MLD Relations of Pisot Substitution Tilings

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Abstract. We consider 1-dimensional, unimodular Pisot substitution tilings with three intervals, and discuss conditions under which pairs of such tilings are locally isomorhphic (LI), or mutually locally derivable (MDL). For this purpose, we regard the substitutions as homomorphisms of the underlying free group with three generators. Then, if two substitutions are conjugated by an inner automorphism of the free group, the two tilings are LI, and a conjugating outer automorphism between two substitutions can often be used to prove that the two tilings are MLD. We present several examples illustrating the different phenomena that can occur in this context. In particular, we show how two substitution tilings can be MLD even if their substitution matrices are not equal, but only conjugate in $GL(n, \mathbb{Z})$. We also illustrate how the (in our case fractal) windows of MLD tilings can be reconstructed from each other, and discuss how the conjugating group automorphism affects the substitution generating the window boundaries.

1. Introduction

In this article, we consider substitutions σ on an alphabet of three letters, whose abelianisation matrix (substitution matrix) M is primitive and unimodular, has irreducible characteristic polynomial, and a leading (Perron-Frobenius, PF) eigenvalue which is a Pisot number. The words generated by such a substitution can be regarded as elements of a free group with three generators, and automorphisms of the free group give rise to transformations of words. Alternatively, we can work with a geometric realisation of the substitution, by letting it act on three intervals, whose lengths are chosen proportional to the components of the left eigenvector associated with the leading PF-eigenvalue λ of M. Each tile is then substituted with a sequence of tiles, whose total length is equal to λ times the original length. Such a geometric realisation generates a tiling of the line, instead of a sequence of symbols, or a word in a free group.

The matrix M represents a linear mapping A of $\mathbb{R}^3$, expressed with respect to some basis $\{b_i\}$. As M is unimodular, this mapping is an automorphism of the lattice L generated by this basis. We choose the geometry of L such that the expanding and contracting eigenspaces of A are perpendicular to each other, so that A commutes with the orthogonal projections on these eigenspaces. This can be realised as follows. After appropriate rescaling, the tile lengths, being components of the PF-eigenvector, are contained in the algebraic field $Q(\lambda)$, and so are all coordinates of lattice points in the expanding eigenspace of V of A. The corresponding coordinates in the contracting eigenspace W can be chosen as the $d-1$ Galois conjugates of the coordinate in V. We then have a cut-and-project scheme (CPS) defined by the lattice L, and the eigenspaces V and W of A:

$$\begin{array}{ccccc} V \cong \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}^3 & \xrightarrow{\pi_2} & W \cong \mathbb{R}^2 \\ \cup & & \cup & & \cup \\ \Lambda & & L & & \Omega \end{array} \tag{1}$$

One of the formulations of the Pisot conjecture states that the vertex set Λ of a Pisot substitution tiling always is a model set, which means that there exists a window set $\Omega \subset W$ which is the closure of its interior, and which has boundary of measure zero, such that $\Lambda = \{\pi_1(x) \,|\, x \in L, \pi_2(x) \in \Omega\}$. Similarly, the subsets of the left end points of all tiles of a given type in a Pisot substitution tiling are model sets, too, with appropriate subwindows Ω_i. For all examples considered below, the Pisot conjecture can be shown to hold, even though a proof for the general case is still missing. For a more detailed description of Pisot substitution tilings and their associated CPS, we refer to [1]. In particular, we note that one can derive also a dual, contractive substitution acting on window sets in W, whose fixed points are the subwindows Ω_i. We remark that the CPS (1) also admits a canonical projection tiling, whose window Ω is the image under the projection π_2 of the parallelepiped spanned by the basis $\{b_i\}$ of L. The subwindow for tile i, whose length is equal to the length of $\pi_1 b_i$, is simply the parallelogram spanned by the vectors $\pi_2 b_j$ and $\pi_2 b_k$, where j and k are the two indices different from i. This canonical projection tiling is not substitutional in general, but its subwindows can serve as convenient starting points for the dual substitution determining the windows. Acting on the canonical windows as seeds, the dual substitution overlap free.

2. LI and MLD Relations

The CPS (1) does not specify the window Ω yet. Substitutions having the same abelianisation matrix M (but differ in the order of the letters within a substituted word) give rise to the same CPS, but will have different windows in general. As we shall see below, even substitutions with different abelianisation matrices may belong to a common CPS.

In the following, we shall study relations between certain substitution tilings belonging to a common CPS. For this, besides the geometric realisation of a substitution tiling it is also useful to consider the substitution action on the underlying free group with three generators. In our examples, the substitution acts with a group automorphism. If for two substitutions σ_1 and σ_2 there exists a fixed word w in the group, such that $\sigma_1(g) = w^{-1}\sigma_2(g)w$ for every generator g of the group, then the two substitutions produce tilings wich are locally isomorphic (LI), meaning that all their finite subpatterns are the same. This can be seen as follows. One first observes that there exists some power of σ_1, such that σ_1^k has a bi-infinite fixed point, and that σ_1^k and σ_2^k are still conjugate in the same way, with a (longer) word w'. In a second step, one can then show that the fixed point of σ_1^k is also a fixed point of σ_2^k, which implies that the two substitutions generate the same tilings.

A more delicate relation is mutual local derivability (MLD) [2]. Two tilings are MLD, if one can be reconstructed from the other in a *local* way, and vice versa. For this to work, the two tilings must first be brought to the appropriate relative scale and position. A good starting point is to consider two tilings belonging to a common CPS. In fact, two (model set) tilings are MLD if and only if the window of one can be constructed by finite unions and intersections of lattice translates of the window of the other, and vice versa. Looking at the windows can suggest an MLD relation, but for proving such a relation it is very helpful if one substitution can be written as a conjugate of the other, $\sigma_1 = \rho^{-1} \circ \sigma_2 \circ \rho$, where ρ is an outer automorphism of the free group (an inner automorphism would lead to an LI relation). Such an automorphism will make the transformation of one tiling into the other explicit.

If a substitution σ acts invertibly on the underlying free group, the boundaries of its windows are generated by a substitution, too. This boundary substitution is given by $\sigma_b = \tilde{\sigma}^{-1}$, which

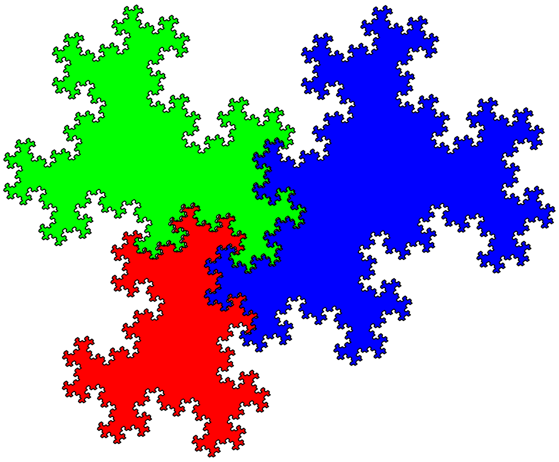

Figure 1. Windows of the substitution $a \to cb$, $b \to c$, $c \to cab$.

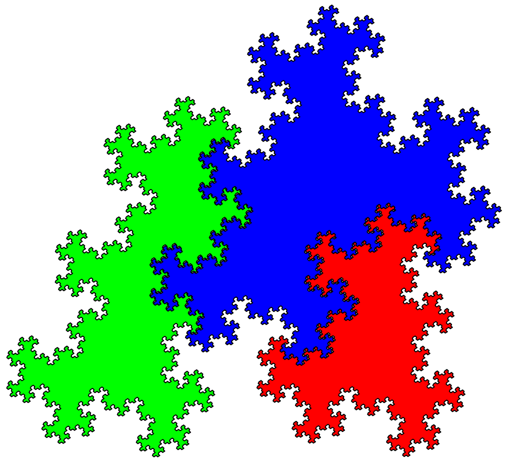

Figure 2. Windows of the substitution $a \to bc$, $b \to c$, $c \to cba$.

is the inverse of σ, read backwards [4, 5]. As a seed for the iteration, it is again convenient to take the windows of the canonical projection tiling belonging to the same CPS. For instance, the canonical window for tile a is bounded by the closed path consisting of the four consecutive segments $\pi_2 b_2$, $\pi_2 b_3$,$-\pi_2 b_2$, and $-\pi_2 b_3$. This path is represented by the word $bcb^{-1}c^{-1}$ in the free group, on which the boundary substitution σ_b acts. In each step, the boundary path is transformed into one with more, but shorter segments, eventually converging to the fractal boundary of the final window.

If we now have two substitutions σ and σ' which are conjugated, $\sigma' = \rho^{-1} \circ \sigma \circ \rho$, their boundary substitutions satisfy $\sigma'_b = \tilde{\rho}^{-1} \circ \sigma_b \circ \tilde{\rho}$. Iterating this, we find $\sigma'^n_b = \tilde{\rho}^{-1} \circ \sigma_b^n \circ \tilde{\rho}$. In the limit of a fully fractalized window, the action of $\tilde{\rho}^{-1}$ can be neglected (it is local at the scale of the then infinitesimally small segments), and σ'_b can be understood as σ_b acting on the seed of σ'_b, transformed by $\tilde{\rho}$. As the dual substitution acting on the canonical windows and their iterates is overlap free, the transformation $\tilde{\rho}$ commutes with the fractalization induced by σ_b, and it becomes manifest that the windows of σ are obtained from those of σ' via the transformation induced by $\tilde{\rho}$. In particular, the windows of the two substitutions have the same fractal structure.

In the following, different phenomena arising in this context are illustrated with a number of examples. As a short-hand notation, we write the action of a substitution σ on a free group as the list of images of the generators, in our case a triple $[\sigma(a), \sigma(b), \sigma(c)]$.

3. Examples

As a first example, we consider the substitutions $\sigma_1 = [cb, c, cab]$ and $\sigma'_1 = [bc, c, cba]$, which have the same abelianisation matrix. These two substitutions are conjugate by the free group automorphism $\rho_1 = [bab^{-1}, b, c]$, with inverse $\rho_1^{-1} = [b^{-1}ab, b, c]$. It is easily checked that indeed we have $\sigma_1 = \rho_1^{-1} \circ \sigma'_1 \circ \rho_1$. In a word generated by σ_1, there is always a b to the right of an a. ρ_1 eats up that b, and adds a b to the left of the a instead, effectively replacing ab pairs by ba pairs. ρ_1^{-1} performs the opposite operation. This is obviously a local operation, no matter whether one works with words in a free group, with symbolic sequences, or with tilings. The LI classes of tilings generated by the two substitutions are MLD. The windows of the two substitutions σ_1 and σ'_1 are shown in Figures 1 and 2, respectively. The windows for the a, b, and c tiles are in red (medium gray), green (light gray), and blue (dark gray). When transforming from Figure 1 to Figure 2, part of the b tiles (green), namely those to the right of an a tile, move to where the

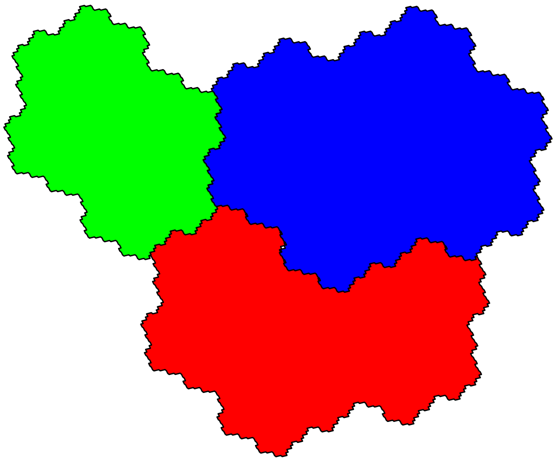

Figure 3. Windows of the substitution $a \to c$, $b \to a$, $c \to cab$.

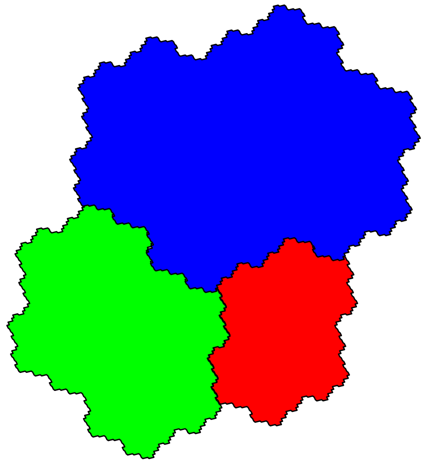

Figure 4. Windows of the substitution $a \to c$, $b \to ca$, $c \to cb$.

a tiles were before. The subwindow of the a tiles (red) thus becomes green, and a congruent copy of it is cut away from the original subwindow of the b tiles in green. The red subwindow of the a tiles instead moves to a different place, because the a tiles are now to the right of a b tile. As a and b tiles have different lengths, the left endpoint of the second tile of ab and ba pairs differs, and so the corresponding subwindows are at different places.

In the second example, we consider two substitutions with different abelianisation matrices, $\sigma_2 = [c, a, cab]$ and $\sigma_2' = [c, ca, cb]$. Again, there is a conjugating automorphism $\rho_2 = [a, a^{-1}b, c]$, with inverse $\rho_2^{-1} = [a, ab, c]$, so that $\sigma_2 = \rho_2^{-1} \circ \sigma_2' \circ \rho_2$. Here, in words produced by σ_2, all b tiles are to the right of an a tile. ρ_2 eats up the a tile to the left of a b tile, effectively replacing all ab pairs by just one b. Other a tiles (not to the left of a b) and all c tiles are left as they are. Conversely, ρ_2^{-1} splits all b in a σ_2'-word into ab pairs. On the tiling level, this operation is local if and only if the length of an ab pair of tiles in the σ_2-tiling is the same as the length of a b tile in the σ_2'-tiling, whereas a and c tiles have the same length for both tilings. With appropriate global scalings, this is indeed the case. $\sigma_2 = \rho_2^{-1} \circ \sigma_2' \circ \rho_2$ implies that the two abelianisation matrices are conjugate in $GL_3(\mathbb{Z})$. In fact, the two substitutions have the same CPS, with the same lattice L. The only difference is, that the linear mapping A is expressed with respect to two different lattice bases, yielding two different matrix representations of A, and different tile lengths (which are the lengths of the projected basis vectors). It is therefore not surprising, that the length of tile b in the σ_2'-tiling is the sum of the lengths of the two tiles a and b of the σ_2-tiling. The windows of the substitutions σ_2 and σ_2' are shown in Figures 3 and 4, respectively, using the same coloring as for the previous example. In Figure 3, part of the a tiles (in red), namely those to the left of a b tile, become the new b tiles in Figure 4 (green), whereas the old b tiles in Figure 3 (green) are discarded. MLD relations can therefore arise also if the two abelianisation matrices are not equal, but conjugate in $GL_3(\mathbb{Z})$, because the two substitutions then share a common CPS. We emphasise, however, that this relation is local only for the tilings with properly sized tiles. This pair of examples had been discussed in detail already in [3]. σ_2' is LI to the Rauzy or Tribonacci substitution.

Finally, as a third example, we consider a quartet of substitutions, all with the same abelianisation matrix M. These substitutions are $\sigma_A = [ca, ab, cab]$, $\sigma_B = [ac, ab, abc]$, $\sigma_C = [ca, ba, bac]$, and $\sigma_D = [ac, ab, bac]$. σ_A and σ_D are conjugate in a way already seen in the first example: $\sigma_D = \rho_3^{-1} \circ \sigma_A \circ \rho_3$, where $\rho_3 = [a, b, a^{-1}ca]$ simply replaces ac pairs by ca pairs. In order to discuss the relations to the other substitutions, we introduce the automorphisms

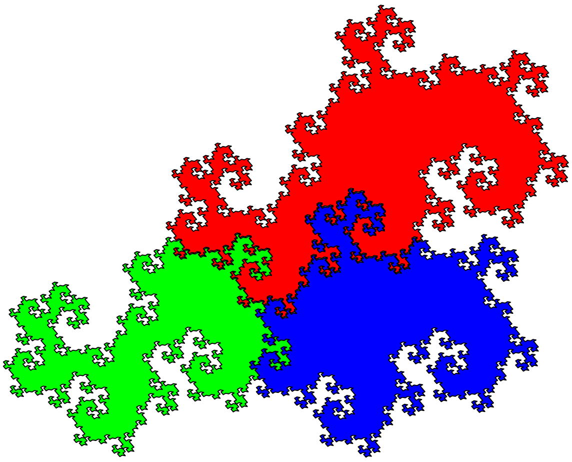

Figure 5. Windows of the substitution $a \to ca$, $b \to ab$, $c \to cab$.

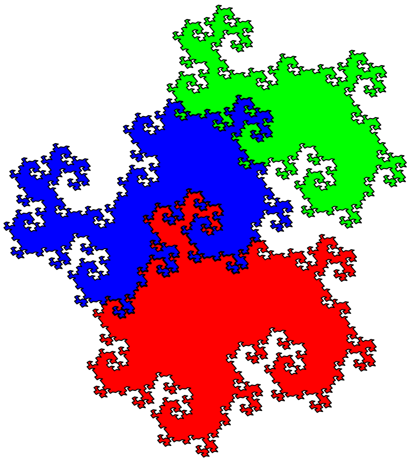

Figure 6. Windows of the substitution $a \to ac$, $b \to ab$, $c \to abc$.

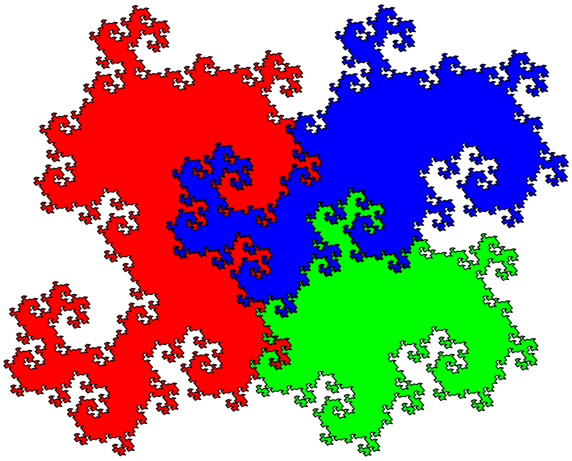

Figure 7. Windows of the substitution $a \to ca$, $b \to ba$, $c \to bac$.

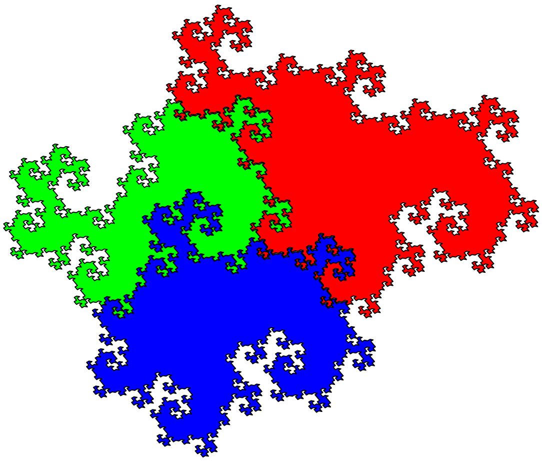

Figure 8. Windows of the substitution $a \to ac$, $b \to ab$, $c \to bac$.

$u_1 = [c, a, ab]$ and $u_2 = [c, a, ba]$. We then have $\sigma_A = u_1 \circ u_1 \circ u_2$, $\sigma_B = u_1 \circ u_2 \circ u_1$, and $\sigma_C = u_2 \circ u_1 \circ u_1$, so that $\sigma_A = u_1 \circ \sigma_B \circ u_1^{-1}$ and $\sigma_C = u_1^{-1} \circ \sigma_B \circ u_1$. The conjugating automorphism u_1 has an abelianisation matrix U which commutes with the common abelianisation matrix M of the substitutions, even though U is non-trivial. This is possible because M is equal to the third power of U. Therefore, M is conjugate to itself by some non-trivial mapping, which acts non-trivially on the lattice L, changing the scale of the tiling by the cubic root of the inflation factor λ of the substitution (or its inverse). Consequently, in order to be MLD, the tilings produced by σ_A, σ_B and σ_C must be at relative scales $\lambda^{-\frac{1}{3}}$, 1, and $\lambda^{\frac{1}{3}}$, respectively. The situtation is in fact similar to the second example, where the substitutions share a common CPS, but different lattice bases of L are used. Here, these different lattice bases still lead to the same abelianisation matrix M, but produce tiles of different sizes. The windows of the substitutions σ_A, σ_B, σ_C, and σ_D are shown if Figures 5 to 8, respectively. We don't discuss the transformations between them in detail here, but it is quite obvious that each subwindow can be obtained as translate or union of translates of subwindows of the other substitutions.

4. Conclusions

The CPS of a Pisot substitution tiling can accommodate many other substitution tilings as well. Some of these are obtained by permuting the letters in the substituted words, but there may be others arising from the choice of a different basis for the lattice of the CPS, as we have seen in the second example. The more complicated a substitution is, the richer is the set of substitution tilings supported by its CPS. Some of the tilings sharing a common CPS can be related in interesting ways, however. In particular, there may local isomorphism or mutual local derivability relations, sometimes in surprising ways. We have discussed some of the phenomena that may arise in this context, and have illustrated them with a number of examples. The key tool was to formulate the substitution as an automorphism of an underlying free group, which allowed to find the relations with algebraic methods, and to make the transformations between the related tilings explicit.

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