ALGEBRAIC K-THEORY OF SPACES, A MANIFOLD APPROACH.

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The basic fact relating the algebraic K-theory of spaces to concordance theory, and hence the geometry of manifolds, is the existence of a natural transformation $A(X) \to Wh^{PL}(X)$ whose homotopy fibre is a homology theory. The purpose of the present paper is to show how this may be described entirely in terms of spaces of manifolds.

Such a manifold description is of interest for several reasons. Most importantly one can now define a map $A(X) \to Wh^{DIFF}(X)$ and this map again turns out to have the property that its homotopy fibre is a homology theory. By combining this fact with known results one obtains that there is a double splitting

$$A(X) \quad \simeq \quad \Omega^{\infty}S^{\infty}(X_+) \times Wh^{DIFF}(X) \times \mu(X) \ .$$

In other words it turns out that $A(X)$ is the product of two well known spaces (well known, that is, from the point of view of their definitions) together with a somewhat mysterious third one. This $\mu(X)$ is a homology theory. Its nature will be briefly discussed in section 2 (after theorem 2).

As another application of the manifold models one can see at once that the composite map $BO \to BG \to A(*) \to Wh^{DIFF}(*)$ is trivial, up to homotopy. This has numerical implications (section 3). In fact the map is canonically trivial, so one obtains a map, well defined up to homotopy,

$$G/O \quad \to \quad \Omega \ Wh^{DIFF}(*) \ .$$

An interesting question is whether or not this map is a rational homotopy equivalence.

As a final application of the manifold description (of which we do not pursue any details here) it should be mentioned that each of the spaces in the basic fibration admits an obvious involution (up to homotopy) and the involutions are compatible with the maps. (This is of interest for example because after localization away from 2 such an involution gives a splitting into eigenspaces, and the eigenspaces of the concordance space have geometric meaning, cf. e.g. [10]).

Here is a brief indication of how the fibration in terms of manifold models is obtained.

Unless explicitly stated otherwise, our arguments with manifolds are supposed to be independent of the category of definition and are thus consistently to be interpreted as being concerned with CAT manifolds where CAT can mean either of TOP, PL, or DIFF (there is an occasional technical point about corners in the DIFF case; such matters are dealt with in the appended section 6). By elementary means, and very explicitly, we will set up a certain diagram in section 1. The main result is then theorem 1 which asserts that this diagram represents a homotopy fibration $\Omega Wh^{CAT}(X) \to h^{CAT}(X) \to A(X)$ and that $h^{CAT}(X)$ is a homology theory. Most of the assertions of theorem 1 can be proved relatively easily, and at any rate complete proofs are meant to be given in this paper (in section 5). The one exception is the assertion that $h^{CAT}(X)$ is a homology theory.

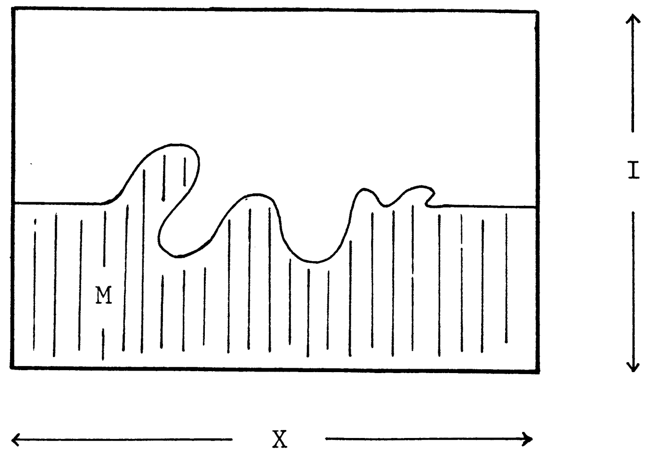
Briefly, this is handled in the following way. By using results from triangulation theory, resp. smoothing theory, one reduces first of all to proving the assertion in the PL case only. $h^{PL}(X)$ now may be re-expressed, up to homotopy, in terms of polyhedra rather than PL manifolds. This is the kind of translation achieved by the *parametrized h-cobordism theorem* in the sense of Hatcher [9]; actually we do not refer to Hatcher's theorem but to a version of it (re-)proved in [23]. After such translation it is then possible to apply suitable parts of the machinery of [24].

Here is a list of section headings.

1. The manifold models.

As pointed out in the introduction already, everything in this section is to be interpreted in terms of CAT manifolds where CAT means either one of DIFF, PL, or TOP. In the DIFF case some of the constructions will create corners. But such corners may be ignored entirely if one adopts the reformulation of the DIFF case described in the appended section 6.

Let X be a compact manifold, with boundary ∂X. Let I be an interval, say $I = [a,b]$. We are interested in submanifolds M of $X \times I$ as in the following picture.



Let N denote the closure of the complement of M, and $F = M \cap N$. As the picture suggests, we want F to be disjoint from the bottom $X \times a$ and top $X \times b$ and we want it to be standard near $\partial X \times I$ in the sense that there exists a neighborhood of the latter whose intersection with F is equal to its intersection with $X \times t$, for some $t \in I$.

We refer to such a triple (M,F,N) as a *partition*; F will be called its *frontier*.

The partitions may be regarded as the 0-simplices of a simplicial set $\mathcal{P}(X)$. A k-simplex in this simplicial set $\mathcal{P}(X)$ is, by definition, a locally trivial family of partitions parametrized by the simplex Δ^k (note that the number t is not required to be constant in such a family; note also that the local triviality is to be understood in the CAT sense here).

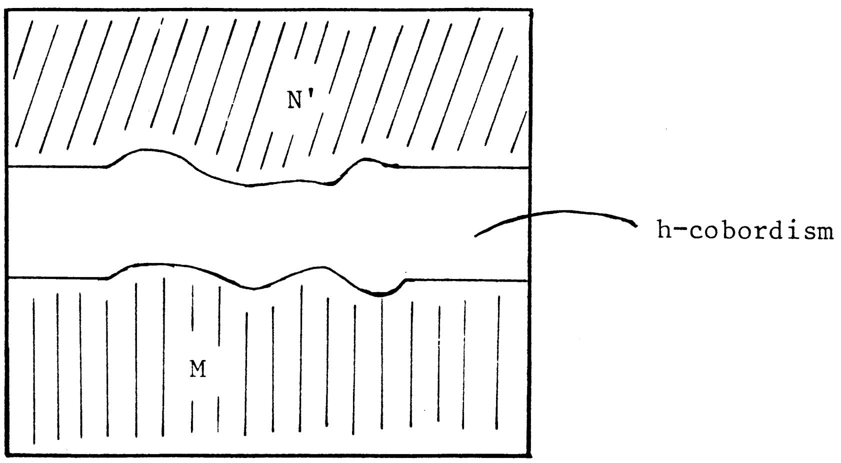
In particular we have the *h-cobordism space* $H(X)$. It is the simplicial subset of $P(X)$ of those (M,F,N) where M is an h-cobordism (rel. boundary) between $X\times a$ and F . We note that *the h-cobordism space* $H(X)$ *is a classifying space for the concordance space* $C(X)$. In fact, by definition $C(X)$ is the simplicial group of those automorphisms of $X\times[0,1]$ which are the identity near $X\times 0$ and $\partial X\times I$. It will suffice to know that the connected component of $H(X)$ containing the trivial h-cobordism $X\times[a,a']$, say, may be identified to the space of orbits of a free action of $C(X)$ on some contractible space. Such a contractible space is given by a *space of collars* of $X\times I$, namely by the simplicial set of embeddings $X\times[0,1] \to X\times[a,b]$ which take $X\times 0$ to $X\times a$ by the identity map on X , and which are standard near $\partial X\times[0,1]$ in a suitable sense.

The only other partitions that we will be eventually interested in are those (M,F,N) where each of M and N looks like a handlebody of a particular type (up to an h-cobordism perhaps). The precise definition will be given in a moment. It is convenient to discuss another general notion first.

The inclusion relation among the M's allows us to consider $P(X)$ as a simplicial partially ordered set, and hence as a simplicial category. We are interested in the simplicial subcategory $h'P(X)$ defined by the condition that the inclusion map $M \to M'$ should be a homotopy equivalence. In fact, we are more interested in a slight refinement of this condition, thus obtaining a simplicial subcategory $hP(X)$ of $h'P(X)$. The refined condition is that each of the two inclusion maps

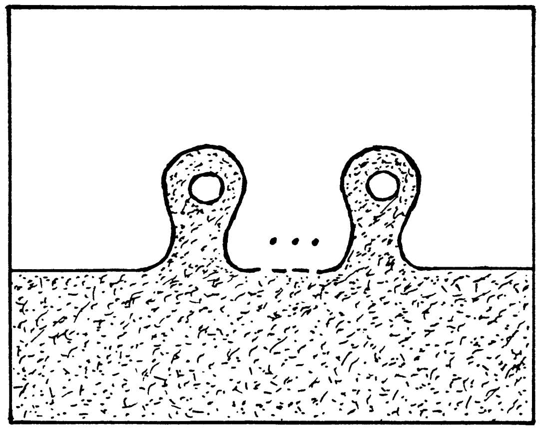
$$F \longrightarrow M' - (M-F) \longleftarrow F'$$

should be a homotopy equivalence. In the special case of general position, that is, where F and F' are disjoint, this amounts to asking that F and F' should cobound an h-cobordism.



Definition $hP_k^m(X)$ is the connected component of $hP(X)$ containing the particular (M,F,N) with

$$M = X \times [a,a'] \cup k \text{ trivial } m\text{-handles.}$$



$P_k^m(X)$ is the simplicial set of objects of $hP_k^m(X)$.

Note that, dually, $hP_k^m(X)$ could also be characterized as the connected component containing the particular (M',F',N') with

$$N' = X \times [b,b'] \cup k \text{ trivial } n\text{-handles,}$$

where $n = \dim(X)-m$.

A first formulation of our main result may now be given, somewhat loosely, as follows.

Theorem 1. (1) Approximately (i.e. up to some connectivity tending to infinity with m and $n = \dim(X)-m$) $H(X)$ may be identified to the homotopy fibre of the inclusion map $P_k^m(X) \to hP_k^m(X)$.

(2) $hP_k^m(X)$ is an approximation to $A(X)$.

(3) $P_k^m(X)$ approximates a homology theory.

The proof will be discussed later, in section 5. We derive more precise formulations of the statements of the theorem now. To do this, we must discuss some general constructions first, namely a *stabilization map* to raise the dimension of X , a *suspension map* to raise m , and a *composition law* by means of which the (suitably stabilized) spaces become infinite loop spaces.

These constructions depend on choices, we therefore discuss such choices first. We fix a subinterval

$$I' = [a',b'] \subset \mathrm{Int}(I)$$

and a submanifold

$$X' \subset \mathrm{Int}(X)$$

so that $\mathrm{Cl}(X-X')$ is a collar on the boundary ∂X. Then $\mathcal{P}'(X)$ is to be the simplicial subset of $\mathcal{P}(X)$ of the partitions (M,F,N) which satisfy

$$F \subset X\times[a',b'] \quad , \quad F \cap (X-X')\times I = (X-X') \times t \quad (\text{for some } t\ ),$$

and $h\mathcal{P}'(X)$ is the corresponding simplicial subcategory of $h\mathcal{P}(X)$. (Note that the second of these conditions involves the half-open collar $(X-X')$, not the collar $\mathrm{Cl}(X-X')$).

Next we add the condition that the number t actually assumes the minimal possible value. Thus we define $\underline{\mathcal{P}}(X)$ as the simplicial subset of $\mathcal{P}(X)$ of the partitions satisfying

$$F \subset X\times[a',b'] \quad , \quad F \cap (X-X')\times I = (X-X') \times a' .$$

It is clear that the inclusions

$$\underline{\mathcal{P}}(X) \to \mathcal{P}'(X) \to \mathcal{P}(X) \quad , \quad \text{resp.} \quad h\underline{\mathcal{P}}(X) \to h\mathcal{P}'(X) \to h\mathcal{P}(X) \ ,$$

are homotopy equivalences.

Likewise we define $\overline{\mathcal{P}}(X)$ by asking that t assumes the maximal possible value, b'. Then $\overline{\mathcal{P}}(X) \to \mathcal{P}'(X)$ and $h\overline{\mathcal{P}}(X) \to h\mathcal{P}'(X)$ are homotopy equivalences, too.

These choices are needed, first of all, in *stabilizing with respect to dimension*. Given a partition (M,F,N), the idea is to take M to its product with an interval, J say; this is then to define a partition in $(X\times J)\times I$. The idea requires modification. For our notion of partition involves conditions of standard behaviour near $\partial(X\times J)\times I$. As a consequence M should not be multiplied with J but with some subinterval, and some kind of standard choice should be made near $X\times\partial J\times I$. Also it is necessary that the latter standard choice should be compatible with the standard behaviour near $\partial X\times J\times I$. It follows, more or less, that M ought to satisfy the conditions for a partition in $\underline{\mathcal{P}}(X)$.

Thus let J be an interval, equipped with a subinterval

$$J' \subset \mathrm{Int}(J) .$$

The *lower stabilization* is defined as the map

$$\underline{\sigma} : \underline{P}(X) \to \underline{P}(X\times J)$$

which acts on a partition (M,F,N) by taking M to

$$X\times[a,a']\times J \cup_{X\times[a,a']\times J'} M\times J'$$

or what is the same thing, the union of $X\times[a,a']\times J$ and $M\times J'$ as subspaces of $X\times I\times J$. The same procedure describes the lower stabilization map $h\underline{P}(X) \to h\underline{P}(X\times J)$.

Dually, the *upper stabilization* is defined as the map

$$\overline{\sigma} : \overline{P}(X) \to \overline{P}(X\times J)$$

which acts on a partition (M,F,N) by taking N to

$$X\times[b,b']\times J \cup_{X\times[b,b']\times J'} N\times J' .$$

Equivalently $\overline{\sigma}$ takes M to

$$M\times J \cup_{M\times \mathrm{Cl}(J-J')} X\times[a,b']\times \mathrm{Cl}(J-J')$$

which is a kind of fibrewise suspension over X.

In view of the homotopy equivalence $\underline{P}(X) \simeq \overline{P}(X)$ the map $\overline{\sigma}$ can be used to define a map $\Sigma: \underline{P}(X) \to \underline{P}(X\times J)$ which is well defined up to homotopy; similarly we can obtain a map $h\underline{P}(X) \to h\underline{P}(X\times J)$. It would not be difficult, in fact, to write down an explicit representative of Σ. We shall not do this, however. For it is apparently not possible to make a choice which is natural. Concretely, $\underline{P}(X)$ is a functor on the category of compact manifolds X and embeddings of codimension 0 between such, and $\underline{\sigma}$ is a natural transformation of functors. But Σ is only a natural transformation up to homotopy. To make Σ more explicit it would be of little use to just make a choice for Σ on objects. One should go on and choose commuting homotopies for morphisms, then coherence homotopies for commutative triangles, and so on. We shall not enter this matter.

The *stabilization with respect to dimension* is defined as

$$\varinjlim_{n} \underline{P}(X\times J^n)$$

where the maps in the direct system are given by the lower stabilization map.

In order to define a composition law let us say that a partition (M_1,F_1,N_1) of $\underline{P}(X)$ has *support in* X_1 if

$$M_1 \subset X\times[a,a'] \cup X_1\times[a',b'] .$$

We can regard $\underline{P}(X)$ as a *partial monoid* [17] where two partitions are composable if, and only if, they have disjoint support. If (M_1,F_1,N_1) and

(M_2, F_2, N_2) are composable then their sum is defined as the partition given by

$$M = M_1 \cup_{X\times[a,a']} M_2$$

or what is the same, the union of M_1 and M_2 as subspaces of $X\times I$; it is a manifold in view of the assumption of disjointness of support.

Stably it is always possible to move to disjoint support (by a kind of general position homotopy). It follows that the composition law makes $\varinjlim \underline{P}(X\times J^n)$ the *underlying space of a* (special) Γ-*space* in the sense of [18]. The classifying space construction with respect to that structure is the simplicial object

$$[q] \longmapsto \text{simplicial set of composable q-tuples in } \varinjlim \underline{P}(X\times J^n)$$

which we denote

$$N_\Gamma(\varinjlim \underline{P}(X\times J^n)) .$$

Similarly we can make this construction with $h\underline{P}$ instead of $\underline{P}$.

We turn to reformulating theorem 1 now (cf. also section 5). For every m we have a commutative diagram

$$\begin{tikzcd}
\varinjlim_{n} \underline{H}(X\times J^n) \arrow[r] \arrow[d] & \varinjlim_{k,n} \underline{P}^m_k(X\times J^n) \arrow[d] \\
\varinjlim_{n} h\underline{H}(X\times J^n) \arrow[r] & \varinjlim_{k,n} h\underline{P}^m_k(X\times J^n)
\end{tikzcd}$$

where the direct system in the k-variable is given by adding handles in some standard way, and the maps from left to right are induced by the identification $\underline{H}(X) = \underline{P}^m_0(X)$. The lower left term in the diagram is contractible (it is a simplicial object of categories with initial objects), and assertion (1) of the theorem says that (for every k , and hence also in the limit with respect to k) the diagram is homotopy cartesian in the dimension range up to $m-\varepsilon$ where ε is some constant (about 3). This is proved in proposition 5.1.

Except for questions on how to add more handles, the diagram is natural in X (with respect to codimension 0 embeddings). There is a suspension map, well defined up to homotopy, from the m-th diagram to the (m+1)-th. The suspension map can be a map of commutative diagrams, but it is not natural with respect to codimension 0 embeddings, only natural up to homotopy.

From the diagram we obtain another by performing the *plus construction* [14] on the two spaces on the right. The resulting diagram is homotopy cartesian in the same dimension range; this is also proved in proposition 5.1.

The composition law on $\underline{P}(X)$ restricts to one on

$$\underline{P}^m(X) = \coprod_{k\geq 0} \underline{P}^m_k(X)$$

and this in turn restricts to one on $\underline{H}(X)$. Similarly with $h\underline{P}(X)$. We obtain a commutative diagram

$$\begin{array}{ccc} N_\Gamma(\lim\limits_{\vec{n}} \underline{H}(X\times J^n)\ ) & \longrightarrow & N_\Gamma(\lim\limits_{\vec{n}} \underline{P}^m(X\times J^n)\ ) \\ \downarrow & & \downarrow \\ N_\Gamma(\lim\limits_{\vec{n}} h\underline{H}(X\times J^n)\ ) & \longrightarrow & N_\Gamma(\lim\limits_{\vec{n}} h\underline{P}^m(X\times J^n)\ ) \end{array} .$$

This diagram is homotopy cartesian in the same dimension range still (one better, actually). For by taking loop spaces, and restricting to a connected component, we obtain from it a diagram homotopy equivalent to the preceding diagram of plus constructions, thanks to results of Segal [18].

In this diagram the maps from left to right are given by inclusion, not the addition of handles or the like. The construction of the diagram does not therefore involve choices depending on X . Hence the diagram is natural for codimension 0 embeddings.

As the N_Γ construction can be iterated, it also results that theorem 1 really is a theorem about infinite loop spaces.

Part (2) of the theorem says that there is a natural transformation (natural, that is, for codimension 0 embeddings)

$$\Omega\, N_\Gamma(\lim_{\vec{n}} h\underline{P}^m(X\times J^n)\ ) \longrightarrow A(X)$$

which is highly connected, depending on m (it is actually (m-1)-connected). This is proved in proposition 5.4.

Given this we obtain, by looping the fibration of theorem 1, a map $\Omega A(X) \to \lim\limits_{\vec{}} \underline{H}(X\times J^n)$ (or rather an approximation to such a map). To avoid the dimension shift it involves, it is convenient to introduce the *Whitehead space* (depending on the category under consideration) as

$$Wh^{CAT}(X) = N_\Gamma(\lim_{\vec{n}} \underline{H}(X\times J^n)\ ) .$$

By looping the de-looped version of the fibration of theorem 1 we then obtain (an approximation to) a map $A(X) \to Wh^{CAT}(X)$.

Part (3) of the theorem finally says that the functor

$$\Omega\, N_\Gamma(\lim_{\vec{n}} \underline{P}^m(X\times J^n)\ )$$

behaves like a homology theory in a stable range of dimensions (the range is that of dimensions up to $m-c$ where c is some constant, about 3 or 4). This means that, in addition to the homotopy property, the functor also has the *excision property*: if

$$\begin{array}{ccc} X_o & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_1 \cup_{X_o} X_2 \end{array}$$

is a (pushout) diagram of codimension 0 embeddings of manifolds then, denoting the functor by Ψ for short, the induced diagram

$$\begin{array}{ccc} \Psi(X_o) & \longrightarrow & \Psi(X_2) \\ \downarrow & & \downarrow \\ \Psi(X_1) & \longrightarrow & \Psi(X_1 \cup_{X_o} X_2) \end{array}$$

will be homotopy cartesian (in the range in question).

2. The splitting theorem.

We have to begin by recalling a few generalities on functors and their stabilizations. If Φ is a functor from pointed spaces to pointed spaces, with suitable homotopy properties, its *stabilization* is given by

$$\Phi^S(X) \quad = \quad \lim_{\overrightarrow{n}} \Omega^n \text{ fibre}(\ \Phi(S^n \wedge X) \to \Phi(*)\ )$$

where, up to homotopy, the n-th map in the direct system is the map of n-th loop spaces of the vertical homotopy fibres in the following diagram associated to the decomposition of the (n+1)-sphere into its upper and lower hemispheres

$$\begin{array}{ccc} \Phi(S^n \wedge X) & \longrightarrow & \Phi(D^{n+1}_+ \wedge X) \\ \downarrow & & \downarrow \\ \Phi(D^{n+1}_- \wedge X) & \longrightarrow & \Phi(S^{n+1} \wedge X) \end{array} \quad .$$

As formulated, the construction of Φ^S is well defined up to homotopy only since it involves identifications of homotopy equivalent spaces. But we can reformulate it a little to remove that ambiguity. Let us define an operation on functors taking Φ to Φ_1 say, where by definition $\Phi_1(X)$ is the homotopy inverse limit of the diagram

$$\Phi(D^1_- \wedge X) \longrightarrow \Phi(S^1 \wedge X) \longleftarrow \Phi(D^1_+ \wedge X)$$

(recall that $\operatorname{holim}(A \to C \leftarrow B)$ is defined as $A \times_C C^{[0,1]} \times_C B$, the space of paths in C with chosen liftings of endpoints to A and B , respectively). There is a natural transformation $\Phi(X) \to \Phi_1(X)$, and the construction can be iterated, say $\Phi_{n+1}(X) = (\Phi_n)_1(X)$. Letting $\Phi_\infty(X) = \lim_{\rightarrow} \Phi_n(X)$ we can then define Φ^S as an honest functor by

$$\Phi^S(X) \quad = \quad \text{fibre}(\ \Phi_\infty(X) \to \Phi_\infty(*)\ )\ .$$

We adapt the construction to functors from unpointed spaces to pointed spaces in the usual way by adding a basepoint. That is, if F is such a functor then its stabilization is defined as

$$F^S(X) \quad = \quad \Phi^S(X \cup *)$$

where Φ denotes the restriction of F to the category of pointed spaces.

We will assume that it is possible to define a map

$$F(X) \longrightarrow \text{fibre}(\ F(X \cup *) \to F(*)\ )\ .$$

For example in our applications, below, F will take values in group-like H-spaces, so this is certainly possible. It follows then that we can define a map

$$F(X) \longrightarrow F^S(X) .$$

Let F be a functor from spaces to (pointed) spaces. We say F is *connective* if it takes n-connected maps to n-connected maps for every sufficiently large n (in our applications this will be the case for $n \geq 2$). We say F is *strongly connective* if in addition it has the property that for every pushout diagram of cofibrations in which the horizontal and vertical maps are m-connected and n-connected, respectively, the square resulting by application of F will be homotopy cartesian in the dimension range up to $m+n-\varepsilon$ where ε is some constant (about 3 , in our applications). This is an excision condition on F . For example, the *homotopy excision theorem* of Blakers and Massey says that the identity functor is strongly connective in this sense.

If F is connective, resp. strongly connective, then its stabilization F^S is so, too. But more is true: if F is strongly connective then F^S actually is a *homology theory*, that is, when it is applied to a pushout diagram of cofibrations it will produce a homotopy cartesian square. If F itself should happen to be a connective homology theory then the stabilization does not really change it, that is, the natural map $F \to F^S$ is a homotopy equivalence in this case.

We will take the liberty now to speak of the Whitehead space $Wh^{CAT}(X)$ as if it were a functor on the category of topological spaces and continuous maps rather than just a functor on CAT manifolds and their codimension 0 embeddings. This can be justified in two ways.

The first is to actually construct such a functor, homotopy equivalent to $Wh^{CAT}(X)$ if X is a CAT manifold. This is done by a homotopy theoretic version of the left Kan extension: one evaluates Wh^{CAT} on each manifold over the given topological space, and then takes the homotopy direct limit of the resulting diagram. (As a technical point, one should use only manifolds of some fixed dimension and then pass to the limit with respect to dimension; also, because of the way we have defined the stabilization with respect to dimension (i.e. pullback with *trivial* disk bundles) one should use only manifolds of some fixed tangential type over the space, for example the parallelizable ones).

The second justification is in the remark that all the necessary suspending involved in stabilization can be very explicitly represented in terms of manifolds, and the resulting construction can be natural, say, on the partially ordered set of codimension 0 submanifolds of euclidean space (this construction also presupposes that one systematically allows for stabilization with respect to dimension).

There is a concrete reason why the process of stabilizing functors is relevant to concordance theory. Namely a certain property of the Whitehead space functor in the DIFF case implies a statement about stabilization (and is, conversely, more or less implied by that statement). The fact is that for an n-connected space X the map $Wh^{DIFF}(*) \to Wh^{DIFF}(X)$ is not just n-connected as one expects, it is about (2n)-connected [4]. As a consequence, the stabilized functor $(Wh^{DIFF})^S$ is actually trivial, up to homotopy.

To apply this let us suppose that F is a functor, and $F(X) \to Wh^{DIFF}(X)$ a natural transformation having the property that its homotopy fibre, $h(X)$ say, is a connective homology theory. By stabilizing we obtain a fibration

$$h^S(X) \longrightarrow F^S(X) \longrightarrow (Wh^{DIFF})^S(X)$$

and hence, since $(Wh^{DIFF})^S(X) \simeq *$, a homotopy equivalence $h^S \to F^S$. On the other hand, $h \to h^S$ is a homotopy equivalence, too, in view of the assumed fact that h is a connective homology theory. On combining the various stabilization maps we will therefore obtain a homotopy commutative diagram, with homotopy equivalences as indicated,

$$\begin{tikzcd}
h(X) \arrow[r] \arrow[d, "\wr"] & F(X) \arrow[r] \arrow[d] & Wh^{DIFF}(X) \arrow[d] \\
h^S(X) \arrow[r, "\sim"] & F^S(X) \arrow[r] & (Wh^{DIFF})^S(X)
\end{tikzcd}$$

and we conclude that $F(X)$ actually splits as

$$F(X) \simeq F^S(X) \times Wh^{DIFF}(X) .$$

Essentially now theorem 1 provides such a map $A(X) \to Wh^{DIFF}(X)$ whose homotopy fibre is a homology theory. There are a few technical points. First the loop space $\Omega Wh^{DIFF}(X)$ in theorem 1 should be replaced by the homotopy fibre of the right vertical map (in the diagram of theorem 1 after the plus construction). By looping the fibration we obtain a map $\Omega A(X) \to \Omega Wh^{DIFF}(X)$, but it is a map of infinite loop spaces, so the Ω may be suppressed. Finally the spaces in theorem 1 are not quite the correct ones, they are the correct ones only in some finite, though arbitrarily large, range of dimensions. To summarize: after replacing of all the spaces by homotopy equivalent ones, and of these in turn by terms in their Postnikov towers, we can have a map $A(X) \to Wh^{DIFF}(X)$, the map can be natural (not just up to homotopy), and its homotopy fibre is a homology theory.

This homology theory is connective. In fact, each of $A(X)$ and $Wh^{DIFF}(X)$ respects n-connected maps (if $n \geq 2$), by [21] and [4]. It follows that the homology theory, if not connective, is at least (-1)-connective. To rule out that possibility it suffices to know that the map of fundamental groups $\pi_1 A(X) \to \pi_1 Wh^{DIFF}(X)$ is surjective in the case where X is the circle. By the h-cobordism theorem $\pi_1 Wh^{DIFF}(X)$ is isomorphic to the Whitehead group $Wh_1(\pi_1 X)$, and this is well known to be trivial in the case where $\pi_1 X$ is an infinite cyclic group. The surjectivity is thus clear.

We conclude that we can split $A(X)$, at least after passage to any term in its Postnikov tower. Putting these splittings together, we obtain one of $A(X)$ itself, natural up to weak homotopy,

$$A(X) \simeq A^S(X) \times Wh^{DIFF}(X) .$$

Remark. The construction of this splitting was indicated in [21], modulo a verification that the map $A(X) \to Wh^{PL}(X)$ could be factored through $Wh^{DIFF}(X)$ in a sufficiently natural way (which implies that theorem 1 holds in the DIFF case). It seemed at the time that the verification required the use of yet another functor, the *combinatorial Whitehead space* [21]. As the present account shows it is not however necessary to use the functor $Wh^{Comb}(X)$ for that purpose. This makes the verification a lot easier.

The functor $A^S(X)$ in turn also splits [22],

$$A^S(X) \simeq \Omega^\infty S^\infty(X_+) \times \mu(X)$$

say. (A different account of such a splitting will be given below). By combining the two splittings we obtain

<u>Theorem</u> 2. There exists a splitting, natural up to weak homotopy,

$$A(X) \simeq \Omega^\infty S^\infty(X_+) \times Wh^{DIFF}(X) \times \mu(X) .$$

Remark. Given a double splitting such as this, with little apparent reason for even a single splitting, one may wonder if perhaps the third factor is trivial. It is known that $\mu(X)$ is rationally trivial - this is equivalent [21] to the theorem of Farrell-Hsiang [7] and Borel [1] on the vanishing of

$$H_*(GL(Z), M'(Q)) ,$$

the homology of $GL(Z)$ acting by conjugation on rational matrices of trace 0. The argument does not extend to prove the triviality of $\mu(X)$ because mod p versions of that vanishing theorem are not currently known. It turns out, however, that the triviality of $\mu(X)$ can be proved directly. We will not discuss any of the proof here. But let us note the following addendum.

Addendum. The following four statements are mutually equivalent to each other:

(1) $\mu(X)$ is trivial,

(2) the composite map $\Omega^\infty S^\infty(X_+) \to A(X) \to A^S(X)$ is a homotopy equivalence,

(3) the map $Wh^{Comb}(X) \to Wh^{DIFF}(X)$ is a homotopy equivalence (i.e. concordances can be 'handled', stably),

(4) $A(*) \simeq \varinjlim_n \Omega^n \, Top_{n+1}/Top_n$.

In fact, the equivalence of (1) and (2) results from the definitions: $\mu(X)$ is the cofibre (in the sense of stable homotopy theory) of the composite map in (2). As to (3), we will just say that $Wh^{Comb}(X)$ has the property that there exists a homotopy fibration $\Omega^\infty S^\infty(X_+) \to A(X) \to Wh^{Comb}(X)$ (the proof is an elementary, though non-trivial, application of the additivity theorem [21], suffice it to say that $Wh^{Comb}(X)$ was designed so that it has this property); the equivalence of (2) and (3) then results by comparing this fibration with the splitting of theorem 2. To obtain the equivalence of (1) and (4) finally, one compares the splitting of theorem 2 with the main result of Kuiper and Lashof [13] a known reformulation of which says that $\varinjlim \Omega^n \, Top_{n+1}/Top_n \simeq Wh^{DIFF}(*) \times \Omega^\infty S^\infty$, cf. [3], [10]. Note that the implication (4) $\Rightarrow$ (1) is valid even without reference to any particular map to $\varinjlim \Omega^n \, Top_{n+1}/Top_n$, thanks to Dwyer's theorem that the homotopy groups of $A(*)$ are finitely generated [6]. Nevertheless an explicit description of such a map is desirable, and we will give one now.

This map, as well as the splitting map on $A^S(X)$, can be described in terms of a *derivative*.

To define that, let Q^d denote the *space of germs of normally oriented d-planes in* R^{d+1} . In the DIFF case this space is homotopy equivalent to $O_{d+1}/O_d \approx S^d$. In the TOP case it is homotopy equivalent (in view of the theorem that microbundles are represented by bundles) to the analogous Top_{d+1}/Top_d where $Top_d = Top(R^d)$ denotes the space of homeomorphisms of R^d (preserving the origin, say). In either case Q^d is (d-1)-connected (cf. the *stability theorem* 5.2 of Kirby and Siebenmann [12] in the TOP case).

The Q^d form a spectrum. To describe the map $Q^d \to \Omega Q^{d+1}$ it will suffice to describe two nullhomotopies of the map $Q^d \to Q^{d+1}$ obtained by adding a common factor R^1 to both the d-plane and the ambient space R^{d+1} . To describe these we replace the space Q^d of germs of normally oriented d-planes by the homotopy equivalent space of germs of d-planes with a collar on one side. Given such a germ of half-collar then, after taking the product with a closed interval, the front (d+1)-face may be moved to either side by pushing it around the corner. This gives two ways of moving it to standard position, and thus two nullhomotopies (cf. the end of section 6 for a reformulation in the DIFF case, making it clear that the spectrum is indeed the sphere spectrum in that case).

Let X be a manifold of dimension d. For simplicity we assume that X is a codimension 0 submanifold of euclidean space R^d. Let $\mathcal{P}^m_k(X)$ be the simplicial set of the partitions of type (m,k), as in section 1. There is a tautological bundle over $\mathcal{P}^m_k(X)$: the fibre over the partition (M,F,N) is given by the triple of manifolds (M,F,N). The *derivative* now is, by definition, a bundle map from this bundle to the bundle with fibre Q^d (which is the trivial bundle in view of our assumption that X is a codimension 0 submanifold of R^d, and hence framed). The map is first defined on the frontier F; it takes each point of F to the germ of normally oriented d-plane represented by F at that point. Extending of the map requires a connectivity consideration. Namely M can be obtained from X by attaching of trivial m-handles (up to h-cobordism), by definition of what $\mathcal{P}^m_k(X)$ is; up to homotopy therefore N can be obtained from F by attaching of (m+1)-cells. Since Q^d is (d-1)-connected, it follows that the map can be extended to N over the (d-m-2)-skeleton of $\mathcal{P}^m_k(X)$. Similarly the map can be extended to M over the (m-2)-skeleton. Putting the two extensions together we obtain, in the stable range of dimensions up to $\min(m,d-m)-2$, a bundle map from the trivial bundle $X\times[a,b]$ to the trivial bundle Q^d. The assumed fact that all partitions are standard near $\partial X\times[a,b]$ implies that the map can be trivial near $\partial X\times[a,b]$. By restricting to $X\times a$, say, the derivative thus provides a map, in a stable range, from $\mathcal{P}^m_k(X)$ to the space of maps $X/\partial X \to Q^d$.

This map is compatible with the stabilization with respect to dimension. That is, if J denotes an interval, and $\underline{\mathcal{P}}^m_k(X)$ the simplicial subset on which the (lower) stabilization map is defined (section 1) then the diagram

$$\begin{array}{ccc} \underline{\mathcal{P}}^m_k(X) & \longrightarrow & \mathit{Map}(X/\partial X, Q^d) \\ \downarrow & & \downarrow \\ \underline{\mathcal{P}}^m_k(X\times J) & \longrightarrow & \mathit{Map}(X\times J/\partial(X\times J), Q^{d+1}) \end{array}$$

commutes up to homotopy where the vertical map on the right is obtained by means of the isomorphism $X\times J/\partial(X\times J) \approx X/\partial X \wedge J/\partial J$ from the structural map

$J/\partial J \wedge Q^d \to Q^{d+1}$; this results at once from the definitions of these maps.

The map is also compatible with stabilization with respect to m . To see this, one simply uses that the two kinds of stabilization correspond to each other under the flip map which interchanges the M and N of a partition.

It results that we can obtain a map from

$$\lim_{\vec{n}} \underline{P}_k^m(X \times J^n) ,$$

or better, from the corresponding homotopy direct limit, to

$$\mathit{Map}(X/\partial X, \lim_{\rightarrow} \Omega^n Q^{d+n})$$

or what is the same up to homotopy, by Poincaré duality,

$$\lim_{\rightarrow} \Omega^n (X_+ \wedge Q^n) .$$

Now in the TOP case if X is a disk then $\underline{P}_k^m(X)$ approximates $A(*)$ by theorem 1 since $Wh^{TOP}(*)$ is trivial. So the procedure gives a map

$$A(*) \longrightarrow \lim_{\rightarrow} \Omega^n \ Top_{n+1}/Top_n .$$

In the DIFF case $\underline{P}_k^m(X)$ approximates $A^S(X)$ by theorems 1 and 2. So we obtain a map

$$A^S(X) \longrightarrow \lim_{\rightarrow} \Omega^n (S^n \wedge X_+) .$$

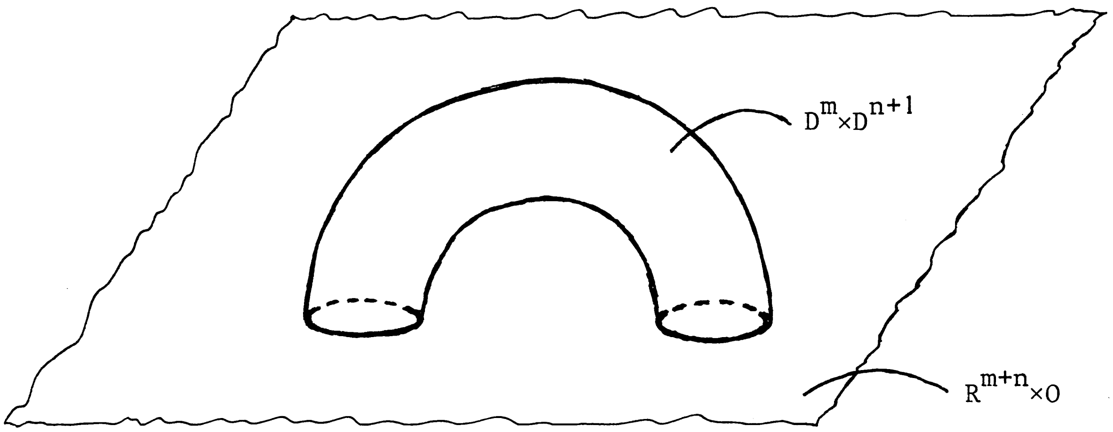
To see that this map is a retraction, up to homotopy, we note that in the above definition of the map it is not really necessary for the derivative to be extended from the frontier F of a partition to *both* of the complementary parts M and N . It would suffice to extend the derivative to N , say (and restrict to $X \times b$ subsequently). But this means that the map can be regarded as being defined on $P_k^m(X)$ in the case where $m = 0$. Taking X to be a high-dimensional disk now, it is not difficult to check that the latter map is equivalent to the standard map [17] inducing the homotopy equivalence

$$(B\Sigma_\infty)^+ \longrightarrow (\Omega^\infty S^\infty)_{(o)} .$$

To conclude we note that it may be difficult to prove directly that the retraction here constructed is related to that of [22] in any particular way. But if we assume the fact that $\Omega^\infty S^\infty(X_+) \to A^S(X)$ is a homotopy equivalence (the remark after theorem 2) then it follows of course that the two retractions are the same, up to homotopy.

3. Properties of the map $BG \to A(*)$.

Let a *tube of type* (m,n) mean a codimension 0 submanifold of euclidean space R^{m+n+1} which contains the lower halfspace $R^{m+n}\times(-\infty,0]$ and which, up to an isotopy with compact support, is obtainable from this lower halfspace by the attaching of an unknotted m-handle.



We let $\mathcal{T}^{m,n}$ denote the simplicial set in which a k-simplex is a locally trivial family of such tubes, parametrized by the simplex Δ^k .

Up to a technical modification, this space of tubes is really the same thing as the space $\underline{P}_k^m(D^{m+n})$ of section 1 in the case $k = 1$. In fact $\mathcal{T}^{m,n}$ can be obtained from the latter by a limiting procedure, namely by letting $D^{m+n}\times[a,b]$ increase to $R^{m+n}\times R^1$, and at any rate the two are homotopy equivalent.

As in section 1 there are stabilization maps from $\mathcal{T}^{m,n}$ to $\mathcal{T}^{m,n+1}$ and to $\mathcal{T}^{m+1,n}$. We let

$$\mathcal{T} = \varinjlim_{m,n} \mathcal{T}^{m,n} .$$

If we want to emphasize the framework we will write $\mathcal{T}^{CAT}$ instead.

Proposition 3.1. There is a homotopy equivalence $\mathcal{T}^{TOP} \simeq BG$. Further there is a fibration up to homotopy

$$\Omega\, Wh^{DIFF}(*) \to \mathcal{T}^{DIFF} \to BG ,$$

and there is a homotopy cartesian square

$$\begin{array}{ccc} T^{\mathrm{DIFF}} & \longrightarrow & BG \\ \downarrow & & \downarrow \\ A^S(*) & \longrightarrow & A(*) \end{array} .$$

Remark. If we assume the fact that $\Omega^\infty S^\infty \to A^S(*)$ is a homotopy equivalence (cf. the remark after theorem 2) we may reformulate the last assertion to say that T^{DIFF} is the homotopy pullback of the diagram of the natural maps

$$\Omega^\infty S^\infty \longrightarrow A(*) \longleftarrow BG .$$

Proof of proposition. This results from the DIFF and TOP versions of theorem 1. For each m and k we have a sequence

$$\lim_{\vec{n}} \underline{H}(D^{m+n}) \longrightarrow \lim_{\vec{n}} \underline{P}^m_k(D^{m+n}) \longrightarrow \lim_{\vec{n}} h\underline{P}^m_k(D^{m+n})$$

which is a homotopy fibration in a certain stable range, depending on m . The first two assertions of the proposition are obtained from this by taking $k=1$. The base of the fibration then represents BG (in a stable range), regardless of the category, the total space approximates T^{DIFF} , resp. T^{TOP} , and the fibre is $\Omega \mathrm{Wh}^{\mathrm{DIFF}}(*)$, resp. $\Omega \mathrm{Wh}^{\mathrm{TOP}}(*) \simeq *$.

To obtain the third assertion we compare the fibration in the case $k=1$ with the one that results by letting k tend to infinity, and applying the plus construction. In either of the two fibrations the fibre is $\Omega \mathrm{Wh}^{\mathrm{DIFF}}(*)$, so the square formed by the total spaces and bases is (approximately) homotopy cartesian. The square of the proposition is obtained from it by letting m tend to infinity, and by rewriting of two of the terms as $A(*)$ and $A^S(*)$, respectively, using theorems 1 and 2.

Proposition 3.2. There is a map $BO \to T^{\mathrm{DIFF}}$ whose composite with $T^{\mathrm{DIFF}} \to BG$ is the J-homomorphism $BO \to BG$.

Proof. The map may conveniently be described as an inclusion map of a subspace of rigid tubes (this formulation is due to Goodwillie, it simplifies another less direct one). Let G^{m+n}_m denote the Grassmannian manifold of m-planes in R^{m+n} . We define an inclusion map $G^{m+n}_m \to T^{m,n}$ by associating to each m-plane a *rigid tube* in some standard way, for example like this. Take the m-plane to its unit sphere and this in turn to the half-sphere which it bounds in the half-space $R^m \times [0,\infty)$. Then take the half-sphere to its tubular neighborhood of radius $1/2$, in R^{m+n+1} , and add that as an m-handle to $R^{m+n} \times (-\infty,0]$.

Corollary 3.3. The composite map $BO \to BG \to A(*) \to Wh^{DIFF}(*)$ is nullhomotopic.

Proof. $BO \to BG$ factors through T^{DIFF} (the preceding proposition), and $T^{DIFF} \to A(*)$ factors through $A^S(*) \simeq \mathrm{fibre}(A(*) \to Wh^{DIFF}(*))$ (by proposition 3.1 above).

Corollary 3.4. There is a map $G/O \to \Omega\, Wh^{DIFF}(*)$.

Proof. Using propositions 3.1 and 3.2 this is obtained as the map of homotopy fibres, $\mathrm{fibre}(BO \to BG) \to \mathrm{fibre}(T^{DIFF} \to BG)$.

An interesting question is whether or not the map

$$G/O \to \Omega\, Wh^{DIFF}(*)$$

is a rational homotopy equivalence.

Remark. A map of this kind has been constructed earlier by Hatcher in the framework of the Cerf function space approach to concordance theory. In particular, the question goes back to Hatcher.

Remark. It should not be assumed that the map is a homotopy equivalence. For by using the triviality of $\mu(*)$ (the remark after theorem 2) one obtains that for every prime p there is a stable range $j < 2p-3$ in which $\pi_j Wh^{DIFF}(*)$ is isomorphic, at p , to $\pi_j A(*)$ and therefore also to $K_j(Z)$. According to Soulé [19] now $K_{22}(Z)$ contains an element of order 691 ; but $\pi_{21}(G/O)$ does not.

The following result is due to Tom Goodwillie.

Proposition 3.5. The following diagram is homotopy commutative

$$\begin{tikzcd} BO \arrow[r] \arrow[d] & BG \arrow[r] & A(*) \arrow[d] \\ O \arrow[r] & G \arrow[r] & \Omega^\infty S^\infty \end{tikzcd}$$

where the vertical map on the left is the Bott map.

Goodwillie has proved this by using a version of the splitting map from $A(*)$ to $\Omega^\infty S^\infty$ that he can construct by fixpoint methods. The argument below uses the splitting map described in the preceding section.

Remark. The Bott map $BO \to O$ can be extended to a map $BG \to G$ (it may be defined in terms of *multiplication by* η, where $\eta \in \pi_1^S$) and it has been more or less verified, by Marcel Bökstedt and myself, that the diagram remains commutative if that map is filled in and if the splitting map is that of [22].

Proof of proposition. We use the following description of the composite of the Bott map $BO \to O$ with $O \to G \to \Omega^\infty S^\infty$. We represent a point of BO by an m-plane in R^{m+n}. The map then takes this to (the one-point-compactification of) the self-map of R^{m+n} which is the antipodal map on the m-plane, and the identity map on its orthogonal complement. We will show that the other composite map in the diagram admits the same description.

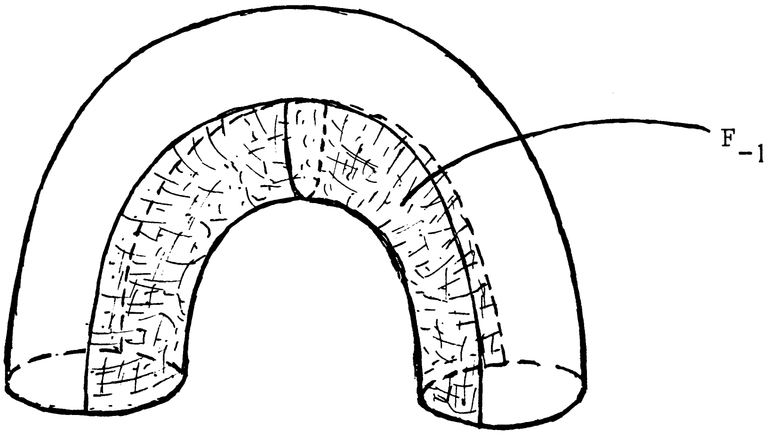
We can rewrite this map somewhat by using the following diagram provided by theorem 2 and propositions 3.1 and 3.2,

$$\begin{tikzcd}
BO \arrow[r] \arrow[dr] & T^{DIFF} \arrow[r] \arrow[d] & A^S(*) \arrow[d] \arrow[dr, "\simeq"] & & \\
& BG \arrow[r] & A(*) \arrow[r] & A^S(*) \arrow[r] & \Omega^\infty S^\infty .
\end{tikzcd}$$

By definition of the splitting map $A^S(*) \to \Omega^\infty S^\infty$ (the preceding section) the map from T^{DIFF} to $\Omega^\infty S^\infty$ is given by the *derivative* in the sense that we must take the actual derivative (Gauß map) on the frontier of the tube, and then extend to the tube itself, and its complement, in more or less arbitrary fashion. The map $BO \to T^{DIFF}$ in turn is represented, in a stable range, by a map which takes each m-plane in R^{m+n} to a *rigid tube* in a certain standard way (cf. the proof of proposition 3.2). Our task is thus to show that by evaluating the derivative on rigid tubes we do recover the Bott map, or rather its composite with $O \to \Omega^\infty S^\infty$.

We proceed in two steps. In the first step we show that the Bott map may be recovered by means of a map on the space of rigid tubes which is closely related to the derivative. In a second step we then verify that this map is actually equivalent to the derivative.

The frontier F of a rigid tube may be naturally decomposed into three parts. The *flat* part F_0 is that part of F which lies in the plane $R^{m+n} \times 0$ (it is all of that plane except of the part where the handle is attached). The *upper* part F_1 is that part on the boundary of the handle which is visible from very high up. And the *lower* part F_{-1} finally is the rest. This turns out to be the part which is of most interest to us.



Let S^{m+n} denote the unit sphere in R^{m+n+1}, and D_+ and D_- its upper and lower hemispheres, respectively. The Gauß map takes both F_o and F_1 into D_+. Hence, for homotopy purposes, the derivative may be identified to the map of quotient spaces

$$F_{-1}/\partial F_{-1} = F/(F_o \cup F_1) \xrightarrow{v} S^{m+n}/D_+ = D_-/\partial D_-$$

(we are ignoring here the corner at the place where F_o meets F_1 or F_{-1}; taking it into account, à la section 6, does not alter anything in an essential way).

The projection $R^{m+n+1} \to R^{m+n}$ induces an embedding $u\colon F_{-1} \to R^{m+n}$. We obtain a map

$$R^{m+n} \cup \infty \longrightarrow D_-/\partial D_-$$

by using the map $v\,u^{-1}$ on the image $u(F_{-1})$, and then extending by the constant map.

Identifying the Grassmannian manifold G_m^{m+n} with the space of rigid tubes we obtain thus a map $G_m^{m+n} \to \Omega^{m+n}S^{m+n}$. This map is homotopic to the composite map

$$G_m^{m+n} \longrightarrow O_{m+n} \longrightarrow \Omega^{m+n}S^{m+n} .$$

In fact this is clear from the remark that the map $F_{-1}/\partial F_{-1} \to D_-/\partial D_-$ is composed of an identification and a reflection: an identification in the n-direction, and a reflection in the m-direction.

We are left to show that, up to homotopy, this map represents the derivative on the space of rigid tubes. To do this it will suffice to show that, in a stable range of dimensions, the map $F \to D_-/\partial D_-$ can be extended to a map on the complement of the tube in such a way that at the slice $R^{m+n} \times 3$ (say) it restricts to the above map $R^{m+n} \to D_-/\partial D_-$. We will show that such an extension can be found over the n-skeleton of G_m^{m+n} .

The easiest way of obtaining the extension is by an appeal to the Thom-Pontriagin construction. We think of G_m^{m+n} as the simplicial set in which a k-simplex is a parametrized family, over Δ^k , of rigid tubes of the type we are discussing. By means of the derivative we have for each such family a locally trivial family of maps $F/(F_0 \cup F_1) \to D_-/\partial D_-$. The family is transverse to the south pole in D_- , and by taking the pre-image of the south pole we obtain a k-parameter family of framed points, one such point for each point of Δ^k . We now choose a k-parameter family of framed intervals with the following properties: the above family of points gives the initial points of the intervals, the family of endpoints is in the slice $R^{m+n} \times 3$, and each of the intervals is disjoint to F (and also $R^{m+n} \times 3$) except for its endpoints. Such a choice is possible, relative to an earlier choice over $\partial\Delta^k$, if k is not bigger than n . The required extension of the map is now obtained in the usual way: a neighborhood of the line in question is mapped to a neighborhood of the south pole, using the framing, and its complement is mapped (arbitrarily) into the complement.

As a final point, note that each of those lines has to bend around: it goes down from F first, and then up to $R^{m+n} \times 3$. This affects the framing, and therefore it would contribute a minus sign if it were not for the fact that the Bott map is concerned with 2-torsion phenomena only. □

As an application we can obtain information on how the image of the J-homomorphism is mapped to the K-theory of the integers. Quillen has shown that most of it injects, cf. [16]. We show here that in the cases not covered by Quillen the map is in fact trivial. (In the argument we have to know that $\mu(*)$ is trivial, cf. the remark after theorem 2).

<u>Corollary</u> 3.6. If $j = 8k$ or $8k+1$, where $k \geq 1$, then the map

$$\pi_j O \to \pi_j^S \to K_j(Z)$$

is trivial.

Proof. If $y \in \pi_j O$ where $j = 8k$ or $8k+1$ then y is in the image of the Bott map $b\colon BO \to O$, say $y = b_*(x)$. The image z of x under the map

$$BO \to BG \to A(*)$$

is in the kernel of the map to $K_j(Z)$, provided that $j \neq 0, 1$. Using theorem 2 we can decompose z as $z_1 + z_2 + z_3$ where

$$z_1 \in \pi_j^S \quad , \quad z_2 \in \pi_j \mathrm{Wh}^{\mathrm{DIFF}}(*) \quad , \quad z_3 \in \pi_j \mu(*) \ .$$

Now $z_3 = 0$ if we assume that $\mu(*)$ is trivial (the remark after theorem 2), and by corollary 3.3 we have $z_2 = 0$. It results that z_1 is in the kernel of the map to $K_j(Z)$. But $z_1 = J(b_*(x)) = J(y)$ by proposition 3.5.

As a final application we show how for every prime p the first deviation of $A(*)$ and $K(Z)$ may be determined exactly (this also presupposes the vanishing of $\mu(*)$).

Corollary 3.7. For every prime p ,

$$\mathrm{coker}(\ \pi_{2p-1}A(*) \to K_{2p-1}(Z)\ )_{(p)} \ \approx\ Z/p \ .$$

Proof. The first p-torsion in BG occurs in π_{2p-2} , and the map

$$Z/p \ \approx\ \pi_{2p-2}BG \longrightarrow \pi_{2p-2}\mathrm{fibre}(\ A(*) \to K(Z)\ )$$

is bijective on p-torsion by an elementary computation [21]. On the other hand the map $\pi_{2p-2}BG \to \pi_{2p-2}A(*)$ is trivial as we now show. Let $x \in \pi_{2p-2}BG$ and let z denote its image in $\pi_{2p-2}A(*)$. Put $z = z_1 + z_2 + z_3$ where

$$z_1 \in \pi_{2p-2}^S \quad , \quad z_2 \in \pi_{2p-2}\mathrm{Wh}^{\mathrm{DIFF}}(*) \quad , \quad z_3 \in \pi_{2p-2}\mu(*) \ .$$

Then $z_3 = 0$ if we assume $\mu(*)$ is trivial, and $z_2 = 0$ by corollary 3.3 since x is in the image of the J-homomorphism. If p is odd then z_1 must be zero since it is in the image of the Bott map and hence 2-torsion (alternatively we could use that π_{2p-2}^S has no p-torsion). If $p = 2$ then z_1 must be zero because on the one hand it is in the kernel of the map to K-theory (because z is) and on the other hand the map $\pi_2^S \to K_2(Z)$ is injective.

Remark. In the case $p = 2$ the result is not new. There must be at least three other proofs in that case. The map

$$K_3(Z) \longrightarrow \pi_2 BG$$

is what Kiyoshi Igusa calls the *Grassmann invariant*. There was an erroneous belief at one time that it was the zero map. This belief led to the conclusion that $\pi_1 C^{\mathrm{DIFF}}(*)$ should map onto $K_3(Z)/\pi_3^S$ [10, section 3] which in turn led to a contradiction as explained in [10, section 8]. Closer scrutiny has subsequently led Igusa to the discovery of his famous *picture* describing an element of $K_3(Z)$ with non-zero Grassmann invariant.

4. Technical tools.

In the next section it will be convenient to use versions of Quillen's theorems A and B [15] for simplicial categories. We record these here.

Recall that if $f\colon \mathcal{A} \to \mathcal{B}$ is a map of categories then for each object $B \in \mathcal{B}$ the *left fibre* f/B is defined as the category of pairs (A,a) where $A \in \mathcal{A}$ and $a\colon f(A) \to B$ is a morphism in $\mathcal{B}$. Dually the *right fibre* B/f is defined as the category of pairs (A,a) where $a\colon B \to f(A)$.

Suppose now that $f\colon \mathcal{A} \to \mathcal{B}$ is a map of simplicial categories. Let $([n],B)$ be an object of $\mathcal{B}$, that is, $B \in \mathcal{B}_n$. We define the *left fibre*

$$f/([n],B)$$

as the simplicial category

$$[m] \longmapsto \coprod_{u\colon [m] \to [n]} f_m / u^*(B)$$

where, as the notation suggests, the coproduct (disjoint union of categories) is indexed by the set of (monotone) maps u from $[m]$ to $[n]$.

Example. Let $f = \mathrm{Id}_{\mathcal{B}}$. Then $f/([n],B)$ is contractible for every $([n],B)$.

Indeed, each of the categories $\mathrm{Id}_{\mathcal{B}_n}/u^*(B)$ is contractible since it has a terminal object. In view of the realization lemma therefore $\mathrm{Id}_{\mathcal{B}}/([n],B)$ maps by homotopy equivalence to

$$[m] \longrightarrow \coprod_{u\colon [m] \to [n]} *$$

which is the n-simplex considered as a simplicial category in a trivial way.

Returning to the general case we note that a map $b\colon B \to B'$ in $\mathcal{B}_n$ induces a map

$$([n],b)_* : f/([n],B) \to f/([n],B') ;$$

similarly a map $v\colon [n] \to [p]$ induces, for every $B'' \in \mathcal{B}_p$, a map

$$v_* : f/([n],v^*(B'')) \to f/([p],B'') .$$

These two kinds of maps will be referred to as *transition maps*.

Theorem A'. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a map of simplicial categories. If for every object $([n],B)$ of $\mathcal{B}$ the left fibre $f/([n],B)$ is contractible then the map f is a homotopy equivalence.

Theorem B'. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a map of simplicial categories. If all transition maps of left fibres are homotopy equivalences then for every object $([n],B)$ the square

$$\begin{array}{ccc} f/([n],B) & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow f \\ \mathrm{Id}_{\mathcal{B}}/([n],B) & \longrightarrow & \mathcal{B} \end{array}$$

is homotopy cartesian.

The theorems admit dual formulations in terms of right fibres.

Before coming to the proof, we note an addendum which is useful in applications of the theorems.

Addendum. If every object of $\mathcal{B}$ is 0-dimensional (up to isomorphism) then the hypotheses of theorem A' or B' need be checked only in the case $[n] = [0]$.

Indeed, suppose that $([n],B)$ is 0-dimensional, that is, $B = v^*(B')$ say, where v is the unique map from $[n]$ to $[0]$. Then $u^*(B) = (vu)^*(B')$ is independent of u , so we obtain an isomorphism of $f/([n],B)$ with the product

$$\Big([m] \longmapsto \coprod_{u\colon [m] \to [n]} * \Big) \times \big([m] \longmapsto f_m/(vu)^*(B') \big)$$

$$= \quad \Delta^n \times f/([0],B') \ .$$

Hence the transition map $v_*\colon f/([n],B) \to f/([0],B')$ is a homotopy equivalence. It follows that if w denotes any of the maps $[0] \to [n]$ then $w_*\colon f/([0],B') \to f/([n],B)$, being a section of v_* , is a homotopy equivalence, too. In view of the assumption that each object is isomorphic to one which is 0-dimensional, we can therefore conclude that every left fibre is homotopy equivalent to one of the type $f/([0],B'')$, and that every transition map is homotopy equivalent to one of the type $([0],b)_*$ where b is a morphism in $\mathcal{B}_0$.

Proof of theorems A' *and* B'. We reduce to theorems A and B of Quillen. Namely to a simplicial category $\mathcal{A}$ one can associate a category $\mathrm{simp}(\mathcal{A})$ whose objects are the pairs $([m],A)$, $A \in \mathcal{A}_m$; the morphisms from $([m],A)$ to

$([m'],A')$ are the pairs (u,a) where $u\colon [m] \to [m']$ and $a\colon A \to u^*(A')$. To a map of simplicial categories $f\colon \mathcal{A} \to \mathcal{B}$ is associated a natural isomorphism

$$\mathrm{simp}(f) \,/\, ([n],B) \;\approx\; \mathrm{simp}(\, f/([n],B)\,) \,.$$

In view of this fact, theorems A' and B' follow at once from theorems A and B of Quillen [15] together with the following lemma.

Lemma. There is a natural chain of homotopy equivalences between a simplicial category $\mathcal{A}$ and the associated category $\mathrm{simp}(\mathcal{A})$.

A proof of this lemma may be found in the (unpublished) thesis of Thomason. (The published excerpt [20] contains a closely related result called there the *homotopy colimit theorem*. When specialized to the simplicial category $\mathcal{A}$, that theorem says that $\mathrm{simp}(\mathcal{A})$ is homotopy equivalent to a kind of barycentric subdivision of $\mathcal{A}$. It is not so difficult then to relate the latter to $\mathcal{A}$ itself; the requisite arguments may be found in the appendix to [18].)

5. Proof of theorem 1.

In the propositions to follow we will take up the parts of theorem 1, one after the other.

Proposition 5.1. The diagram

$$\begin{tikzcd} \mathcal{H}(X) \arrow[r] \arrow[d] & \mathcal{P}^m_k(X) \arrow[d] \\ h\mathcal{H}(X) \arrow[r] & h\mathcal{P}^m_k(X) \end{tikzcd}$$

is homotopy cartesian in the dimension range up to $q - 3$ where

$$q = \min(m,n) \quad , \quad n = \dim(X) - m \ .$$

When stabilized with respect to dimension, the diagram will remain homotopy cartesian, in that range, under the plus construction.

Note that in order to stabilize with respect to dimension, in the second part, one should really replace $\mathcal{P}(X)$ by $\underline{\mathcal{P}}(X)$ (section 1). In the DIFF case the stabilization in the sense of section 1 creates corners, so one should work with manifolds with general corners as in section 6. The plus construction does not create additional trouble, the diagram can be kept strictly commutative (of course the plus construction need be applied to the two terms on the right only since it would not alter the terms on the left anyway).

Proof of proposition. We begin by showing that theorem B' (the preceding section) applies, in its version for right fibres, to the inclusion map

$$j : \mathcal{P}^m_k(X) \longrightarrow h\mathcal{P}^m_k(X) \ .$$

The argument involves an application of the isotopy extension theorem (cf. [11] and [12] for the PL and TOP cases, respectively). We first check that although the addendum to theorem B' does not apply directly, still its conclusion holds true. An object of $h\mathcal{P}^m_k(X)$ in degree p is, by definition, a locally trivial family of partitions over the simplex Δ^p . Such a family is trivializable since Δ^p is contractible, but in general the trivializing isomorphism is not in the category (the only isomorphisms in $h\mathcal{P}^m_k(X)$ are the identity maps). Fortunately however the isotopy extension theorem tells us that there is a p-parameter family of automorphisms of $X \times I$ to trivialize the family. This family of automorphisms of $X \times I$ induces an isomorphism from the right fibre under consideration to the right fibre over some totally degenerate object. As in the addendum we can now conclude that it suffices to check the hypotheses of theorem B' in degree 0 only.

Since $\mathcal{P}_k^m(X)$ is really a simplicial set, not simplicial category, the same is true of the right fibre $([0],(M,F,N))/j$, say. Unravelling of the definitions shows it is the simplicial set of the partitions (M',F',N') having the property that $M \subset M'$ and that each of the inclusions

$$F \longrightarrow M' - (M-F) \longleftarrow F'$$

is a homotopy equivalence. Applying a general position argument in a collar neighborhood of F we obtain that this simplicial set contains as a deformation retract the simplicial subset of the partitions having the additional property that F' is disjoint to F ; the region between F and F' is thus an h-cobordism.

Fixing arbitrarily some collar neighborhood of F , we define a still smaller simplicial subset by insisting that F' should be contained in that collar neighborhood. This simplicial subset is a deformation retract, too, as we see by application of the theorem of uniqueness of collars (i.e. the theorem which says that the space of collars is contractible). We must apply the theorem twice, in fact. Namely h-cobordisms are invertible (in dimensions at least 5, say), so a first application of the uniqueness of collars theorem shows that it is a contractible choice to pick for each F' a collar containing it; a second application then moves this collar to the chosen one.

We conclude that $([0],(M,F,N))/j$ is homotopy equivalent to the h-cobordism space $\mathcal{H}(F)$.

To show that the transition map $([0],b)^*$ induced from a morphism $b\colon (M_0,F_0,N_0) \to (M_1,F_1,N_1)$ in $h\mathcal{P}_k^m(X)$ is a homotopy equivalence, it suffices to treat the case where F_0 and F_1 are disjoint (for if necessary we could find a suitable F_2 disjoint to both). Using the invertibility of h-cobordisms again, we can further reduce to the case where the h-cobordism between F_0 and F_1 is trivial. In this case the transition map corresponds, under the above homotopy equivalence, to the map $\mathcal{H}(F_1) \to \mathcal{H}(F_0)$ obtained by adding to each h-cobordism some fixed external collar from below. Certainly this map is a homotopy equivalence.

Having verified the hypotheses of theorem B' we conclude that for every (M,F,N) the square

$$\begin{tikzcd}
([0],(M,F,N)) \,/\, j \arrow[r] \arrow[d] & \mathcal{P}^m_k(X) \arrow[d] \\
([0],(M,F,N)) \,/\, \mathrm{Id}_{h\mathcal{P}^m_k(X)} \arrow[r] & h\mathcal{P}^m_k(X)
\end{tikzcd}$$

is homotopy cartesian. To obtain the first part of the proposition we compare this square with a corresponding one for the case $k = 0$. Assuming as we may that X has non-empty boundary we have a map from $\mathcal{H}(X) = \mathcal{P}^m_o(X)$ to $\mathcal{P}^m_k(X)$ which adds handles in some standard way near the boundary (if the boundary is empty it is still possible to apply essentially the same argument, by puncturing X first). Choose any pair of objects which correspond under this handle addition process, with frontiers $F_{(o)}$ and $F_{(k)}$, say. The corresponding homotopy cartesian squares are then mapped to each other, and the map of upper left terms is the same, up to homotopy, as the map

$$\mathcal{H}(F_{(o)}) \longrightarrow \mathcal{H}(F_{(k)})$$

induced from an inclusion $F_{(o)} \subset F_{(k)}$. This inclusion is (q-1)-connected where $q = \min(m,n)$, $n = \dim(X)-m$, because $F_{(k)}$ is a connected sum of $F_{(o)}$ with k copies of $S^m \times S^n$. A fundamental property of the concordance space functor now says that the functor essentially preserves connectivity [4]. So $\mathcal{H}(F_{(o)}) \to \mathcal{H}(F_{(k)})$ is (q-2)-connected (provided that $q \geq 3$). From this connectivity of the map of upper left terms, and the fact that the lower left terms are contractible, we conclude that the square formed by the right columns is homotopy cartesian in the range stated.

We are left to check the behaviour of the square under the plus construction. Let the right vertical map of the square be denoted $V \to W$, for short. Then $\mathrm{fibre}(V \to W)$ is (q-2)-equivalent to an H-space (namely $\mathcal{H}(X)$), and $\mathrm{fibre}(V^+ \to W^+)$ is an H-space (because $V^+ \to W^+$ is a map of infinite loop spaces; cf. the end of section 1). It suffices therefore to show that the map $\mathrm{fibre}(V \to W) \to \mathrm{fibre}(V^+ \to W^+)$ induces an isomorphism in homology (in the range stated). Now $V \to V^+$ and $W \to W^+$ are isomorphisms in homology. So we can draw the desired conclusion from the comparison theorem for spectral sequences provided we know that, for each of the fibrations, the fundamental group of the base acts trivially on the homology of the fibre (in that range). This is clear in the case of $V^+ \to W^+$. In the case of the map $V \to W$ we proceed as follows.

Returning to our earlier discussion we have a graph (namely the 1-skeleton of $h\mathcal{P}^m_k(X)$) and a local coefficient system over it (namely the homology of $\mathcal{H}(F)$) and we want to show that the local coefficient system is trivial in the range in question. To show this it will suffice to show that for *every* F (and not just certain assorted ones) we can define a (q-2)-connected map $\mathcal{H}(X) \to \mathcal{H}(F)$ in such a way that these maps are compatible with transition maps (up to homotopy, and in the range in question).

F is homotopy equivalent to the wedge of X with some spheres of dimensions m and n, and it comes equipped with a structural map to X (in view of the inclusion in $X\times I$). We may assume that X is homotopy equivalent to a complex of dimension m-2 (for if necessary we could have replaced X by a neighborhood of an (m-2)-skeleton). As n may be assumed to be very large anyway, we obtain that the map $F \to X$ has a section (up to homotopy) and the section itself is unique up to homotopy. But it is the stable case which we are discussing here, so the homotopy class of maps is represented by a unique isotopy class of codimension 0 embeddings (the stable tangent bundle of F pulls back from X, using immersion theory therefore the homotopy class lifts to a unique regular homotopy class; that in turn lifts to a unique isotopy class by general position). It results that we obtain a map $\mathcal{H}(X) \to \mathcal{H}(F)$ which is unique up to homotopy. We conclude with the remark that the uniqueness of the map automatically implies that its construction is compatible with transition maps. □

For the purposes of the next proposition it will be useful to know how the simplicial category $\underline{\mathcal{P}}(X)$ relates to other simplicial categories of manifolds. Recall that an object in $\underline{\mathcal{P}}(X)$ is a partition (M,F,N) (resp. parametrized family of such) subject to the technical condition that M contain a certain standard part $X\times[a,a']$ of $X\times[a,b]$ and that, in a specified neighborhood of $\partial X\times[a,b]$, M is not bigger than that standard part.

We define $\mathcal{M}(X)$ to be the simplicial category of the manifolds M (resp. parametrized families of such) which arise when a partition (M,F,N) is stripped of everything but M as an abstract manifold containing $X\times[a,a']$. In other words, an object of $\mathcal{M}(X)$ is a manifold M (resp. parametrized family of such) containing $X\times[a,a']$, subject to the condition that there exists at least one embedding of M in $X\times[a,b]$ (rel. $X\times[a,a']$) making it the M-part of some partition (M,F,N) in $\underline{\mathcal{P}}(X)$. The morphisms in $\mathcal{M}(X)$ are the embeddings (resp. parametrized families of such) restricting to the identity map on $X\times[a,a']$; the prefix h will be used to denote the subcategory of those which are homotopy equivalences. In analogy to our earlier notation, $h\mathcal{M}^m_k(X)$ will denote the connected component containing a handlebody of type (m,k), and $\mathcal{M}^m_k(X)$ will denote its simplicial set of objects.

Finally suppose X is a *euclidean manifold*, that is, it is a codimension 0 submanifold of euclidean space R^d , $d = \dim(X)$. In this case we can define a simplicial category of euclidean manifolds $EM(X)$ which is about halfway between $\underline{P}(X)$ and $M(X)$. An object is a codimension 0 submanifold M of $R^d\times[a,b]$ containing $X\times[a,a']$ (resp. a parametrized family of such). As before it is convenient to ask the technical condition that, in a specified neighborhood of ∂X , M be not bigger than $X\times[a,a']$; and we also ask that M be disjoint to $(R^d-X)\times[a,a']$. Morphisms in $EM(X)$ are given by inclusion, and the prefix h singles out those which are homotopy equivalences. $hEM_k^m(X)$ denotes a certain connected component, and $EM_k^m(X)$ its simplicial set of objects.

We are interested in the forgetful maps between these simplicial categories in the stable case, that is, the maps

$$\lim_{\vec{n}} h\underline{P}_k^m(X\times J^n) \xrightarrow{f} \lim_{\vec{n}} hEM_k^m(X\times J^n) \xrightarrow{g} \lim_{\vec{n}} hM_k^m(X\times J^n) .$$

We denote by f' and g' the maps of simplicial sets obtained by restricting f and g , respectively, to objects. If $M \in M_k^m(X)$ we denote by

$$\mathrm{fibre}(g,M)$$

the homotopy fibre of g at M .

Lemma 5.2. Let X be a euclidean manifold, and $M \in M_k^m(X)$. There are homotopy equivalences

$$\begin{aligned} \mathrm{fibre}(g,M) &\simeq \mathrm{fibre}(g',M) \\ &\simeq \text{space of stable framings of } M \text{ (rel. } X\text{)} , \end{aligned}$$

and

$$\begin{aligned} \mathrm{fibre}(gf,M) &\simeq \mathrm{fibre}(g'f',M) \\ &\simeq \text{product of this space of framings with} \\ &\qquad \text{the space of retractions } M \to X . \end{aligned}$$

Proof. We treat only the case of the map gf . The three other cases result by straightforward modification and some omission of detail.

We apply theorem B' (the preceding section) in its version for right fibres. The addendum does not apply directly (because a locally trivial family of M's over Δ^p is not necessarily trivializable relative to the constant family $X\times[a,b]$) but as in the proof of the preceding proposition we can still justify the conclusion of the addendum by an appeal to the isotopy extension theorem.

Let $M \in h\mathcal{M}_k^m(X \times J^n)$ represent an object of degree 0 in the direct limit on the right. Given an object of the right fibre $([0],M)/gf$ then, after enlarging of n if necessary, the object is represented by a partition (M',F',N') of $h\underline{P}_k^m(X \times J^n)$ (resp. parametrized family of such) together with an embedding $i\colon M \to M'$ (rel. $X \times [a,a']$) which is a homotopy equivalence.

Now stabilization kills knotting phenomena. After enlarging n some more, if necessary, we can therefore assume firstly that the image $i(M)$ defines a partition in $h\underline{P}_k^m(X \times J^n)$, and secondly that the inclusion $i(M) \to M'$ defines a morphism in $h\underline{P}_k^m(X \times J^n)$.

By taking $(M',F',N';i)$ to $(i(M),\ldots;\mathrm{Id})$ we can thus define a functor of $([0],M)/gf$ into a certain subcategory, and the latter is a deformation retract in view of the homotopy given by the natural transformation $i(M) \to M'$.

But the simplicial category in question is really a simplicial set, namely the simplicial set of embeddings $M \to X \times [a,b]$ (rel. $X \times [a,a']$). Actually there is a technical condition here, namely the image of the embedding should be contained in a certain subspace $X \times [a,a'] \cup X' \times [a,b']$ (cf. section 1). A more than technical condition is also required at first glance, namely that the image of the embedding define a partition; however, as noted before, this unknotting condition is automatically satisfied because of stabilization.

Stably, the space of embeddings $M \to X \times [a,b]$ may be replaced by the space of immersions which in turn, in view of the Smale-Hirsch theorem and its PL and TOP analogues, may be replaced by the space of tangential maps. By using the canonical framing of $X \times [a,b]$ we can decompose the latter space as the product of the two spaces given in the lemma.

From this translation, up to homotopy, of right fibres $([0],M)/gf$ it is immediate that all transition maps are homotopy equivalences. For any map resulting from restricting a mapping space to a deformation retract is a homotopy equivalence. Thus theorem B' applies, and the right fibre represents the homotopy fibre, as claimed. □

Let $\mathcal{R}_k^m(X)$ be the category of *retractive spaces over* X of type (m,k). An object is a triple (Y,r,s) where $r\colon Y \to X$ is a retraction with section s , and Y is a topological space which can be obtained, up to homotopy equivalence relative to the subspace $s(X)$, by the attaching of k m-cells to X . Let the prefix h denote the subcategory of the (weak) homotopy equivalences.

We shall admit that $h\mathcal{R}_k^m(X)$ is an approximation to $A(X)$: a connected component of $A(X)$ can be obtained by the plus construction on

$$\lim_{\substack{\rightarrow \\ m,k}} h\mathcal{R}^m_k(X) \ ,$$

equivalently $A(X)$ arises as the loop space of the classifying space

$$N_\Gamma(\lim h\mathcal{R}^m(X))$$

where

$$h\mathcal{R}^m(X) \;=\; \coprod_{k\geqslant 0} \; h\mathcal{R}^m_k(X)$$

and where the composition law is given by gluing at X. The numerical value for this approximation mentioned earlier, amounts to the fact that the map $h\mathcal{R}^m_k(X) \to h\mathcal{R}^{m+1}_k(X)$ is $(m-1)$-connected. To see this one can, say, translate into the simplicial monoids used in [22], and the connectivity then results from the Freudenthal suspension theorem.

In order to define a forgetful map from $h\underline{\mathcal{P}}^m_k(X)$ to $h\mathcal{R}^m_k(X)$ we must first replace the latter by a suitable simplicial category. Such a simplicial category may be obtained from $h\mathcal{R}^m_k(X)$ by admitting parametrized families (over simplices) which are locally fibre homotopy trivial; we denote it by $h\mathcal{R}^m_k(X)_\bullet$.

<u>Lemma</u> 5.3. The inclusion $h\mathcal{R}^m_k(X) \to h\mathcal{R}^m_k(X)_\bullet$ is a homotopy equivalence.

Proof. It will suffice to show that, for every p , the (degeneracy) map

$$d\colon h\mathcal{R}^m_k(X) \longrightarrow h\mathcal{R}^m_k(X)_p$$

is a homotopy equivalence. Let f be the map in the other direction obtained by restricting to the last vertex of Δ^p . Then $d\,f$ is an identity map, so it will suffice to show that $f\,d$ is homotopic to the identity map on $h\mathcal{R}^m_k(X)_p$. The required homotopy will be represented by a functor from $h\mathcal{R}^m_k(X)_p$ to itself together with two natural transformations to this functor: one from the identity map, and one from the map $f\,d$. To obtain the functor, let $\Delta^p\times\Delta^1 \to \Delta^p$ be a homotopy from the identity map on Δ^p to the projection into the last vertex. Pullback with this map takes any parametrized family over Δ^p to one over $\Delta^p\times\Delta^1$. The latter family may be regarded, in turn, as one over Δ^p (it is necessary at this point that we are working with fibre-homotopy-triviality rather than just local triviality). This defines the functor. The two natural transformations are induced by the two maps $\Delta^p \to \Delta^p\times\Delta^1$. □

<u>Proposition</u> 5.4. The maps

$$\begin{array}{ccccc} \lim\limits_{\rightarrow n} h\underline{\mathcal{P}}^m_k(X\times J^n) & \longrightarrow & \lim\limits_{\rightarrow n} h\mathcal{R}^m_k(X\times J^n)_\bullet & \longleftarrow & h\mathcal{R}^m_k(X) \\ (M,F,N) & \longmapsto & M & & \end{array}$$

are homotopy equivalences.

Proof. The map on the right is included here for book-keeping only. To show it is a homotopy equivalence, it suffices to show that the map $g: hR_k^m(X) \to hR_k^m(X\times J)$ given by product with the interval J is a homotopy equivalence (actually the stabilization process uses a technical modification of the map g, cf. the discussion of stabilization in section 1). We admit that the map $j: hR_k^m(X) \to hR_k^m(X\times J)$ given by pushout with an inclusion $X \to X\times J$ is a homotopy equivalence. But there is a natural transformation from j to g, namely $Y\cup_X X\times J \to Y\times J$. Therefore g is homotopic to j and thus a homotopy equivalence, too.

To handle the map on the left, we first reduce to working with euclidean manifolds. There exists a closed disk bundle over X whose total space X' is a euclidean manifold (to obtain X' embed X in high-dimensional euclidean space and take a tubular, resp. regular, neighborhood; in the PL case this needs the Haefliger-Wall theorem [8] on the existence of stable normal disk bundles; similarly in the TOP case [12]). Pullback with the disk bundle (or rather a technical modification, as in the definition of the stabilization map) defines a map $\varinjlim hP_{\underline{k}}^m(X\times J^n) \to \varinjlim hP_{\underline{k}}^m(X'\times J^n)$, and this map is a homotopy equivalence in view of the fact that disk bundles admit inverses.

In proving the proposition there is therefore no loss of generality if we assume that X itself is a euclidean manifold. We assume this from now on. The simplicial category $\varinjlim hEM_k^m(X\times J^n)$ is then defined.

Let $hS_k^m(X)$ denote the simplicial category obtained by stripping the objects (Y,r,s) of $hR_k^m(X)$. of their structural retractions r. An object of $hS_k^m(X)$ is thus a pair (Y,s) (resp. a parametrized family subject to a condition of local fibre homotopy triviality). An argument with right fibres, as in the proof of lemma 5.2, shows that the homotopy fibre at (Y,s) of the forgetful map $hR_k^m(X) \to hS_k^m(X)$ is homotopy equivalent to the space of retractions $Y \to X$. On comparing with lemma 5.2 we obtain that the homotopy fibres of the vertical maps in the diagram

$$\begin{array}{ccc}
\lim\limits_{\vec{n}} hP_{\underline{k}}^m(X\times J^n) & \longrightarrow & \lim\limits_{\vec{n}} hR_k^m(X\times J^n). \\
\big\downarrow & & \big\downarrow \\
\lim\limits_{\vec{n}} hEM_k^m(X\times J^n) & \longrightarrow & \lim\limits_{\vec{n}} hS_k^m(X\times J^n)
\end{array}$$

are mapped to each other by homotopy equivalence. Hence the upper horizontal map in the diagram will be a homotopy equivalence if and only if the lower horizontal map is. We have thus reduced to showing that the forgetful map

$$\lim_{\vec{n}} hEM_k^m(X\times J^n) \longrightarrow \lim_{\vec{n}} hS_k^m(X\times J^n)$$

is a homotopy equivalence.

Call this map f. By theorem A' (the preceding section) f will be a homotopy equivalence if for every object of the simplicial category on the right, the right fibre over this object is contractible. We will show this.

Let such an object be represented by a family (Y,s) in $hS_k^m(X\times J^n)_q$, say. An object of $([q],(Y,s))/f$ is then a tuple

$$u\colon [p] \to [q] \quad , \quad M \quad , \quad \Delta^p \times_{\Delta^q} Y \xrightarrow{\ t\ } M$$

where $M \in EM_k^m(X\times J^n)_p$ (perhaps after enlarging of n) and t is a map of p-parameter families, and in fact a homotopy equivalence.

We note at this point that we are free to deform the structure maps t; it is for this purpose that it was necessary to get rid of the structural retractions $Y \to X$, i.e. to replace $hR_k^m(X)$. by $hS_k^m(X)$.

As a first application of this remark, let us suppose that Y' is a deformation retract of Y, as q-parameter family. The deformation induces a deformation of the structural maps t which, in turn, we may re-interpret as a simplicial homotopy of the identity map on $([q],(Y,s))/f$. This shows that the latter contains $([q],(Y',s'))/f$ as a deformation retract.

But the category $hS_k^m(X\times J^n)_q$ is connected and (what is slightly stronger) any two objects can be related by a chain in which consecutive members are related by deformation retraction. It follows that the contractibility of $([q],(Y,s))/f$ need only be checked in the case of a single object.

As we are free to choose this object as we please we may in particular assume that it is totally degenerate. The addendum to theorem A' therefore applies, reducing our checking to degree 0. We conclude that it is enough to pick a single object (Y,s), in degree 0, and show the right fibre over this object is contractible.

Again we can pick the object (Y,s) as we please. In particular we can pick it as an object of $hEM_k^m(X)$. There is a little trick here. Namely we can focus attention to things far out in the direct system $n \mapsto hEM_k^m(X\times J^n)$, which amounts to working with manifolds of very large dimension. But still (Y,s) can be in $hEM_k^m(X)$, that is, a manifold of small dimension.

By a general position homotopy it can therefore be achieved that the structural maps are embeddings in a certain range of parameters (depending on how far out the above n has been chosen); also it can be assumed that the images of the structure maps are disjoint to the frontiers of the manifolds.

If, after this step, we compose a structural map $Y\times\Delta^p \to M$ with the inclusion $M \subset R^{d+n}\times[a,b]\times\Delta^p$ we obtain a p-parameter family of embeddings of Y in $R^{d+n}\times[a,b]$ (rel. $X\times[a,a']$). But stably the space of these embeddings is contractible. In a stable range we can therefore deform into the natural inclusion of Y (by assumption Y is a submanifold of $R^d\times[a,b]$). The deformation can be done by ambient isotopy, in view of the isotopy extension theorem. It induces therefore a further deformation retraction of our simplicial category.

With the latter deformation we have achieved that, in a stable range of dimensions, the structure maps $Y\times\Delta^p \to M$ are precisely *the same* as the natural inclusions. The simplicial subcategory we have deformed into now admits the following description: in a stable range it is the full simplicial subcategory of $h\mathcal{EM}^m_k(X\times J^n)$ of the partitions having the property that M contains Y, the image of Y misses the frontier, and the inclusion of Y is a homotopy equivalence.

By thickening Y a little we can obtain an object of $h\mathcal{EM}^m_k(X\times J^n)$. We conclude that (in an appropriate range of parameters $p \leqslant p'$, say) any finite subcategory $\mathcal{C}$ of the category in degree p may be equipped with an initial object. Thus the inclusion of $\mathcal{C}$ into the category in degree p is nullhomotopic. This is true for every finite $\mathcal{C}$, so the category in degree p is contractible. It results that our simplicial category $([0],(Y,s))/f$ is p'-connected. But p' is as large as we please, so the simplicial category is contractible. We are done. □

Let us put

$$h^{CAT}(X)^m = \Omega\, N_\Gamma(\lim_{\vec{n}} \underline{\mathcal{P}}^m(X\times J^n)) .$$

It is a functor on the category of codimension 0 embeddings of CAT manifolds. The functor respects homotopy equivalences (in a stable range); this results, for example, from proposition 5.1 together with the fact that the other functors there have this property.

<u>Proposition</u> 5.5. The functor $h^{CAT}(X)^m$ has the excision property in a stable range of dimensions up to $m-c$ (here c is a constant, about 3 or 4).

As pointed out in the introduction, the proof uses many things. First the TOP and DIFF cases are reduced to the PL case by using results from triangulation theory and smoothing theory, respectively. The proof in the PL case then uses all of the results of [23] and almost all of those in [24].

Here is an outline of the argument.

It is known [2] that for PL manifolds X the forgetful map from $Wh^{PL}(X)$ to $Wh^{TOP}(X)$ is a homotopy equivalence. By using propositions 5.1 and 5.4 in the PL and TOP cases, and comparing the diagrams of 5.1, it results then that (in a stable range) the forgetful map from $h^{PL}(X)^m$ to $h^{TOP}(X)^m$ is a homotopy equivalence, too.

Similarly it is known [3] that, for DIFF manifolds X , the functor

$$\text{fibre}(\ Wh^{DIFF}(X) \to Wh^{TOP}(X)\ )$$

has the excision property. By using propositions 5.1 and 5.4 in the DIFF and TOP cases, and comparing the diagrams of 5.1, it follows then that (in a stable range) the functor

$$\text{fibre}(\ h^{DIFF}(X)^m \to h^{TOP}(X)^m\ )$$

has the excision property, too. It results that, in a stable range, $h^{DIFF}(X)^m$ will have the excision property as soon as $h^{TOP}(X)^m$ has.

We have thus reduced to the PL case.

This case, too, is handled very indirectly. The first step is to translate into non-manifolds. Let $\mathcal{R}(X).$ denote the simplicial category of the retractive spaces (Y,r,s) considered above (the proof of proposition 5.4). But let us switch here to the PL viewpoint, that is, we suppose Y is a compact polyhedron (resp. locally trivial parametrized family of such) and r and s are PL maps. Let a *simple map* in $\mathcal{R}(X).$ denote a map $(Y,r,s) \to (Y',r',s')$ having the property that $Y \to Y'$ has contractible point inverses, and let $s\mathcal{R}(X).$ denote the simplicial category of the simple maps. By taking a PL submanifold of $X \times I$ to its underlying polyhedron, one obtains an inclusion map from the simplicial set $\underline{P}(X)$ to the simplicial category $s\mathcal{R}(X)$, and one shows that stably there results a homotopy equivalence

$$\lim_{\vec{n}} \underline{P}(X \times J^n) \longrightarrow \lim_{\vec{n}} s\mathcal{R}(X \times J^n). \ .$$

This is very closely related to the main result of Hatcher [9]. A proof independent of Hatcher's is given in [23]. By restricting the homotopy equivalence to a union of connected components one obtains a homotopy equivalence

$$\lim_{\vec{n}} \underline{P}^m_k(X \times J^n) \longrightarrow \lim_{\vec{n}} s\mathcal{R}^m_k(X \times J^n). \ ,$$

from which in turn one obtains another

$$\Omega\ N_\Gamma(\lim_{\vec{n}} \underline{P}^m(X \times J^n)) \longrightarrow \Omega\ N_\Gamma(\lim_{\vec{n}} s\mathcal{R}^m(X \times J^n).) \ .$$

Note that the direct limit on the right serves only a book-keeping function. It is very easy to get rid of.

Next, it is necessary to get out of the polyhedral framework (because the category of polyhedra does not admit quotient space constructions). Leaving aside some technicalities with the retractions (in essence they are to be gotten rid of, as in the beginning of the proof of 5.4), this involves two steps. The first step is to pass from polyhedra to triangulated polyhedra (this step is surprisingly difficult), the second step is to admit mild singularities now, i.e. to pass from triangulated polyhedra to simplicial sets (this step is not so difficult). These matters are dealt with in [23], too.

Let us keep the notation $\mathcal{R}(X).$, but let it be understood that we are dealing with simplicial sets now. Then a certain bisimplicial category $sS.\mathcal{R}(X).$ is defined [21], and it is possible to show that this *does* satisfy the excision property [24]. Leaving aside some technicalities with suspensions, one can define a natural map

$$N_\Gamma(\lim_{\vec{m}} s\mathcal{R}^m(X).) \longrightarrow sS.\mathcal{R}(X). \;,$$

so it will suffice to show this is a homotopy equivalence.

This is proved indirectly again. Namely one does not just translate $h^{PL}(X)^m$, as we have been doing up to now, one translates in effect the whole diagram of proposition 5.1. By stabilizing, and de-looping, one obtains in this way a homotopy cartesian square

$$\begin{tikzcd} N_\Gamma(s\mathcal{R}^h(X).) \arrow[r] \arrow[d] & N_\Gamma(\lim_{\vec{m}} s\mathcal{R}^m(X).) \arrow[d] \\ N_\Gamma(h\mathcal{R}^h(X).) \arrow[r] & N_\Gamma(\lim_{\vec{m}} h\mathcal{R}^m(X).) \end{tikzcd}$$

where the superscript h on the left indicates that we are dealing with a certain bisimplicial subcategory, the condition on (Y,r,s) is that $s\colon X \to Y$ should be a weak homotopy equivalence. For general reasons on the other hand [21] the square

$$\begin{tikzcd} sS.\mathcal{R}^h(X). \arrow[r] \arrow[d] & sS.\mathcal{R}(X). \arrow[d] \\ hS.\mathcal{R}^h(X). \arrow[r] & hS.\mathcal{R}(X). \end{tikzcd}$$

is also homotopy cartesian, and the former square maps to it. To show the map of upper right terms is a homotopy equivalence it will therefore suffice to show that each of the other three maps is a homotopy equivalence. This is trivial in one case (the lower left terms are contractible) but it is certainly not trivial in the other two cases.

In fact, that the map

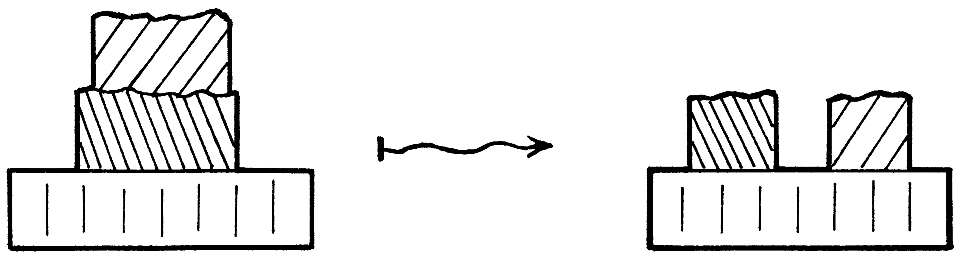
$$N_\Gamma(\lim_{\vec{m}} hR^m(X).) \longrightarrow hS.R(X).$$

is a homotopy equivalence, is really the main result of [24], and most of the material in that paper is used to prove it.

The case of the map

$$N_\Gamma(sR^h(X).) \longrightarrow sS.R^h(X).$$

on the other hand is much easier. In this case one shows directly that cofibration sequences may be moved to split ones [24]. This is closely related to a geometric fact concerning h-cobordisms in the stable case, namely that a *composition* of such may be moved to a *sum*.



6. Appendix: Smooth manifolds with general corners.

Since some of the constructions used in this paper take one out of the category of smooth manifolds, it is necessary to either modify the constructions or else enlarge the category. The latter seems to be the lesser evil, and the purpose of this section is to describe a simple way of doing it.

The method is really well known, it is closely related to the usual process of *smoothing the corners* [5]. Whitehead has used a more elaborate version of the same method, with different aims [25].

What we achieve is that the spaces of smooth manifolds of interest to us may be blown up to larger spaces involving manifolds which are by no means smooth anymore, but the enlarged spaces are still homotopy equivalent to the original spaces. After explaining this we indicate how our earlier constructions are to be adapted, if any.

Let Y be a smooth manifold (think of it as the $X \times I$ of section 1), and let F be a topological submanifold of Y of codimension 1 (think of it as the frontier of a partition). Let $x \in F$ and let v be a non-zero tangent vector of Y at x. We say that v is *normal to* F if for one (and hence for every) smooth chart of Y around x the following is true: there exist constants $c > 0$ and $C > 0$ so that for all s with $|s| \leq C$ the distance function satisfies a Lipschitz inequality

$$d(x+sv, F) \geq c\,|s|\ ;$$

in other words, near x the points on the line in direction v stay well away from F.

Remark. Whitehead [25] uses a slightly more restrictive notion: a normal vector in his sense is to be normal still when translated to nearby points of F. We do not ask this condition here (we could, though).

By a *smooth normal field to* F will be meant a smooth vector field on Y (all of it, not just a neighborhood of F) so that for every $x \in F$ the vector at x is non-zero, and normal to F in the above sense.

Example. Suppose that F is a smooth submanifold of Y, and suppose it is normally oriented. Then the smooth normal fields (in the above sense) subordinate to the given normal orientation, form a convex set. In particular the space of such vector fields is contractible.

Let a *normalized submanifold of* Y denote a pair consisting of a topological submanifold F of codimension 1 in Y, and a smooth normal field to F in the sense just defined. There is a notion of locally trivial family of such data (the vector field must vary smoothly), so we obtain a space (i.e. simplicial set).

Remark. If we were using a more restrictive notion of normal field, such as Whitehead's, we could relax the notion of normalized submanifold a little. Namely it would be enough to ask the continuity of the vector field rather than its smoothness since it would be possible to approximate a continuous normal field by a smooth one.

We say that a normalized submanifold is *smooth* if the underlying manifold F is a smooth submanifold of Y. The forgetful map which takes a smooth normalized submanifold to the underlying normally oriented manifold F, is a homotopy equivalence; this is a slightly more elaborate version of the remark in the above example.

We now show how a normalized submanifold may be moved to one which is smooth. We shall ignore, for simplicity, that F may have a boundary. We pick for each point $x \in F$ a smoothly embedded disk D_x in Y of the same dimension as F which has x as its center and which is transverse to the vector v_x. If D_x is sufficiently small it is transverse everywhere to the vector field, so we can construct a flowbox B_x by integrating to distance ε_x in either direction. After restricting ε_x and D_x some more, if necessary, we can assume that the flowbox B_x is transverse to F, that is, every flowline in it meets F exactly once (in an interior point).

By projecting D_x along the flow we obtain an embedding $p_x \colon D_x \to F$. Given two charts as this then, if D' denotes the overlap $p_x(D_x) \cap p_y(D_y)$, and $D = p_x^{-1}(D')$, we obtain two functions on D, namely the bijection $D \to p_y^{-1}(D')$ on the one hand, and the real-valued function on the other which measures the distance of D_x to D_y along the flow. Both these functions are smooth. In the first case this means that the (D_x, p_x) define an atlas of a smooth structure on F. In the second case we can conclude that, by tapering off such distance functions to zero, we can construct a smooth embedding of that particular smooth manifold near F.

Given two smooth embeddings obtained in this fashion we can measure their distance along the flow. By restricting to embeddings sufficiently close to the original topological embedding (to avoid pathologies arising from moving too far) and by selecting one of the embeddings as a basepoint, we can thus establish a bijection of the set of embeddings to a convex set of real-valued smooth functions. Since the construction is also compatible with parameters, we conclude that *the space of normalized submanifolds contains as a deformation retract the subspace of those which are smooth.*

Here is a summary of how the constructions of section 1 may be formulated in the DIFF case. Let a *partition* mean a triple (M,F,N) as before, but it is to be understood now that its frontier F is a normalized submanifold of $X\times I$ in the sense discussed above. In other words, we include as additional data a smooth vector field on $X\times I$, and we suppose it is normal to F in the above sense (as a technical condition we have to ask that the vector field is standard in some neighborhood of $\partial X\times I$, that is, equal there to the velocity vector field of the I coordinate). In speaking of the *partial ordering* which underlies the construction of the simplicial category $h\mathcal{P}(X)$ we insist that, for any two partitions to be related at all, it is necessary that the associated vector fields are the same. To define the *lower stabilization map* $\underline{\sigma}$ we assume that the interval J is equipped with a smooth vector field which is outward normal at the boundary of the subinterval J' and which is zero near the boundary of J. The map $\underline{\sigma}$ is defined just as before, the requisite vector field on $X\times J\times I$ is obtained by adding the vector fields on the $X\times I$ and J factors. A further condition is needed on the objects of $\underline{\mathcal{P}}(X)$ in order for $\underline{\sigma}$ to make sense, namely at the points of the slice $X\times a'$ the vector field must be pointing upward in the I-direction. In connection with the *composition law*, finally, we include in the definition of when X_1 is a support of (M,F,N) that the vector field should be standard outside of X_1.

As the target of the derivative map, in sections 2 and 3, we used the space Q^d of germs of normally oriented d-planes in R^{d+1}. Such a thing here means the germ of a d-plane together with an (everywhere defined) vector field normal to it. Equivalently, if the germ is at x, the data consist of a smooth vector field which is non-zero at x together with the germ of a transversal to the associated flow (a topological transversal, that is; with a Lipschitz condition). By convexity arguments we see that we may forget first the plane and then all of the vector field except for its value at x. This gives a homotopy equivalence $Q^d \to S^d$. Under this homotopy equivalence the two nullhomotopies of the map $Q^d \to Q^{d+1}$ in section 2 correspond to the two nullhomotopies induced from the two inclusions $S^d \to D_+^{d+1}$ and $S^d \to D_-^{d+1}$. In fact, what we did amounts to taking the image of a vector of R^{d+1} and tilt it either to left or right in $R^{d+1}\times R^1$, thus moving it to the velocity vector of the added R^1 factor or to its inverse, respectively.

References.

1. A. Borel, *Stable and L^2-cohomology of arithmetic groups*, Bull. A.M.S. 3 (1980), 1025-1027.

2. D. Burghelea and R. Lashof, *The homotopy type of the space of diffeomorphisms*. I, II, Trans. A.M.S. 196 (1974), 1-50.

3. ————, *Stability of concordances and the suspension homomorphism*, Ann. of Math. 105 (1977), 449-472.

4. D. Burghelea, R. Lashof, and M. Rothenberg, *Groups of automorphisms of manifolds*, Springer Lecture Notes in Math. 473 (1975).

5. A. Douady et L. Hérault, *Arrondissement des variétés à coins*, Comm. Math. Helv. 48 (1973), 484-489.

6. W.G. Dwyer, *Twisted homological stability for general linear groups*, Ann. of Math. 111 (1980), 239-251.

7. F.T. Farrell and W.C. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of disks, spheres, and aspherical manifolds*, Proc. Symp. Pure Math. vol. 32, part I, A.M.S. (1978), 325-337.

8. A. Haefliger and C.T.C. Wall, *Piecewise linear bundles in the stable range*, Topology 4 (1965), 209-214.

9. A. Hatcher, *Higher simple homotopy theory*, Ann. of Math. 102 (1975), 101-137.

10. ————, *Concordance spaces, higher simple homotopy theory, and applications*, Proc. Symp. Pure Math. vol. 32, part I, A.M.S. (1978), 3-21.

11. J.F.P. Hudson, *Piecewise linear topology*, Benjamin, New York, 1969.

12. R.C. Kirby and L.C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Math. Studies, Nr. 88 (1977).

13. N.H. Kuiper and R.K. Lashof, *Microbundles and bundles*. II, Invent. math. 1 (1966), 243-259.

14. D.G. Quillen, *Cohomology of groups*, Actes, Congrès Intern. Math. 1970, t. 2, 47-51.

15. ————, *Higher algebraic K-theory*. I, Springer Lecture Notes in Math. 341 (1973), 85-147.

16. ————, (Letter to Milnor), Springer Lecture Notes in Math. 551 (1976), 182-188.

17. G. Segal, *Configuration spaces and iterated loop spaces*, Invent. math. 21 (1973), 213-221.

18. ————, *Categories and cohomology theories*, Topology 13 (1974), 293-312.

19. C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. math. 55 (1979), 251-295.

20. R.W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Camb. Phil. Soc. 85 (1979), 91-109.

21. F. Waldhausen, *Algebraic K-theory of topological spaces*. I, Proc. Symp. Pure Math. vol. 32, part I, A.M.S. (1978), 35-60.

22. ————, *Algebraic K-theory of topological spaces*. II, Springer Lecture Notes in Math. 763 (1979), 356-394.

23. ————, *Spaces of PL manifolds and categories of simple maps*, to appear.

24. ————, *Algebraic K-theory of spaces*, to appear.

25. J.H.C. Whitehead, *Manifolds with transverse fields in euclidean space*, Ann. of Math. 73 (1961), 154-212; Collected works, vol. IV, Pergamon Press (1962).

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