ALGEBRAIC K-THEORY OF SPACES, LOCALIZATION, AND THE CHROMATIC FILTRATION OF STABLE HOMOTOPY.

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This paper represents a first step in applying localization techniques to the computation of the algebraic K-theory of spaces, and in particular to the task of reducing that computation to the computation of the algebraic K-theory of rings.

In order not to obscure the essential points by great generality we shall restrict ourselves to the special case of the space $A(*)$, the algebraic K-theory of a point. What we would like to do is to reduce the computation of $A(*)$ to that of $K(Z)$, the algebraic K-theory of the ring of integers, and in particular to compute $\mathrm{fibre}(A(*) \to K(Z))$, the homotopy fibre of the natural map.

That task is not easy. For, as will be explained in an appendix, it follows from the Lichtenbaum-Quillen conjecture (which is regarded as rather respectable among experts in the algebraic K-theory of rings) that $\mathrm{fibre}(A(*) \to K(Z))$ must in some way or other account for all of that formidable object, the cokernel of J .

Here is an outline of what is done in this paper. The space $A(*)$ may be constructed according to a certain recipe out of the category of pointed spaces of finite (homotopy) type; alternatively one could use spectra of finite type for the purpose (these matters are explained in section 1 below). The recipe is fairly general and can be applied in the same way to other categories of spaces or spectra. In particular if p is a prime, the recipe can be applied to the category of p-local spectra of finite type.

Let us denote the result of this construction by $A(*,p)$. Let $Z_{(p)}$ denote the ring of integers localized at p . There is a natural map

$$A(*,p) \longrightarrow K(Z_{(p)})$$

and we shall show that its homotopy fibre may be identified to the p-local part of $\mathrm{fibre}(A(*) \to K(Z))$. In this sense the task of computing the latter has been broken up into its p-local parts now.

In contradistinction to what one might expect by analogy with the algebraic K-theory of the ring Z , it is possible here to continue fracturing by localization methods. This is where the *chromatic filtration* comes in (there is one such for each prime p). By definition, the chromatic filtration is a particular sequence of localization functors in stable homotopy. The characteristic feature of these localization functors, as opposed to localization functors in general, is that they may be defined in terms of *acyclic spaces of finite type* (these matters are explained in section 2 below). The existence of the sequence is still conjectural beyond the first few terms; the relevant conjectures are due to Bousfield and Ravenel.

As will be explained (in section 3) the existence of the chromatic filtration implies the existence of a *localization tower* (whose maps are induced by localization functors)

$$A(*,p) = A(*,p,\infty) \longrightarrow \ldots \longrightarrow A(*,p,2) \longrightarrow A(*,p,1) \longrightarrow A(*,p,0) .$$

The bottom term $A(*,p,0)$ turns out to be the same (up to homotopy) as $K(Q)$, the algebraic K-theory of the ring of rational numbers; the next term $A(*,p,1)$ is in some sense the algebraic K-theory of the *non-connective* J (image-of-J-theory at the prime p). The layers of the tower (the homotopy fibres of the maps of consecutive terms) represent the contributions of what in Ravenel's terminology are the *monochromatic* phenomena in stable homotopy theory.

There is a second tower associated to the chromatic filtration, an *integral* (or *connective*) analogue of the former tower,

$$A(*,p) = \tilde{A}(*,p,\infty) \longrightarrow \ldots \longrightarrow \tilde{A}(*,p,2) \longrightarrow \tilde{A}(*,p,1) \longrightarrow \tilde{A}(*,p,0) .$$

The bottom term $\tilde{A}(*,p,0)$ here is $K(Z_{(p)})$, the algebraic K-theory of the ring of p-local integers, and the next term $\tilde{A}(*,p,1)$ is the algebraic K-theory of the *connective* J . The construction of the spaces $\tilde{A}(*,p,n)$ is very much like that of the algebraic K-theory of rings in the framework of the plus construction. This means that a certain amount of explicit computation is possible in low degrees. There does not however seem to exist a direct description of the layers in the tower. This suggests to try reducing to the former tower in order to obtain information about the layers.

There is a natural transformation $\tilde{A}(*,p,n) \to A(*,p,n)$. Modulo certain technical assumptions we can give an explicit description of the fibre, by localization methods again. For $n = 0$ the map is the natural map $K(Z_{(p)}) \to K(Q)$, and in that case our description of the fibre reduces to a case of Quillen's localization theorem.

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1. Review of algebraic K-theory.

We recall the definition of $A(*)$ from the category viewpoint [14], [5], [16]. Let C be the category of pointed spaces of finite type, that is, pointed spaces having the homotopy type of a finite CW complex (as a technical point, C is not a 'small' category, but we can replace it by one). Then $A(*)$ is defined as the loop space of the CW complex

$$\bigcup_{m,n} w_m S_n C \times \Delta^m \times \Delta^n \;/\sim\;,$$

the geometric realization of the bisimplicial set $[m],[n] \mapsto w_m S_n C$, where $w_m S_n C$ is the set of commutative diagrams in C,

$$\begin{array}{ccccccccc}
* = Y^0_{0,0} & \rightarrowtail & Y^0_{0,1} & \rightarrowtail & Y^0_{0,2} & \rightarrowtail \cdots \rightarrowtail & Y^0_{0,n} \\
\sim\downarrow & & \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\
* = Y^1_{0,0} & \rightarrowtail & Y^1_{0,1} & \rightarrowtail & Y^1_{0,2} & \rightarrowtail \cdots \rightarrowtail & Y^1_{0,n} \\
\sim\downarrow & & \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\sim\downarrow & & \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\
* = Y^m_{0,0} & \rightarrowtail & Y^m_{0,1} & \rightarrowtail & Y^m_{0,2} & \rightarrowtail \cdots \rightarrowtail & Y^m_{0,n}
\end{array}$$

in which the horizontal arrows $\rightarrowtail$ denote cofibrations, and the vertical arrows $\xrightarrow{\sim}$ denote (weak) homotopy equivalences.

The face and degeneracy maps in the vertical direction are given by omission and reduplication of data. This may conveniently be summarized by saying that the bisimplicial set arises as the nerve of a simplicial category; namely of $[n] \mapsto wS_nC$, the category of the diagrams

$$* = Y_{0,0} \rightarrowtail Y_{0,1} \rightarrowtail Y_{0,2} \rightarrowtail \cdots \rightarrowtail Y_{0,n}$$

and their weak homotopy equivalences.

The face structure in the horizontal direction is slightly more complicated. All but one of the face maps are still given by omission of data, but the face map numbered 0 involves a quotient space construction. It takes the above object to

$$* = Y_{1,1} \rightarrowtail Y_{1,2} \rightarrowtail \dots \rightarrowtail Y_{1,n} \quad , \qquad \text{where} \quad Y_{1,k} = Y_{0,k} / Y_{0,1} \ .$$

(As a technical point, quotient spaces are only well defined up to canonical isomorphism. This need not concern us very much, however. One just rearranges the construction a little by including the choices of quotients $Y_{i,j} = Y_{0,j}/Y_{0,i}$ in the data of the diagrams, cf. [14], [16]).

The construction is formal in the sense that it uses little knowledge about the category C. Indeed, the only thing required (apart from the technical point concerning the existence of an object in C which is both initial and terminal) is the fact that there are singled out two particular kinds of morphisms which are called *cofibrations* and *weak equivalences*, respectively, and which have suitable properties (e.g. cofibrations have quotients, and the weak equivalences satisfy a gluing lemma).

This suggests defining the notion of a *category with cofibrations and weak equivalences*. This is a category C equipped with subcategories $co(C)$ and $w(C)$, and the data are subject to a short list of plausible axioms (which will not be repeated here, cf. [14], [16]). The definition of the simplicial category $[n] \mapsto wS_nC$ (or $wS.C$, for short) now carries over word for word. We think of this simplicial category (or rather of the loop space of its geometric realization) as the *algebraic* K-*theory of the category* C or better, to be precise, as the K-*theory of* C *with respect to the chosen notions of cofibration and weak equivalence*.

In practice it turns out that the notion of cofibration is usually fixed once and for all. That is, it just doesn't occur in practice that some category C is considered as a category with cofibrations in more than one way. In particular, for the spaces and spectra in the present paper the term cofibration will always have its usual meaning. By contrast, it is not at all pathological nor even exceptional that some category C is considered as a category with weak equivalences in more than one way. For example if E is a spectrum, and C the category of pointed spaces (resp. of spectra) then the notion of E-equivalence is a perfectly acceptable notion of weak equivalence in C. In fact, the interplay between different notions of weak equivalence arising in this way is one of the things that localization theory is going to be about.

It may be appropriate to say a word about the ever recurring *finite type* condition. One could take it as one of the facts of life that in connection with algebraic K-theory there is always some finiteness condition around, be it explicit or implicit. But one can also give a simple explanation: in the absense of a finiteness condition algebraic K-theory just isn't interesting and therefore is not considered. For as soon as, say, infinite sums are allowed in the category C one can go through a version of the Eilenberg swindle. Namely if the endofunctor $F(A) = A \vee A \vee \dots$ is defined then one certainly has an isomorphism $Id \vee F \approx F$. On the other hand the sum in C induces a composition law on $|wS.C|$ making it an infinite loop space in the manner of Segal [10] and in particular therefore a group-like H-space (cf. [16] for details). In the homotopy $Id \vee F \simeq F$ one can then cancel F to conclude that the identity map on $|wS.C|$ is null-homotopic.

There is one general computation that is easy to do. This is the determination of K_0, the class group, in terms of generators and relations. By definition this group is $\pi_0\Omega|wS.C|$ or what is the same thing, the fundamental group of the CW complex $|wS.C|$. There is a well known recipe on how to compute the fundamental group of a reduced CW complex in terms of the cells of dimension 1 and 2. Applying the recipe in the case at hand one obtains that the class group is the abelian group generated by the objects $A \in C$, and subject to two kinds of defining relations,

$$[A^0] = [A^1] \text{ if there is a weak equivalence } A^0 \xrightarrow{\sim} A^1 , \quad \text{and}$$

$$[A_{01}] + [A_{12}] = [A_{02}] \text{ if there is a cofibration sequence } A_{01} \rightarrowtail A_{02} \twoheadrightarrow A_{12} .$$

In particular, in the case of the pointed spaces of finite type and their weak homotopy equivalences one obtains the group $\pi_0 A(*) \approx Z$, and the integer represented by a space is just its (reduced) Euler characteristic. Other cases will be considered later.

To conclude this review we shall outline an argument now to justify the fact that the space $A(*)$ may not only be defined in terms of pointed spaces of finite type but also in terms of spectra of finite type. We will need to know about some general results for this.

A functor between categories with cofibrations and weak equivalences, say $F: C \to C'$, is called *exact* if it preserves all the relevant structure. In that case it induces a map $wS.F: wS.C \to wS.C'$.

A *weak equivalence* between exact functors $C \to C'$ is a natural transformation $F \to F'$ so that for every $A \in C$ the map $F(A) \to F'(A)$ is a weak equivalence in C'. Not very surprisingly there results a homotopy between $wS.F$ and $wS.F'$ in this case. For example the cone functor on the category of pointed spaces is exact, so it induces a self-map on $A(*)$, and it is weakly equivalent to the trivial map, so the self-map is null-homotopic.

A *cofibration sequence* of exact functors $C \to C'$ is a sequence of natural transformations $F' \to F \to F''$, or $F' \rightarrowtail F \twoheadrightarrow F''$ as we shall write, having the property that for every $A \in C$ the map $F'(A) \to F(A)$ is a cofibration in C', and $F(A) \to F''(A)$ represents the associated quotient map. A basic technical tool about the construction $C \mapsto wS.C$ is the *additivity theorem*. One of several equivalent formulations says if $F' \rightarrowtail F \twoheadrightarrow F''$ is a cofibration sequence of exact functors then there exists a homotopy between $wS.F$ and the sum of the maps $wS.F'$ and $wS.F''$.

To come back to the situation at hand, there is a cofibration sequence of exact functors on the category of pointed spaces,

$$\text{identity} \rightarrowtail \text{cone} \twoheadrightarrow \text{suspension} .$$

In view of the additivity theorem therefore the self-map $\mathrm{Id} \vee \Sigma$ of $A(*)$ is null-homotopic, thus the suspension represents a homotopy inverse for the additive H-space structure on $A(*)$. In particular the suspension induces a homotopy equivalence of $A(*)$ to itself.

Now $C \mapsto wS.C$ is compatible with direct limits, so we obtain that (up to homotopy) $A(*)$ is also definable in terms of the category with cofibrations and weak equivalences $\overline{C}$ say,

$$\overline{C} = \varinjlim C_n ,$$

where each C_n is the category of pointed spaces of finite type, and $C_n \to C_{n+1}$ is the suspension map. $\overline{C}$ is a category of spectra containing the full subcategory of the finite spectra but it is somewhat smaller than $\widetilde{C}$, say, the category of spectra of finite (homotopy) type. We will therefore want to know that the inclusion $wS.\overline{C} \to wS.\widetilde{C}$ is a homotopy equivalence. While this is certainly plausible it is not self-evident, and an argument is required. The argument is provided by the following useful criterion whose applicability in the present situation is straightforward to check.

The criterion gives a sufficient condition for an exact functor $F: C \to D$ to induce a homotopy equivalence $wS.C \to wS.D$. We refer to it as the *approximation theorem*. The idea behind is that the homotopy type of $wS.C$ should only depend on the 'homotopy theory underlying C ' (whatever that may be). The approximation theorem makes this precise in the form of three axioms [16]. The first axiom says, roughly, that the general setup should be as in homotopy theory (in particular this rules out some fancy notions of weak equivalence and asks that mapping cylinder constructions should be available). The second axiom says if $A \to A'$ is a map in C then if $F(A) \to F(A')$ is a weak equivalence in D it follows that $A \to A'$ is a weak equivalence in C (the converse is implied by the exactness of F , of course). The third axiom finally insists that objects of D are 'homotopy equivalent' to objects coming from C , and morphisms too; the precise formulation is that given objects $A \in C$ and $B \in D$, and a map $f: F(A) \to B$ in D , then there exist a cofibration $g: A \to A'$ in C and a weak equivalence $h: F(A') \to B$ in D so that the resulting triangle commutes, i.e. $f = hF(g)$.

2. Review of localization.

The main references are to papers by Adams [1], Bousfield [2], and Ravenel [8].

Let E be a spectrum. A spectrum X is called E-*acyclic* if the E-homology groups $E_*X = \pi_*(E \wedge X)$ are trivial. Likewise a map $X' \to X''$ is called an E-*equivalence* if it induces an isomorphism $E_*X' \to E_*X''$. A spectrum Y is said to be E-*local* if it does not admit any non-trivial map from an E-acyclic spectrum; an equivalent condition is that for every E-equivalence $X' \to X''$ the induced map of sets of homotopy classes $[X'',Y] \to [X',Y]$ is an isomorphism.

By an E-*localization* of a spectrum X is meant any E-local spectrum Y together with an E-equivalence $X \to Y$. It follows from the definitions that the E-localization is unique up to (weak) homotopy equivalence under X. Bousfield has shown that it always exists, in fact that there exists an E-*localization functor* L_E [2].

There is a correspondence between localization functors and *acyclicity types*. For on the one hand the E-localization depends only on the class of the E-acyclic spectra: if E' and E'' happen to have the same acyclic spectra then their associated localization functors are the same, by definition. And on the other hand the E-acyclic spectra may be recovered from the localization functor L_E as the 'pre-image of zero'; that is, the E-acyclic spectra are precisely the ones whose E-localization is trivial (up to homotopy). The correspondence allows us to formulate a finite type condition on the localization functor L_E in terms of the associated acyclicity type. The condition is simply that $\mathrm{Cl}(L_E)$, the class of the E-acyclic spectra, is in some sense generated by finite spectra.

To make this precise let us say that a class of spectra is *saturated* if it is closed under

- homotopy equivalence and shifting (suspension and de-suspension)
- the formation of (possibly infinite) wedges
- the formation of mapping cones.

For any spectrum E the class of the E-acyclic spectra is saturated. Conversely it is known [2] that any saturated class occurs in this fashion from a suitable E. If M is any collection of spectra let the *saturation of* M mean the smallest saturated class of spectra containing M ; we denote it sat(M). We will say that a localization functor L , resp. the associated acyclicity type Cl(L) , is *generated* by a collection of spectra M if Cl(L) = sat(M) . And we will say that a localization functor is of *finite type*, or that it is a *finite localization functor*, if it is generated by some collection M any member of which is a finite spectrum. (Note that the number of spectra in M may well be infinite, however).

A finite localization functor has an important property which we refer to as the *convergence property*. It says that for every X the localization $L_E(X)$ may be obtained, up to homotopy, as the direct limit of a sequence of E-equivalences each of which has *finite* homotopy cofibre. In particular if X is finite then $L_E(X)$ is the direct limit (up to homotopy) of a sequence of finite spectra E-equivalent to X .

The proof may first of all be reduced to the assertion that the E-acyclic spectrum $L_E(X)/X$, the (homotopy-)cofibre of $X \to L_E(X)$, is the direct limit (up to homotopy) of a sequence of finite E-acyclic spectra. (For $L_E(X)$ can be reconstructed by attaching $L_E(X)/X$ to X). By hypothesis now $Cl(L_E)$ is generated by some collection M any member of which is finite and therefore certainly has the property asserted of $L_E(X)/X$. Inspection of the individual constructions permitted in generating sat(M) out of M now shows that each member of sat(M) must have the property also; in particular therefore $L_E(X)/X$ does.

The following properties of a spectrum E and of the associated localization functor L_E are particularly desirable. It is known that these four properties are mutually equivalent [8].

- Every direct limit of E-local spectra is E-local,
- L_E commutes with direct limit (up to homotopy),
- $L_E = L_T$ where $T = L_E(S)$, the localization of the sphere spectrum,
- $L_E(X) = X \wedge T$ (up to homotopy), in particular $T = L_E S = L_E L_E S = T \wedge T \wedge S = T \wedge T$.

A spectrum (resp. localization functor) having these properties is called *smashing* [8].

Finite localization functors are smashing. For if L_E is any such then for every X the localization $L_E(X)$ is obtainable from X by repeated attaching of finite E-acyclic spectra (the convergence property). It follows that $L_E(X)$ is the direct limit of the localizations of the finite subspectra of X , thus L_E commutes with direct limit and is therefore smashing.

It has been conjectured by Bousfield [2] and Ravenel [8] that, conversely, all smashing localization functors should be of finite type. Furthermore Ravenel has formulated some spectacular conjectures which assert a complete classification of the smashing localization functors. We shall discuss these conjectures below.

One defines a partial ordering on localization functors by saying that $L' \geqslant L''$ if L' retains at least as much information as L'' does; in other words if every L'-trivial spectrum is also L''-trivial. One knows that, up to homotopy, $L'L'' = L'' = L''L'$ in this situation.

If a smashing localization is not trivial it is $\geqslant L_{(0)}$, the rationalization. On the other hand every rationally trivial spectrum decomposes into its p-primary parts. There is therefore no essential loss of generality in restricting attention to localization functors which are $\leqslant L_{(p)}$, the localization at a prime p. The conjectures of Ravenel, below, assert that there is precisely a sequence of smashing (or indeed, finite) localization functors between $L_{(p)}$ and $L_{(0)}$,

$$L_{(p)} = L(p,\infty) > \dots > L(p,2) > L(p,1) > L(p,0) = L_{(0)} ;$$

this (conjectural) sequence is the *chromatic filtration*.

Following Ravenel, but adapting the notion a little, let us say that a spectrum is *disharmonic* (at p, to be precise) if it is trivial with respect to all finite localization functors $< L_{(p)}$. Examples of disharmonic spectra are provided by the bounded-above p-torsion spectra (I am indebted to Bökstedt for pointing out this fact and for contributing the following argument):

Let L be a finite localization functor $< L_{(p)}$. Then L is smashing and it trivializes at least one bounded-below spectrum X not trivialized by $L_{(p)}$. Since X is bounded below the Hurewicz theorem applies, and $X \wedge Z/p$ contains as a summand a (shifted) copy of the Eilenberg-MacLane spectrum Z/p. The triviality of $L(X) = T \wedge X$ thus not only entails that of $T \wedge X \wedge Z/p$ but also that of $T \wedge Z/p = L(Z/p)$. We conclude by a cofibration argument that L trivializes every p-torsion spectrum bounded both above and below, i.e. having only finitely many non-zero homotopy groups. A bounded-above spectrum, finally, is a direct limit of such, so it is trivialized by L, too.

Here is an interesting special case. Let L be a finite localization functor, and $S_L = L(S)$ the localization of the sphere spectrum. Then $S \to S_L$ is a S_L-equivalence since L is smashing. Let $\tilde{S}_L$ be the connected cover of S_L. Then $S_L/\tilde{S}_L$ is bounded above and hence disharmonic. It follows that $\tilde{S}_L \to S_L$ and $S \to \tilde{S}_L$ are also S_L-equivalences.

To conclude this review we will now describe in more detail the conjectures of Ravenel [8] as far as they are relevant to the present context. The conjectures were motivated by the manifestation of certain algebraic phenomena in the context of the Adams-Novikov spectral sequence associated to the Brown-Peterson spectrum BP. The conjectures seek to say that the algebraic phenomena are there for geometric reasons.

Let $BP_{(p)}$ denote the p-localization of BP; it is a ring spectrum (in the sense of stable homotopy theory - no coherence conditions asserted) and its homotopy groups form a polynomial ring $Z_{(p)}[v_1,v_2,..,v_n,...]$ where the generator v_n has grading $2p^n-2$; it is convenient to let $v_0 = p$, the prime at hand. The multiplication by v_n gives a (graded) self-map of $BP_{(p)}$, and one defines $BP_{(p)}[v_n^{-1}]$ as the telescope of this self-map; that is, the homotopy direct limit of the sequence

$$BP_{(p)} \xrightarrow{\cdot v_n} BP_{(p)} \xrightarrow{\cdot v_n} \cdots \quad .$$

The spectrum $BP_{(p)}[v_n^{-1}]$ admits the multiplication by v_n as an automorphism, it is thus a periodic spectrum (if $n>0$).

Following Ravenel we let L_n denote the localization functor associated to $BP_{(p)}[v_n^{-1}]$, the prime p being understood.

The *smashing conjecture* [8] asserts that L_n is smashing. This is known to be true for $n \leqslant p-2$ as well as for $n = 1$ if $p = 2$ [8].

When combined with the *finiteness conjecture* of Bousfield and Ravenel (that smashing localizations are necessarily finite) it asserts that L_n is finite. This is known to be true for L_1 [2] (and of course for L_0). The situation is slightly better with regard to the existence of finite L_n-trivial spectra. Such spectra have been obtained for small values of n in connection with the construction of the so-called periodic families in the stable homotopy of spheres [8], [3].

The *class invariance conjecture* [8] finally asserts that, as far as finite spectra are concerned, there are no acyclicity types beyond those provided by the L_n .

It is known [8] that the functors L_n form a sequence with respect to the partial ordering of the localization functors, namely $L_n > L_{n-1}$. The three conjectures taken together then say that the sequence of the L_n is the aforementioned chromatic filtration.

Independently of the conjectures one knows that all finite spectra X are *harmonic* [8], that is, they are local for the homology theory given by the wedge of all the $BP_{(p)}[v_n^{-1}]$; in particular if X is finite and non-trivial then $L_n(X)$ is non-trivial for sufficiently large n .

On the other hand one also knows many (infinite) X which are *dissonant*, that is, they are trivialized by each of the L_n (if the conjectures are true then "dissonant" is the same as "disharmonic"). For example the p-torsion Eilenberg-MacLane spectra are known to be dissonant [8].

3. The local counterparts of $A(*)$.

Let C denote the category of spectra. Let $L\colon C \to C$ be a localization functor. Associated to L there is a category of weak equivalences wC where, by definition, a map in C is in wC (or is a *w-map*, as we shall say) if the homotopy cofibre is trivialized by L .

A spectrum is *finite up to w-equivalence* if it is in the same connected component, in wC , as some finite spectrum; we denote the subcategory of the w-finite spectra by C_{wf} . Let $C_{(L)}$ denote the category of the L-local spectra, and

$$C_{(L)f} = C_{(L)} \cap C_{wf} .$$

If L' is a second localization functor, coarser than L , we let $C^{L'}$ denote the category of the L'-trivial spectra, and

$$C^{L'}_{(L)} = C_{(L)} \cap C^{L'} .$$

Let the *h-maps*, finally, mean the weak homotopy equivalences.

Localization theorem. Let L and L' be localization functors of finite type, and $L > L'$. There is a homotopy cartesian square

$$\begin{array}{ccc} hS.C^{L'}_{(L)f} & \longrightarrow & hS.C_{(L)f} \\ \downarrow & & \downarrow \\ hS.C^{L'}_{(L')f} & \longrightarrow & hS.C_{(L')f} \end{array}$$

where the term on the lower left is contractible.

In other words, if one considers the K-theories of the L-local and of the L'-local spectra, respectively, then their difference (i.e. the homotopy fibre of the natural map) is explicitly describable, namely it is represented by the K-theory of the category of those L-local spectra which are L'-trivial.

Proof. There is a similar looking result which is valid in a much more general context. In the situation at hand we check that the terms may be re-written in the desired form.

Namely if a category with cofibrations is equipped with two notions of weak equivalence, one finer than the other, then under rather general hypotheses which we will not spell out here, there results a homotopy cartesian square of the associated K-theories [14], [5], [16]. In particular there is such a square in the case of the category C_{hf} of the homotopy-finite spectra, equipped with the two notions of weak equivalence w and w' given by L and L' , respectively. It reads

$$\begin{array}{ccc} wS.C^{w'}_{hf} & \longrightarrow & wS.C_{hf} \\ \downarrow & & \downarrow \\ w'S.C^{w'}_{hf} & \longrightarrow & w'S.C_{hf} \end{array} .$$

In order to put this square into the desired form we will need to know of the finiteness of the localization functors, and of the ensuing smashing property (section 2).

Since L is smashing we can replace it, if necessary, by the functor given by smash-product with a L-localization T of the sphere spectrum. The L-localization can thus be an exact functor in the technical sense, so it induces a map in K-theory.

Similarly L' can be replaced, if necessary, by smash-product with T' . But it can also be replaced by smash-product with $T \wedge T'$ (since $L > L'$). It results that we can define a natural transformation from the above square to the square of the theorem: On the upper terms the map is induced by smash-product with T , and on the lower terms it is induced by smash-product with $T \wedge T'$. (We are using here that $hS.C_{(L)f} = wS.C_{(L)f}$ in view of the fact that h-maps and w-maps are the same in $C_{(L)}$; and similarly with the other terms).

To conclude we check that the map of squares is a homotopy equivalence on each term. We treat only the case of the map $wS.C_{hf} \to hS.C_{(L)f}$. The other cases are similar.

The map factors as

$$wS.C_{hf} \longrightarrow wS.C_{wf} \longrightarrow hS.C_{(L)f} \quad ,$$

so it suffices to show that these two maps are homotopy equivalences.

The inclusion $wS.C_{hf} \to wS.C_{wf}$ is a homotopy equivalence because of the approximation theorem (section 1) which applies in view of the convergence property (section 2) of the finite localization functor L .

The localization map $C_{wf} \to C_{(L)f}$ is left inverse to the inclusion $C_{(L)f} \to C_{wf}$ up to a natural transformation which is a w-equivalence. It results that the localization map induces a deformation retraction from $wS.C_{wf}$ to $wS.C_{(L)f} = hS.C_{(L)f}$. This completes the proof of the localization theorem. ■

Let now P be a set of primes. We denote by $A(*,P)$ the analogue of $A(*)$ constructed from P-local spaces or spectra; that is, $\Omega|hS.C_{(P)f}|$.

<u>Lemma</u> 1. There is a natural map $A(*,P) \to K(Z_{(P)})$ which is an equivalence away from P . More precisely, the homotopy groups of the homotopy fibre are P-torsion, and the first p-torsion, $p \in P$, occurs in dimension $2p-2$.

<u>Proof</u>. The map is given by *linearization* (this involves a definition of the algebraic K-theory of rings analogous to that of the algebraic K-theory of spaces, but in terms of abelian-group-objects, resp. module-objects, cf. [16]). To obtain the numerical statement we have to know that $A(*,P)$ can also be defined in other terms. This is one of the main results about the algebraic K-theory of spaces, the argument is given in [16] for the case where P is the set of all primes, i.e. the case of $A(*)$. It is not difficult to modify the argument so as to apply to the case of general P . The outcome is that $A(*,P)$ may be redefined, up to homotopy, as

$$Z \times \varinjlim BH(V^k S_{(P)})^+$$

where $V^k S_{(P)}$ denotes a wedge of k P-local sphere spectra, $H(..)$ is the simplicial monoid of homotopy equivalences, $BH(..)$ its classifying space, and $(..)^+$ denotes the plus construction of Quillen. Given that, under the translation, the map $A(*,P) \to K(Z_{(P)})$ corresponds to the natural map $BH(V^k S_{(P)}) \to BGl_k(Z_{(P)})$, the asserted numerics now follows easily from the fact that the higher homotopy of $S_{(P)}$ is P-torsion only and the first p-torsion occurs in dimension $2p-3$. ■

<u>Lemma</u> 2. The map $A(*,(0)) \to K(Q)$ is a homotopy equivalence.

<u>Proof</u>. This is the special case $P = \emptyset$ of the preceding lemma. ■

Let $F(*,P)$ denote the K-theory of the P-local torsion spaces, or what is the same, the P-torsion spaces.

<u>Lemma</u> 3. There is a homotopy equivalence

$$F(*,P) \simeq \prod'_{p \in P} F(*,p)$$

where Π' denotes the restricted product, the direct limit of the products indexed by the finite subsets of P .

<u>Proof</u>. Every P-torsion spectrum decomposes, up to homotopy, into its p-primary parts, and only finitely many of these parts are non-trivial because of the finite type condition on the spectrum. This shows that the approximation theorem (section 1) applies to the reconstruction map $\Pi'_{p\in P}\, C_f^p \to C_f^P$ which takes a finite collection of p-primary spectra to the wedge of these spectra. ■

Lemma 4. There is a diagram of homotopy fibrations

$$\begin{array}{ccccc} \prod'_{p\in P} F(*,p) & \longrightarrow & A(*,p) & \longrightarrow & A(*,(0)) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \prod'_{p\in P} K(Z/p) & \longrightarrow & K(Z_{(P)}) & \longrightarrow & K(Q) \end{array} \quad .$$

In particular the square on the left is homotopy cartesian.

Proof. The upper row is given by the localization theorem applied to the rationalization map $A(*,P) \to A(*,(0))$, together with the rewriting provided by lemma 3. The lower row is the analogous case of Quillen's localization theorem for the map $K(Z_{(P)}) \to K(Q)$. To obtain the map from top to bottom it is necessary to rewrite the lower row suitably, namely as the analogue of the upper row in the framework of abelian-group-objects, cf. [16]. The map on the right is a homotopy equivalence by lemma 2. ∎

Theorem. The square

$$\begin{array}{ccc} A(*) & \longrightarrow & \prod_p A(*,p) \\ \downarrow & & \downarrow \\ K(Z) & \longrightarrow & \prod_p K(Z_{(p)}) \end{array}$$

is homotopy cartesian, and for every prime p there is a homotopy equivalence

$$\text{fibre}(\ A(*) \to K(Z)\ )_{(p)} \quad \simeq \quad \text{fibre}(\ A(*,p) \to K(Z_{(p)})\ )\ .$$

Proof. By lemma 4 there are homotopy cartesian squares

$$\begin{array}{ccc} \prod'_p F(*,p) & \longrightarrow & A(*) \\ \downarrow & & \downarrow \\ \prod'_p K(Z/p) & \longrightarrow & K(Z) \end{array} \qquad\qquad \begin{array}{ccc} F(*,p) & \longrightarrow & A(*,p) \\ \downarrow & & \downarrow \\ K(Z/p) & \longrightarrow & K(Z_{(p)}) \end{array}$$

and the localization at p induces a map from the former to the latter. We take the product of all these maps. Then the square formed by the right hand columns gives the square of the theorem. To show it is homotopy cartesian it suffices to show that the square formed by the left hand columns is homotopy cartesian. That is, we want to show that the map

$$\text{fibre}(\ \textstyle\prod'_p F(*,p) \to \prod'_p K(Z/p)\ ) \longrightarrow \text{fibre}(\ \prod_p F(*,p) \to \prod_p K(Z/p)\ )$$

is a weak homotopy equivalence; equivalently (by lemma 4 and since the homotopy fibre commutes with products and direct limits, up to homotopy) that the inclusion map

$$\prod_p{}' \text{ fibre}(A(*,p) \to K(Z_{(p)})) \longrightarrow \prod_p \text{ fibre}(A(*,p) \to K(Z_{(p)}))$$

is one. But by lemma 1 the homotopy group π_n fibre($A(*,p) \to K(Z_{(p)})$) is zero for sufficiently large p (depending on n). So the map induces an isomorphism on homotopy groups.

The second part of the theorem follows from the first by taking p-localizations of the vertical fibres and noting that

$$\prod_q \text{ fibre}(A(*,q) \to K(Z_{(q)}))_{(p)} \quad \simeq \quad \text{fibre}(A(*,p) \to K(Z_{(p)}))$$

in view of lemma 1. ■

Let us fix a prime p now. Recall from section 2 the localization functors

$$L_{(p)} = L_\infty > \dots > L_n > \dots > L_1 > L_0$$

where L_n is associated to $BP_{(p)}[v_n^{-1}]$ (and L_0 is the same as rationalization). Following the conjectures of Bousfield and Ravenel discussed in section 2 we make the

Hypothesis. L_n is a *finite* localization functor.

Let us denote the category of the L_n-local spectra by $C_{(p,n)}$. We define $A(*,p,n)$ to be its K-theory,

$$A(*,p,n) \quad = \quad \Omega|hS.C_{(p,n)f}| \ ,$$

where as usual the subscript f indicates the finite type condition. Localization induces maps between these spaces, so we obtain a tower of spaces and maps,

$$A(*,p) \quad = \quad A(*,p,\infty) \longrightarrow \dots \longrightarrow A(*,p,n) \longrightarrow \dots \longrightarrow A(*,p,0) \ ,$$

interpolating between $A(*,p)$ and the K-theory of the rational numbers.

Next, let $C^{n-1}_{(p,n)}$ be the subcategory of $C_{(p,n)}$ of the spectra which are L_{n-1}-trivial; this is what Ravenel calls the n-*th monochromatic category* [8]. By the localization theorem its K-theory

$$M(*,p,n) \quad = \quad \Omega|hS.C^{n-1}_{(p,n)f}|$$

represents the n-th layer in the localization tower,

$$M(*,p,n) \quad \simeq \quad \text{fibre}(A(*,p,n) \to A(*,p,n-1)) \ .$$

The following argument, due to Bökstedt, can be used to prove the non-triviality of $M(*,p,n)$ in certain cases. Suppose h_* is a homology theory coarser than L_n

(that is, L_n-triviality implies h_*-acyclicity). Suppose further that for finite L_{n-1}-trivial X the groups h_iX are finite and periodic of period $2s$, say. Let c_iX denote the order of h_iX. Then, as one checks, the rational number given by the alternating product

$$cX = c_0X \cdot (c_1X)^{-1} \cdot c_2X \cdot \ldots \cdot c_{2s-2}X \cdot (c_{2s-1}X)^{-1}$$

is multiplicative for cofibration sequences. It results that c defines a homomorphism from the class group $\pi_0M(*,p,n)$ to the multiplicative group of rational numbers.

The argument applies in the case of $\pi_0M(*,p,1)$ and shows that this group is not trivial. For it is known [8] that the localization functor L_1 is definable in terms of p-local complex K-theory, and KU_i applied to a finite torsion spectrum is certainly finite and periodic. It suffices then to note that the number cX is not 1 in the case of the Moore spectrum S/p.

It is likely that a similar argument can be applied to show that $\pi_0M(*,p,2)$ is not trivial, and more specifically that the Toda spectrum $V(1)$ represents an element of infinite order. (Recall that $V(1)$ is the mapping cone of a certain graded self-map on the Moore spectrum S/p; the self-map induces multiplication by (a power of) v_1 in BP-homology). Assuming this is so, we can deduce a strange looking consequence. Namely the element $[V(1)]$ in $\pi_0M(*,p,2)$ projects to zero in $\pi_0A(*,p,2)$ because the cofibration sequence $\Sigma^k(S/p) \rightarrowtail S/p \twoheadrightarrow V(1)$ (where k is even) implies a relation $[V(1)] = [S/p] - [S/p]$. Therefore $[V(1)]$ must be the image of some element υ_1, say, in $\pi_1A(*,p,1)$. Thus the periodicity operator $v_1 \in \pi_*BP_{(p)}[v_1^{-1}]$ somehow corresponds to a 'phantom unit' υ_1 in algebraic K-theory.

As to a general attack on the spaces $M(*,p,n)$, the first (and perhaps main) step should be the search for a *devissage theorem*. Its content would be that for the purpose of constructing $M(*,p,n)$ one does not really need *all* of the monochromatic category $C^{n-1}_{(p,n)}$ but only a subcategory of *elementary* objects. A good candidate for the elementary objects would seem to be the spectra in $C^{n-1}_{(p,n)}$ which are periodic of minimal period.

We proceed to the construction of the *integral localization tower* $\tilde{A}(*,p,n)$.

Recall our standing hypothesis that L_n is a finite localization functor. As a consequence L_n is smashing (section 2), and $S_{(n)}$, the L_n-localization of the sphere spectrum, satisfies $S_{(n)} \wedge S_{(n)} \simeq S_{(n)}$ and is thus a very particular kind of ring spectrum. In particular the associated infinite loop space $QS_{(n)}$ is a ring space.

Let $M_k(QS_{(n)})$ denote the space of $k \times k$ matrices. It is a multiplicative H-space and, if $n \geq 1$, the monoid of connected components is $M_k(Z_{(p)})$. Define

$\widehat{GL}_k(QS_{(n)})$ as the union of connected components given by pullback with the inclusion of $GL_k(Z_{(p)})$ in $M_k(Z_{(p)})$.

Lemma. The H-space $\widehat{GL}_k(QS_{(n)})$ has a canonical (up to homotopy) classifying space.

Proof. $QS_{(n)}$ may be defined as the space (or better, simplicial set) of maps $S \to S_{(n)}$, and $M_k(QS_{(n)})$ may be identified to the mapping space $Map(V^k S, V^k S_{(n)})$. The latter is homotopy equivalent to $Map(V^k S_{(n)}, V^k S_{(n)})$ which is a monoid by composition of maps; the requisite homotopy equivalences are given by restriction along $V^k S \to V^k S_{(n)}$ on the one hand and by smash product with $S_{(n)}$ on the other, using that $S_{(n)} \wedge S_{(n)} \simeq S_{(n)}$. It results that $\widehat{GL}_k(QS_{(n)})$ is homotopy equivalent, as H-space, to a monoid. ■

We define

$$\tilde{A}(*,p,n) = Z \times \varinjlim_k B\widehat{GL}_k(QS_{(n)})^+ .$$

The factor Z is the class group of the ring $\pi_0 QS_{(n)}$, it has to be taken care of in this artificial way since the class group is invisible to the plus construction. The case $n = 0$ is exceptional from the present point of view, we can include it by defining $\tilde{A}(*,p,0)$ as $Z \times \varinjlim BGL_k(Z_{(p)})^+$.

By exploiting the plus construction one can arrive at a certain amount of numerics (as in [14], [16]). There is one general result which can be obtained in this way, namely the fact that the map

$$\tilde{A}(*,p,n) \longrightarrow \tilde{A}(*,p,n-1)$$

is an equivalence away from p (this uses that $QS_{(n)} \to QS_{(n-1)}$ is an equivalence away from p , as well as 1-connected). Note this is in sharp distinction from the situation with the other localization tower.

Beyond that it is possible to obtain quantitative results in (very) low dimensions. For example the first homotopy in fibre($\tilde{A}(*,p,1) \to \tilde{A}(*,p,0)$) occurs in dimension $2p-2$ and is cyclic of order p . But it seems unreasonable to expect that one can go much further in this way.

Perhaps the best approach eventually will be to compare the two localization towers. The idea is that in order to obtain information about

$$\text{fibre}(\ \tilde{A}(*,p,n) \to \tilde{A}(*,p,n-1)\ )$$

one should first try to compute with fibre($A(*,p,n) \to A(*,p,n-1)$) as well as the fibres of a natural transformation

$$\tilde{A}(*,p,n) \longrightarrow A(*,p,n) .$$

There is no problem in defining a map $\tilde{A}(*,p,n) \to A(*,p,n)$. Briefly, one can also construct $\tilde{A}(*,p,n)$ out of $\coprod_k B\widehat{GL}_k(QS_{(n)})$ by *group completion* (with respect to block sum). And $\coprod_k B\widehat{GL}_k(QS_{(n)})$ is practically contained in $|hS_1C_{(p,n)f}|$ (there are some technicalities; in particular the category $hS_1C_{(p,n)f}$ should be blown up to a homotopy equivalent simplicial category in order that one can have an honest inclusion, cf. corresponding constructions in [16]). The inclusion of $\coprod_k B\widehat{GL}_k(QS_{(n)})$ into $|hS_1C_{(p,n)f}|$, the geometric realization of the category in degree 1, now induces an inclusion of the suspension $\Sigma(\coprod_k B\widehat{GL}_k(QS_{(n)}))$ into $|hS.C_{(p,n)f}|$, the geometric realization of the full simplicial category. The adjoint of the latter inclusion then extends, by the group completion principle, to the desired map of $\tilde{A}(*,p,n)$ into the loop space $\Omega|hS.C_{(p,n)f}|$.

We will describe a localization theorem for the map $\tilde{A}(*,p,n) \to A(*,p,n)$ now. We need a further hypothesis. In fact we need the further hypothesis even for formulating the theorem.

The hypothesis is that there exists a category of modules over the ring spectrum $\tilde{S}_{(n)}$, the connected cover of $S_{(n)}$ (for $n \geq 1$). The hypothetical part about it is that the morphisms in the category should be actual maps, not homotopy classes of maps. (There has been done some work on module spectra in this sense by Robinson [9]; recent unpublished work of Schwänzl and Vogt is also relevant). Let the hypothetical category be denoted $\mathrm{Mod}(\tilde{S}_{(n)})$. It will be a category with cofibrations and weak equivalences in the technical sense of section 1. In fact there are two notions of weak equivalence, the h-*maps* and the w-*maps*, where the former are the weak homotopy equivalences and the latter are the maps which become equivalences upon changing the ground ring from $\tilde{S}_{(n)}$ to $S_{(n)}$ (or what amounts to the same, cf. below, the maps which become homotopy equivalences by L_n-localization).

An object of $\mathrm{Mod}(\tilde{S}_{(n)})$ is said to be *finite* if there is a finite filtration (sequence of cofibrations, that is) any quotient of which is free of rank 1, i.e. a perhaps shifted copy of $\tilde{S}_{(n)}$. Somewhat more generally we can also speak of *finiteness up to* h-*equivalence* (resp. w-*equivalence*); we indicate this by the subscript hf (resp. wf). The coarser notion of weak equivalence gives rise to the subcategory $\mathrm{Mod}(\tilde{S}_{(n)})^w$ of the w-trivial modules, or *torsion modules* as we will say.

The desired localization theorem says that the homotopy fibre

$$\mathrm{fibre}(\ \tilde{A}(*,p,n) \to A(*,p,n)\ )$$

is represented by the K-theory of the category of torsion modules over $\tilde{S}_{(n)}$.

The argument of proof is similar to that given in the proof of the localization theorem in the beginning of this section. Namely for general reasons there is a homotopy cartesian square

$$\begin{array}{ccc}
\Omega|hS.\mathrm{Mod}(\widetilde{S}_{(n)})^{w}_{hf}| & \longrightarrow & \Omega|hS.\mathrm{Mod}(\widetilde{S}_{(n)})_{hf}| \\
\downarrow & & \downarrow \\
\Omega|wS.\mathrm{Mod}(\widetilde{S}_{(n)})^{w}_{hf}| & \longrightarrow & \Omega|wS.\mathrm{Mod}(\widetilde{S}_{(n)})_{hf}|
\end{array}$$

in which the lower left term is contractible. The upper left term is the K-theory of the category of torsion modules over $\widetilde{S}_{(n)}$. It only remains to be shown, therefore, that the map on the right may be identified to the map $\widetilde{A}(*,p,n) \to A(*,p,n)$.

The identification of the upper right term with $\widetilde{A}(*,p,n)$ comes from the main result of [16]; cf. the proofs of lemmas 1 and 4 above for similar points.

The identification of the lower right term with $A(*,p,n)$ is similar to the argument at the end of the proof of the localization theorem (the last three paragraphs). Two points deserve mentioning. The first is that one can construct a L_n-localization of a given $\widetilde{S}_{(n)}$-module by (infinitely) repeated attaching of *finite* L_n-acyclic modules; this uses Bökstedt's lemma (section 2) that $S \to \widetilde{S}_{(n)}$ is a $S_{(n)}$-equivalence. It results that there exists a L_n-localization functor on $\mathrm{Mod}(\widetilde{S}_{(n)})$ which is of finite type (in view of its construction) and therefore also has the convergence property (section 2). The second point is that a L_n-local spectrum has a *unique* $S_{(n)}$-module structure which may therefore be suppressed or resurrected according to the need of the moment.

It is a matter of checking the definitions, finally, to see that under these identifications the two maps correspond as desired.

4. Appendix: An implication of the Lichtenbaum-Quillen conjecture.

We give a quick review of the Lichtenbaum-Quillen conjecture, a homotopy theoretic reformulation, and finally the application to obtaining a kind of lower bound on the difference of $A(*)$ and $K(Z)$.

The content of LQC is that for many rings (and schemes) the algebraic K-theory ought to be expressible in terms of etale cohomology and thereby computable. With the advent of the *etale K-theory* of Dwyer and Friedlander [4] a simpler, and more explicit, formulation became possible. The new formulation is that the natural transformation

$$K_*(R,Z/p) \longrightarrow K_*^{et}(R,Z/p)$$

should be an isomorphism for suitable R . Actually this is conjectured only for odd primes p , and for sufficiently high degrees; it is known that some such restriction is necessary, cf. [12].

As usual here $K_*(R,Z/p)$ denotes the *K-theory of* R *with coefficients in* Z/p . We think of it in terms of spectra, namely as the homotopy of $K(R,Z/p)$, the smash product of the K-theory spectrum $K(R)$ and the Moore spectrum S/p .

The necessity of working with finite coefficients comes from the fact that the etale homotopy, and therefore also the etale K-theory, does not behave properly unless one restricts to working with finite coefficients.

We will not define the etale K-theory here. We don't have to, in fact. For Thomason has proved the amazing result that etale K-theory is the same, in many cases, as "Bott periodic" algebraic K-theory [13]. In view of this result LQC translates into the conjecture that the map

$$K_*(R,Z/p) \longrightarrow K_*(R,Z/p)[\beta^{-1}]$$

is an isomorphism (for suitable R , odd p , and in sufficiently high degrees).

As to the *Bott periodic algebraic K-theory*, we find it convenient to use the definition given by Snaith [11]. Namely the Moore spectrum S/p supports a self-map known as the *Adams map*; if p is odd the map is of degree $2p-2$. It induces a graded self-map of $K(R,Z/p)$, and $K(R,Z/p)[\beta^{-1}]$ is now defined as the mapping telescope of the latter, the homotopy direct limit of the sequence

$$K(R,Z/p) \longrightarrow K(R,Z/p) \longrightarrow \dots$$

in which each map is the map in question.

Actually Snaith's procedure is slightly different in that he defines $K(R,Z/p)$ as the spectrum of maps $S/p \to K(R)$, so the self-map on $K(R,Z/p)$ is given by composition with the Adams map. However the distinction is minor since the Moore spectrum and the Adams map are self-dual with respect to Spanier-Whitehead duality.

At any rate, the definition is equivalent to letting

$$K(R,Z/p)[\beta^{-1}] \;=\; K(R) \wedge S/p[\beta^{-1}]$$

where $S/p[\beta^{-1}]$ is the mapping telescope of the Adams map.

Recall the localization functor L_1 (section 2). It is known [2] that

$$S/p[\beta^{-1}] \;=\; L_1(S/p) \;.$$

Since L_1 is smashing (section 2) we obtain

$$K(R,Z/p)[\beta^{-1}] \;=\; K(R) \wedge S/p \wedge L_1(S) \;=\; L_1(K(R)) \wedge S/p \;.$$

So LQC translates into a conjecture saying that the homotopy cofibre, F say, of the localization map

$$K(R) \longrightarrow L_1(K(R))$$

is annihilated by smash product with S/p (for suitable R and odd p, that is, and in sufficiently high degrees). In view of the cofibration sequence

$$S \xrightarrow{\cdot p} S \longrightarrow S/p$$

this means that the self-map of F given by multiplication by p is an equivalence (in high degrees), so F may be identified (in high degrees) to the telescope of the self-map; that telescope is $F[p^{-1}]$, the localization away from p.

Replacing $K(R)$ by $K(R)_{(p)}$ now (the localization at p) we conclude that the homotopy cofibre of

$$K(R)_{(p)} \longrightarrow L_1(K(R)_{(p)})$$

is unchanged (in high degrees) by inverting p, that is, by the rationalization functor L_0. Since $L_0 = L_0 L_1$ it follows that the homotopy cofibre is trivial (in high degrees).

We have thus translated LQC into a conjecture saying that, for suitable R, and odd p, the localization map

$$K(R)_{(p)} \longrightarrow L_1(K(R)_{(p)})$$

should be an equivalence of sufficiently highly connected covers; in other words that, apart from some bounded piece, the p-local $K(R)_{(p)}$ should already be L_1-local; in still other words that, in terms of the chromatic filtration, $K(R)_{(p)}$ should support first order phenomena only.

Before discussing any implications of LQC we must briefly comment on which rings R are supposed to be 'suitable'. Etale homotopy requires all coefficients to be finite, as pointed out before, but it also requires them to be prime to the residue characteristics at hand. As a result the etale K-theory $K_*^{et}(R,Z/p)$ is only defined if p is invertible in R, and there can't possibly be any conjecture about it otherwise.

On the other hand the homotopy theoretical reformulation of LQC makes perfect sense for general R. A standard argument shows that for some R the validity of LQC in this sense is equivalent to its validity for the ring of fractions $R[p^{-1}]$. In particular this is so for Z, the ring of integers. Namely by the theorems of Quillen, the difference of $K(Z)$ and $K(Z[p^{-1}])$ is given by $K(Z/p)$, and that is trivial at p except in degree 0.

By naturality of localization applied to the map $QS^0 \to K(Z)$ now there is a commutative diagram

$$\begin{array}{ccc} QS^0_{(p)} & \longrightarrow & K(Z)_{(p)} \\ \downarrow & & \downarrow \\ L_1(QS^0_{(p)}) & \longrightarrow & L_1(K(Z)_{(p)}) \end{array} .$$

If the right hand vertical map is assumed to be an equivalence it follows that, at p, the map $QS^0 \to K(Z)$ factors through J, the connective cover of $L_1(QS^0)$.

On the other hand the map $QS^0 \to K(Z)$ factors through $QS^0 \to A(*)$ which is known to be a split injection [15], [17]. If one assumes the validity of LQC it thus follows that (at least for odd p and in sufficiently high degrees) the difference between $A(*)$ and $K(Z)$ must in some way or other account for the difference between QS^0 and J.

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