Stabilization by Additive Noise

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Stabilization by Noise

Well known phenomenon due to Multiplicative Noise. For example:

1. **By Itô noise,** due to Itô-Stratonovic correction:
    - **For SDE:** [Arnold, Crauel, Wihstutz '83], [Pardoux, Wihstutz '88 '92].....
    - **For SPDE:** [Kwiecinska '99],[Caraballo, Mao et.al. '01], [Cerrai '05], [Caraballo, Kloeden, Schmallfuß '06]....
2. **By Rotation:**[Baxendale et.al.'93], [Crauel et.al.'07].....

Consider here:

- Degenerate additive noise
- effect of noise transported by the nonlinearity
- Stabilization effects on dominating behaviour

Introduction

AIM:

- Rigorous error estimates for Amplitude equations
- Understand interplay between noise and nonlinearity

- SPDEs of Burgers-type near a change of stability
- Dominant modes evolve on a slow time-scale
- Stable modes decay on a fast time-scale
- Evolution of dominant modes given by Amplitude eq.
- Formal derivation well known [Cross, Hohenberg, '93]

Examples

1. **Burgers equation**

$$\partial_t u = \partial_x^2 u + \nu u + u\partial_x u + \sigma\xi$$

2. **Surface Growth**

$$\partial_t h = -\partial_x^4 h - \nu\partial_x^2 h - \partial_x^2|\partial_x h|^2 + \sigma\xi$$

3. **Rayleigh Bénard Convection**
   3D-Navier-Stokes coupled to a heat equation

An Equation of Burgers type

For simplicity we consider only a scalar Burgers equation.

Equation of Burgers type

$$\partial_t u = (\partial_x^2 + 1)u + \nu\epsilon^2 u + u\partial_x u + \epsilon^2 \xi \qquad \text{(B)}$$

- $u(t,x) \in \mathbb{R}$, $t > 0$, $x \in [0,\pi]$
- Dirichlet boundary conditions $(u(t,0) = u(t,\pi) = 0))$
- Moving frame $\int_0^\pi u(t,x)dx = 0$
- $\nu\epsilon^2 u$ linear (in)stability
- $|\nu\epsilon^2| \ll 1$ distance from bifurcation
- $\xi(t,x)$ Gaussian white noise

The Linear Operator

The Linear Operator:

$L = -\partial_x^2 - 1$ Dirichlet b. c. on $[0, \pi]$ and $\int_0^\pi u(x)dx = 0$

- Orthonormal system generated by $\sin(kx)$, $k = 1, 2, \ldots$
- Eigenvalues: $\lambda_k = k^2 - 1, \quad k = 1, 2, \ldots$

$$0 = \lambda_1 < \omega < \lambda_2 < \ldots < \lambda_k \to \infty$$

- The dominant mode

$$\mathcal{N} = \mathrm{span}\{\sin\} \text{ – the kernel of } L$$

The Noise

Two cases of noise:

- First:
  White noise acting directly on $\mathcal{N}$
- Later:
  Degenerate noise not acting on $\mathcal{N}$

Wiener Process

$$\partial_t u = (\partial_x^2 + 1)u + \nu\epsilon^2 u + u\partial_x u + \epsilon^2 \xi \qquad \text{(B)}$$

Noise: $\xi(t,x) = \partial_t W(t,x)$

$$W(t,x) = \sum_{k=1}^{\infty} \sigma_k \beta_k(t) \sin(kx)$$

- $\sigma_k \in \mathbb{R}, \quad |\sigma_k| \le C$
- $\{\beta_k\}_{k\in\mathbb{N}}$ i.i.d. Brownian motions

Remark: For space-time white noise $\sigma_k = 1 \ \forall k$.

Question:

How does noise affects the dynamics of dominant modes in $\mathcal{N}$?

The Amplitude Equation

$$\partial_t u = (\partial_x^2 + 1)u + \nu\epsilon^2 u + u\partial_x u + \epsilon^2 \partial_t W \qquad \text{(B)}$$

Ansatz:

$$u(t,x) = \epsilon a(\epsilon^2 t)\sin(x) + \mathcal{O}(\epsilon^2)$$

Result: Amplitude Equation

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta, \qquad \text{(A)}$$

where $\beta(T) = \epsilon\sigma_1\beta_1(\epsilon^{-2}T)$ rescaled noise in $\mathcal{N}$.

Interesting fact:

Nonlinearity $B(u,v) = \frac{1}{2}\partial_x(uv)$ does not map $\mathcal{N}$ to $\mathcal{N}$!
Higher order modes are involved!

Formal Calculation

$$\partial_t u = -Lu + \nu\epsilon^2 u + B(u,u) + \epsilon^2 \partial_t W \qquad \text{(B)}$$

Ansatz:

$$u(t,x) = \epsilon \underbrace{A(\epsilon^2 t)}_{\in\mathcal{N}} + \epsilon^2 \underbrace{\psi(\epsilon^2 t)}_{\perp\mathcal{N}} + \ldots$$

Thus $(T = \epsilon^2 t,\ P_c$ Projection onto $\mathcal{N},\ P_s = I - P_c)$
as $P_c B(A,A) = 0$

$$\partial_T A = \nu A + 2P_c B(A,\psi) + \partial_T P_c \tilde{W} + \mathcal{O}(\epsilon)$$

and

$$\epsilon^2 \partial_T \psi = -L\psi + P_s B(A,A) + \epsilon \partial_T P_s \tilde{W} + \mathcal{O}(\epsilon)\ ,$$

where $\tilde{W}(T) = \epsilon W(\epsilon^{-2} T)$.

Formal Calculation II

Neglecting all small terms leads to

$$\partial_T A = \nu A + 2P_c B(A,\psi) + \partial_T P_c \tilde{W}$$

with $\psi = L^{-1} P_s B(A,A)$.

With $A(T) = a(T)\sin$

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta, \qquad \text{(A)}$$

where $-\frac{1}{12} = 2P_c B(\sin, L^{-1} P_s B(\sin,\sin))$.

The Theorem

$$\partial_t u = (\partial_x^2 + 1)u + \nu\epsilon^2 u + u\partial_x u + \epsilon^2 \partial_t W \qquad \text{(B)}$$

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta \qquad \text{(A)}$$

Theorem – Approximation

u is solution of (B) – a is solution of (A)
$u(0) = \epsilon a(0) \sin + \epsilon^2 \psi_0$ with $\psi_0 \perp \sin$

Then for κ, T_0, p, $\delta > 0$ there is $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t \in [0, T_0 \epsilon^{-2}]} \|u(t) - \epsilon a(t\epsilon^2) \sin \|_\infty > \epsilon^{2-\kappa} \Big)$$

$$< C\epsilon^p + \mathbb{P}(|a(0)| > \delta) + \mathbb{P}(\|\psi_0\|_\infty > \delta)$$

Impact of the Noise

Recall:

Dominant modes driven only by noise acting on $\mathcal{N}$.

No impact of $\beta_2, \ \beta_3, \ \dots$

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta \ , \qquad \text{(A)}$$

where $\beta(T) = \epsilon \sigma_1 \beta_1(\epsilon^{-2} T)$ rescaled noise in $\mathcal{N}$.

The Noise II

Assumption:

No noise on the dominant mode: $\sigma_1 = 0$

$$W(t) = \sum_{k=2}^{\infty} \sigma_k \beta_k(t) \sin(k\cdot)\ , \qquad \xi(t) = \partial_t W(t)$$

Question: How does noise interacts with the nonlinearity?

Two extreme cases:

- Noise only on the second mode
  $\sigma_k = 0$ for $k \neq 2$
- Near white noise
  $\sigma_k = 1$ for $k \geq 2$

Stabilisation by Additive noise – Setting

$$\partial_t u = (1+\partial_x^2)u + \nu\epsilon^2 u + u\partial_x^2 u + \sigma\phi$$

- Dirichlet boundary conditions on $[0,\pi]$, $\int_0^\pi u dx = 0$
- $\mathcal{N} = \text{span}\{\sin\}$ – One dominating mode
- $\phi(t,x) = \partial_t\beta_2(t)\sin(2x)$ – Noise only on 2nd mode

Previous Result

Previous Approximation Result:

If $\sigma = \epsilon^2$, then for $t \in [0, T_0\epsilon^{-2}]$

$$u(t) = \epsilon a(\epsilon^2 t) \sin + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \partial_T a = \nu a - \tfrac{1}{12} a^3$$

No impact of Noise!

Need larger Noise!

Stabilisation by Additive noise – Result

Consider larger noise

$$\partial_t u = (1+\partial_x^2)u + \nu\epsilon^2 u + u\partial_x^2 u + \epsilon\phi \qquad \text{(B2)}$$

Amplitude Equation [DB, Hairer, Pavliotis, 07]

$$da = (\nu - \tfrac{1}{88})a\,dT - \tfrac{1}{12}a^3\,dT + \tfrac{1}{6}a \circ d\tilde\beta_2 \qquad \text{(A2)}$$

in Stratonovic sense, with $\tilde\beta_2(T) = \epsilon\beta_2(\epsilon^{-2}T)$.

- Stabilisation effect for $\nu \in (0, 1/88)$.
- Problem: $u(t) - \epsilon a(\epsilon^2 t)\sin \approx \frac{\epsilon^2}{\lambda_1}\underbrace{\partial_T\tilde\beta_2(T)}_{\text{white noise}}\sin(2\cdot) + \mathcal{O}(\epsilon^2)$

Formal Motivation

$$da = (\nu - \tfrac{1}{88})a\,dT - \tfrac{1}{12}a^3\,dT + \tfrac{1}{6}a \circ d\tilde{\beta}_2 \qquad \text{(A2)}$$

Stabilization effect

Itô to Stratonovic correction is $-\frac{1}{72}a$
Where does the other term comes from?

Consider slow time: $(u(t) = \epsilon\psi(\epsilon^2 t))$

$$\partial_T\psi = -\epsilon^{-2}L\psi + \nu\psi + \epsilon^{-1}B(\psi,\psi) + \epsilon^{-1}\partial_T\tilde{\Phi} \qquad \text{(B2')}$$

Formal Calculation

$$\partial_T \psi = -\epsilon^{-2} L\psi + \nu\psi + \epsilon^{-1} B(\psi,\psi) + \epsilon^{-1}\partial_T \tilde{\Phi}_2$$

Ansatz with $\psi_k \in \text{span}(\sin(kx))$:

$$\psi(T) = \psi_1(T) + \psi_2(T) + \epsilon\psi_3(T) + \mathcal{O}(\epsilon)$$

1st mode: (using $B_n(\psi_k,\psi_l) = 0$ for $k \notin \{n-l, n+l, l-n\}$)

$$\partial_T \psi_1 = \nu\psi_1 + 2\epsilon^{-1} B_1(\psi_2,\psi_1) + 2B_1(\psi_2,\psi_3) + \mathcal{O}(\epsilon)$$

2nd mode: $L\psi_2 = \epsilon B_2(\psi_1,\psi_1) + \epsilon\partial_T \Phi_2 + \mathcal{O}(\epsilon^2)$

3rd mode: $L\psi_3 = 2B_3(\psi_2,\psi_1) + \mathcal{O}(\epsilon)$

New contribution to 1st mode:

$$4\epsilon^2 B_1(L^{-1}\partial_T\Phi_2, L^{-1}B_3(\partial_T\Phi_2,\psi_1))$$

Formal Motivation

New contribution to 1st mode:

$$4\epsilon^2 B_1(L^{-1}\partial_T\Phi_2, L^{-1}B_3(\partial_T\Phi_2,\psi_1)) = c(\epsilon\partial_T\tilde{\beta}_2)^2 A$$

What is noise²?

Instead of $\epsilon\partial_T\tilde{\beta}_2$ we use $Z_\epsilon(T) = \epsilon^{-1}\int_0^T e^{-3(T-s)\epsilon^{-2}} d\tilde{\beta}_2(s)$.

Lemma: Averaging with error bounds

$$\int_0^T A(s)Z_\epsilon(s)^2 ds = \frac{1}{6}\int_0^T A(s)ds + \mathcal{O}(\epsilon)$$

Stabilisation by Additive noise – Theorem

DB, Hairer, Pavliotis, 07

Let u be a continuous $H_0^1([0,\pi])$-valued solution of (B2) with $u(0) = \epsilon a(0) + \epsilon\psi_0$ and $\psi_0 \perp \sin$. Let a be a solution of (A2) and define

$$R(t) = e^{-Lt}\psi_0 + \left(\int_0^t e^{-3(t-s)} d\beta_2(s)\right) \sin(2\cdot),$$

then for all $\delta, \kappa, p, T_0 > 0$ there is a constant C such that

$$\mathbb{P}\left(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t) - \epsilon a(\epsilon^2 t)\sin(\cdot) - \epsilon R(t)\|_{H^1} > \epsilon^{3/2-\kappa}\right)$$

$$\leq C\epsilon^p + \mathbb{P}\left(\|u(0)\|_{H^1} > \delta\epsilon\right) .$$

More Noise – Near White Noise

What about more noise?

More Noise – Near White Noise

$$\partial_t u = (1+\partial_x^2)u + \nu\epsilon^2 u + u\partial_x^2 u + \epsilon\xi \qquad \text{(B3)}$$

with $\xi(t,x) = \partial_t \sum_{k=2}^{\infty} \beta_k(t)\sin(kx)$ (near white noise)

Amplitude Equation

There is a Brownian motion B such that

$$da = \tilde{\nu} a dT - \frac{1}{12}a^3 dT + \sqrt{\sigma_a a^2 + \sigma_b}\, dB \qquad \text{(A3)}$$

for some constants $(\tilde{\nu},\ \sigma_a,\ \sigma_b)$.

Multiplicative AND Additive Noise!

More Noise – Theorem

DB, Hairer, Pavliotis, 07

For $\alpha \in [0, \frac{1}{2})$ let u be a continuous $H_0^\alpha([0,\pi])$-valued solution of (B3) with $u(0) = \epsilon a(0) + \epsilon\psi_0$ and $\psi_0 \perp \sin$. Let a be a solution of (A3) and define

$$R(t) = e^{-tL}\psi_0 + \int_0^t e^{-(t-s)L} dW(s) \, .$$

Then for all $\kappa, \delta, p, T_0 > 0$ there is a constant $C > 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t) - \epsilon a(\epsilon^2 t)\sin(\cdot) - \epsilon R(t)\|_{H^\alpha} > \epsilon^{5/4-\kappa} \right)$$

$$\leq C\epsilon^p + \mathbb{P}\left(\|u(0)\|_{H^\alpha} > \delta\epsilon\right) \, .$$

Summary

- SPDEs of Burgers type near a change of stability
- Approximation of transient dynamics via amplitude equations
- Stabilisation by additive noise
- Effect of noise on dominant modes
- Noise transported by nonlinearity between Fourier-modes

Further results:

- Attractivity results
- Approximation of moments
- Approximation of invariant measures