r(x,\frac{x}{\varepsilon},t)\,dx = 0. \quad (4.5)$$

2. The average dynamics is history dependent due to microscopic density fluctuations

Follow up and future work

1. Compute dynamics for layered and periodic media



2. Adjust bond strength-scaling relations, investigate/ identify limit equations comparisons to heterogeneous materials modeled by classical linear elasticity