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THE MAP $BSG \to A(*) \to QS^0$

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§1. *INTRODUCTION*

Let $A(X)$ be the algebraic K-theory of the space X. This can be defined in various ways, see [9], [10], [11], [12]. Let BG be the space classifying 0-dimensional virtual spherical fiberbundles. There are maps $F : BG \to A(X)$, $l : A(X) \to K(\mathbb{Z})$; $i : QS^0 \to A(*)$. In [10],[11], maps $A(*) \to QS^0$, splitting i up to homotopy are constructed.

In this paper, we construct a splitting map $Tr : A(*) \to QS^0$, and compute the composite $BG \to A(*) \to QS^0$. We apply this construction to show that $\pi_3(Wh^{Diff}(*)) \approx \mathbb{Z}/2$. A further application is [4]. There it is used that the splitting given here agrees with the splitting in [11]; this will be proved in [5].

Recall that

$$A(*) \simeq \lim_{n,k} B\ \mathrm{Aut}\ (v^k S^n)^+$$

where Aut denotes the simplicial monoid of homotopy equivalences, and + denotes the Quillen plus

construction. In this description of $A(*)$, we can define $f : BG \to A(*)$ as the inclusion

$$BG = \lim_n (B\ \mathrm{Aut}\ S^n) \subset \lim_{n,k} B\ \mathrm{Aut}(v^k S^n)^+ = A(*)$$

and $l : A(*) \to K(\mathbb{Z})$ as the linearization map

$$A(*) = \lim_{n,k} B\ \mathrm{Aut}\ (v^k S^n)^+ \to \lim_{n,k} B\ \mathrm{Aut}\ (H_n(v^k S^n))^+ = K(\mathbb{Z}).$$

Let $BSG \subset BG$ classify the oriented spherical fiberbundles. The composite

$$BSG \to BG \xrightarrow{f} A(*) \xrightarrow{l} K(\mathbb{Z})$$

is the trivial map.

In §3 we will show that the composite $BG \xrightarrow{f} A(*) \xrightarrow{Tr} QS^0$ equals a certain map $\eta : BG \to QS^0$, studied in §2. In §2 we show that if $i \geq 3$, then

$$\pi^S_{i-1} \simeq \pi_i(BG) \xrightarrow{\eta} \pi_i(QS^0) \cong \pi^S_i$$

is given by multiplication with $\bar{\eta}$, the generator of $\pi^S_1 \approx \mathbb{Z}/2$. In particular, for $i \geq 3$ the map $\Theta : \pi^S_{i-1} \oplus \pi^S_i \cong \pi_i(BSG) \oplus \pi_i(QS^0) \xrightarrow{f_*+i_*} \pi_i(A(*)) \to \pi_i(QS^0) \cong \pi^S_i$ is given by $\Theta(x,y) = \bar{\eta}x + y$.

The splitting of $A(*)$ induces a splitting $\pi_i(A(*)) \cong \pi_i(QS^0) \oplus C_i$. If $x \in \pi^S_i$, $i \geq 2$, then $\Theta(x,\eta x) = 0$, so that $f_*(x) + i_*(\eta x) \in C_{i+1}$. We want to show that for some choices of x, this element is nontrivial.

The composite $QS^0 \xrightarrow{i} A(*) \xrightarrow{l} K(\mathbb{Z})$ is studied in [7]. There it is shown that

$\pi_{4i+3}(QS^0) = \pi^S_{4i+3} \to \pi_{4i+3}(K(\mathbb{Z}))$ is injective on the image of the J-homomorphism.

Recall from [1] that there are classes $\mu_{8i+1} \in \pi^S_{8i+1}$, $i \geq 1$, so that $\bar{\eta}^2\mu_{8i+1}$ is in the image of the J-homomorphism. Similarly, $\bar{\eta}^3 \in \pi^S_3$ is also in the image of the J-homomorphism. Choose $x = \mu_{8i+1}$, $x = \bar{\eta}\mu_{8i+1}$ or $x = \bar{\eta}^2$. Then $f_*(x) + i_*(\bar{\eta}x) \in C_{8i+3}$, and $l_*(f_*(x) + i_*(\bar{\eta}x)) = l_*i_*(\bar{\eta}x) \neq 0$.

We have proved

THEOREM 1.1. *The kernel of the map* $Tr_n : \pi_n(A(*)) \to \pi_n(QS^0)$ *contains a nontrivial element of order 2 if* $n \equiv 2,3 (\mathrm{mod}\ 8)$; $n \geq 3$.

On the other hand, it is known that $C_3 \leq \mathbb{Z}/2$ [6], so we have

COROLLARY 1.2. $\pi_3 A(*) \cong \pi^S_3 \oplus \mathbb{Z}/2$.

It is known [9], [11] that $A(*)$ splits as a product

$$A(*) \simeq QS^0 \times Wh^{Diff}(*) \times \mu.$$

It will be proved in [13] that $\mu = 0$. We conclude

THEOREM 1.3.

(*i*) $\pi_3\ Wh^{Diff}(*) = \mathbb{Z}/2$

(*ii*) *There are nontrivial two-torsion classes in*

$\pi_{8i+2}(Wh^{Diff}(*))$ *and* $\pi_{8i+3}(Wh^{Diff}(*))$; $i \geq 1$.

§2. *SPHERICAL FIBER BUNDLES AND* η.

In this paragraph we study a certain map $\eta : BG \to G$. We first give a homotopy theoretical definition of η, and calculate the induced maps of homotopy groups. Finally, we show that η agrees with a geometrically defined map, which will be used in §3.

Let $X = \Omega^3 Y$ be a threefold loopspace. Let $\bar{\eta} : S^3 \to S^2$ be the Hopf map.

Definition 2.1. $\eta_X : BX = \Omega^2 Y \to \Omega^3 Y = X$ is the map induced by $\bar{\eta}$.

Example 2.2. $X = \Omega^\infty S^\infty$. We identify $\pi_*(X)$ with the ring π^S_* of stable homotopy groups of spheres. The map $(\eta_X)_* : \pi^S_* = \pi_*(X) \to \pi_*(BX) = \pi^S_{*+1}$ is given by product with $\bar{\eta} \in \pi^S_1$.

Example 2.3. Let $X = \mathbb{Z}\times BG$ be the classifying space of based stable spherical fibrations; X is an infinite loop space [3]. Then $\Omega X = \Omega BG$ can be identified with the space of stable homotopy equivalences of spheres, i.e., $i : \Omega X \xrightarrow{\sim} (\Omega^\infty S^\infty)_{\pm 1}$. This equivalence is not an H-space equivalence, when $\Omega^\infty S^\infty = QS^0$ is given the H-space structure derived from loop sum. But

$$\Omega^3 i : \Omega^4 X \xrightarrow{\sim} \Omega^3(QS^0)$$

is an equivalence of threefold loopspaces, so that

$$(\eta_{\Omega^3 QS^0})\, \Omega^3 i \simeq (\Omega^3 i)(\eta_{\Omega^4 X}).$$

We conclude from the previous example, that for $i \geq 3$

$$(\eta_{\mathbb{Z}\times BG})_* : \pi^S_{i-1} \cong \pi_i(\mathbb{Z} \times BG) \to \pi_i(G) \simeq \pi^S_i$$

is induced by composition with $\bar{\eta} \in \pi^S_1$ for $i \geq 3$. For $i \leq 2$ we do not get any information. Actually, $\Omega^3 X$ is not equivalent to $\Omega^2(QS^0)$ as a threefold loopspace. The induced map $(\eta_{\mathbb{Z}\times BG})_* : \pi^S_2 \to \pi^S_3$ is trivial, whereas multiplication by $\bar{\eta}$ is nontrivial.

Let X be an infinite loopspace. Composition of loops defines an infinite loop map

$$\mu : \Omega^\infty S^\infty \times X \to X.$$

There are structure maps $\Theta_n : E\Sigma_n \times_{\Sigma_n} X^n \to X$, and a commutative diagram

$$(2.4) \qquad \begin{array}{ccc} \coprod_{m\geq 0} B\Sigma_m \times X & \xrightarrow{(id\times_{\Sigma_m}\Delta)} & \coprod_{m\geq 0} E\Sigma_m \times_{\Sigma_m} X^m \\ \downarrow (i_m\times id) & & \downarrow \coprod \Theta_m \\ \Omega^\infty S^\infty \times X & \xrightarrow{\mu} & X \end{array}$$

There are two maps $f_i : S^1 \times X \to B\Sigma_2 \times X$, $f_i = (f'_i \times id)$, where f'_o is the trivial map, and f'_1 represents the generator of $\pi_1(B\Sigma_2) = \mathbb{Z}/2$.

Composition with the square above defines maps $g_i : S^1 \times X \to \coprod_{m\geq 0} B\Sigma_m \times X \to X$. The difference $g_1 - g_o$ defines a map $g : S^1 \wedge X \to X$. This difference is the image under μ of the difference $(i_2 g_1 - i_2 g_o) \times id$. But $i_2 g_1 - i_2 g_o$ is equal to $\bar{\eta} : S^1 \to QS^0$, so that the adjoint of g is the map $\eta_X : X \to \Omega X$.

In particular, the map $\eta_{\mathbb{Z}\times BG} : \mathbb{Z} \times BG \to G$ can be described as the difference between the adjoints of the maps g_i (i=0,1)

$$g_i : S^1 \times (\mathbb{Z}\times BG) \to E\Sigma_2 \times_{\Sigma_2} (\mathbb{Z}\times BG)^2 \xrightarrow{\Theta_2} \mathbb{Z}\times BG.$$

Let ξ be the standard (virtual) spherical fiberbundle on $\mathbb{Z}\times BG$. Let $\wedge$ denote fiberwise smashproduct. Then g_i classifies certain virtual bundles on $S^1\times(\mathbb{Z}\times BG)$. These bundles are the identifications of the bundle $\xi \wedge \xi$ on $I \times (\mathbb{Z}\times BG)$, using certain bundle maps $\tau_i : \xi \wedge \xi \to \xi \wedge \xi$ as clutching function, where $\tau_o = id$, and $\tau_1(x \wedge y) = y \wedge x$.

We reformulate this description as follows.

LEMMA 2.5. *Let* ξ *be the standard bundle over* $\mathbb{Z} \times BG$. *The automorphisms* $\tau_i : \xi \wedge \xi \to \xi \wedge \xi$ $(i = o,1)$ *induce maps*

$$t_i : \mathbb{Z} \times BG \to G.$$

The difference $t_1 - t_o$ *equals* $\eta : \mathbb{Z} \times BG \to G$ *up to homotopy.*

Finally, consider the following situation. Let B be a finite dimensional space. Let ξ be a spherical fibration over B, classified by a map

$$f : B \to \mathbb{Z} \times BG.$$

Let ξ' be a spherical fibration over B, and u a fiber homotopy trivialization:

$$u : \xi' \wedge_B \xi \to S^N \times B.$$

The map u can be interpreted as an S-duality parametrized over B, see [2].

A 2N-dual u' of this map is a map $u' : S^N \times B \to \xi \wedge_B \xi'$ such that the following diagram commutes up to fiber homotopy

$$\begin{array}{ccc}
\xi' \wedge_B \xi \wedge_B (S^N \times B) & \xrightarrow{u \wedge id} & (S^N \times B) \wedge_B (S^N \times B) \\
{\scriptstyle id \wedge u'} \downarrow & & \downarrow \cong \\
\xi' \wedge_B \xi \wedge_B \xi \wedge_B \xi' & \xrightarrow{v} & (S^{2N} \times B)
\end{array}$$

where $v(a,b,c,d) = u(a,c) \wedge_B u(d,b)$. The transfer $Tr : B \to \Omega^N S^N \to QS^o$ is defined as the adjoint of the map

$$t : S^N \times B \xrightarrow{u'} \xi \wedge_B \xi' \xrightarrow{Twist} \xi' \wedge_B \xi \xrightarrow{u} S^N \times B \to S^N.$$

LEMMA 2.6. *The following diagram is homotopy commutative*

$$\begin{array}{ccc}
B & \xrightarrow{f} & \mathbb{Z} \times BG \\
{\scriptstyle Tr} \downarrow & & \downarrow \\
QS^o & \xleftarrow{i} & G
\end{array}$$

where $i : SG \to QS^o$ *is the standard identification of* SG *with the component of* 1 *in* QS^o.

Proof. The map Tr can also be defined as the adjoint of a suspension of t:

$$id \wedge t : S^{2N} \times B \xrightarrow{id \wedge u'} (S^N \times B) \wedge_B \xi \wedge_B \xi' \to$$
$$\xrightarrow{id \wedge (u \circ Twist)} (S^N \times B) \wedge_B (S^N \times B).$$

Let $\xi' \wedge_B \xi \wedge_B \xi \wedge_B \xi' \to \xi' \wedge_B \xi \wedge_B \xi \wedge_B \xi'$ be the map permuting the second and third factor. By assumption, the following diagram commutes up to fiber homotopy

$$\begin{array}{ccccc}
(S^N \times B) \wedge_B (S^N \times B) & \xrightarrow{id \wedge u'} & S^N \wedge_B \xi \wedge_B \xi' & \xrightarrow{id \wedge (u \circ Twist)} & (S^N \times B) \wedge_B (S^N \times B) \\
& {\scriptstyle v} \searrow & \uparrow {\scriptstyle u \wedge id} & & \uparrow {\scriptstyle v} \\
& & \xi' \wedge_B \xi \wedge_B \xi \wedge_B \xi' & \xrightarrow{Tw_{23}} & \xi' \wedge_B \xi \wedge_B \xi \wedge_B \xi'
\end{array}$$

We conclude that $B \to G \subset QS^o$ is the difference $t_1' - t_o'$ between the maps

$$t_i' : B' \to G$$

induced by the automorphisms

$$\tau_i' : \xi' \wedge_B \xi \wedge_B \xi \wedge_B \xi' \to \xi' \wedge_B \xi \wedge_B \xi \wedge_B \xi'$$

$\tau_1' = Tw_{23}$, $\tau_o' =$ identity. The lemma follows from 2.5.

§3. *TRANSFER AND SPLITTING*

In this paragraph we will construct a splitting map $Tr : A(*) \to QS^o$. This splitting map will be used to prove theorem 1.1. In a later paper it will show that this map agrees with the splitting maps in [10] and [11], cf [5].

We recall some properties of the transfer map [2].

Let B be a finite dimensional space. Let $F \to E \to B$ be a fibration with section, and suppose that fiber F is homotopy equivalent to a finite complex. Then there is a transfer map $\tau : B \to \Omega^\infty S^\infty(E_+)$. Let $Tr_E : B \to \Omega^\infty S^\infty$ be the composite of τ with the map $\Omega^\infty S^\infty(E_+) \to \Omega^\infty S^\infty(pt_+) = \Omega^\infty S^\infty$, induced by $E \to pt$.

We will need the following properties of the transfer:

Let $S^1 \underset{B}{\wedge} E \to B$ be the fiberwise double suspension of E.

3.1. $$Tr_E \simeq Tr_{S^2 \underset{B}{\wedge} E} : B \to \Omega^\infty S^\infty.$$

Let E_1, E_2 be two fibrations over B as above. Then we can consider the fiberwise wedge $E = E_1 \vee E_2 \to B$.

3.2. $$Tr_E \simeq Tr_{E_2} + Tr_{E_2} : B \to \Omega^\infty S^\infty$$

These properties will be proved at the end of this section.

Recall that the algebraic K-theory of a point can be defined as

$$A(*) = \underset{\substack{\longrightarrow \\ n,k}}{\lim} B\,Aut(\vee^k S^n)^+.$$

Let $f : B \to B\,Aut(\vee^k S^{2n})$ be a finite dimensional approximation. There is an induced fibration $(\vee^k S^{2n}) \to E \to B$. To this fibration, there is an associated transfer map $Tr_E : B \to \Omega^\infty S^\infty$. Let $\sigma : B\,Aut(\vee^k S^{2n}) \to B\,Aut(\vee^k S^{2n+2})$ be

induced by double suspension. Then the map σf induces the fiberwise double suspension of $E : (\vee^k S^{2n+2}) \to E' \to B$, and because of 3.1 $Tr_E \simeq Tr_{E'}$. By a homotopy colimit argument, these maps extend to a map

$$Tr_k : \underset{n}{\lim} B\,Aut(\vee^k S^n) \to \Omega^\infty S^\infty.$$

The stabilization map

$$B\,Aut(\vee^k S^n) \to B\,Aut(\vee^{k+1} S^n)$$

induced by adding a factor in the wedge, induces by 3.2 a diagram, which is homotopy commutative on all finite subspaces

$$\begin{array}{ccc}
\coprod_{k\geq 0} \underset{\substack{\to \\ n}}{\lim} (B\,Aut(\vee^k S^{2n})) & \longrightarrow & \coprod_{k\geq -1} \underset{\substack{\to \\ n}}{\lim} (B\,Aut(\vee^{k+1} S^{2n})) \\
\Big\downarrow \coprod Tr_k & & \Big\downarrow \coprod Tr_k \\
\Omega^\infty S^\infty & \xrightarrow{\quad *[1] \quad} & \Omega^\infty S^\infty
\end{array}$$

The map $*[1]$ here denotes loop sum with the identity loop. Again, you can extend to a map, defined on finite subcomplexes

$$Tr : \mathbb{Z} \times \underset{n,k}{\lim} B\,Aut(\vee^k S^{2n}) \to \Omega^\infty S^\infty$$

And by the universal property of the plus construction, this finally extends to a map

$$Tr : A(*) \to \Omega^\infty S^\infty.$$

Recall from [8] that $\Omega^\infty S^\infty \cong \mathbb{Z} \times \underset{k}{\lim} B\Sigma_k^+$. The map $\mathbb{Z} \times \underset{k}{\lim} B\Sigma_k \to \mathbb{Z} \times \underset{n,k}{\lim} B\,Aut \vee^k S^n \to \Omega^\infty S^\infty$ actually is the map inducing the equivalence, so $Tr : A(*) \to \Omega^\infty S^\infty$ is a split surjection.

Now, theorem 1.1 follows from the description of $\eta_{\mathbb{Z}\times BG}$ as a transfer in 2.8.

It remains to prove 3.1 and 3.2. Recall from [2] that the transfer τ_E has the following properties:

3.3 Given a fibration $p : E \to B$ as above, and a map $g : X \to B$, we have a pullback diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{g}} & E \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B \end{array}$$

Then $\Omega^\infty S^\infty(\tilde{g}_+) \circ \tau_{\tilde{E}} \simeq \tau_E \circ g$.

3.4 Given fibrations $p_i : E_i \to B_i$ as above, we can form the fiberwise smashproduct

$$p_1 \underset{B_1 B_1}{\wedge} p_2 : E_1 \underset{B_1 B_2}{\wedge} E_2 \longrightarrow B_1 \times B_2 .$$

The following diagram commutes up to homotopy

$$\begin{array}{ccc} B_1 \times B_2 & \xrightarrow{\tau_{B_1} \times \tau_{B_2}} & \Omega^\infty S^\infty(E_{1_+}) \times \Omega^\infty S^\infty(E_{2_+}) \\ & {\scriptstyle \tau_{B_1\times B_2}} \searrow & \downarrow \\ & & \Omega^\infty S^\infty(E_1 \times E_2)_+ \end{array}$$

We can now prove 3.1. If $F \to F \to *$ is a fibration with trivial base, then

$$\tau_F : S^0 \to \Omega^\infty S^\infty$$

is given by the Euler characteristic $\chi(F)$. This is to be understood in the pointed sense here; thus a sphere has Euler characteristic $+1$ or -1 depending on the parity of the dimension.

From 3.3 it follows, that if $F \to F \times B \to B$ is a product fibration, then $\tau_{F\times B} : B \to \Omega^\infty S^\infty(B\times F)_+$ is the composite

$$B \to pt \to pt_+ \to \Omega^\infty S^\infty(pt_+) = \Omega^\infty S^\infty \xrightarrow{\chi(F)} \Omega^\infty S^\infty .$$

Applying 3.4 to $E_1 = E$; $E_2 = S^2 \times B \to B$ and then 3.3 to the diagonal map $B \to B \times B$, the statement 3.1 follows.

In order to prove 3.2, note that if $f_i : S^N \times B \to E_i$ are duality maps of exspaces in the sense of [2], then the fiberwise coproduct followed by fiberwise wedge

$$S^N \times B \to S^N \vee S^N \times B \xrightarrow{f_1 \vee f_2} E_1 \vee E_2$$

is also a duality map. The 2N-dual of this map is the wedge of the 2N-duals of f_1 and f_2 followed by the fold map

$$E_1 \vee E_2 \xrightarrow{Df_1 \vee Df_2} (S^N \vee S^N) \times B \xrightarrow{\text{fold}} S^N \times B$$

The transfer map $Tr_{E_1 \vee E_2}$ is the adjoint of the composite

$$S^N{\times}B \to (S^N{\vee}S^N){\times}B \xrightarrow{f_1 \vee f_2} E_1 \vee E_2 \xrightarrow{Df_1 \vee Df_2} (S^N{\times}S^N){\times}B \to S^N{\times}B$$

which equals the sum $Tr_{E_1} + Tr_{E_2}$.

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