Freezing Waves in Neurons

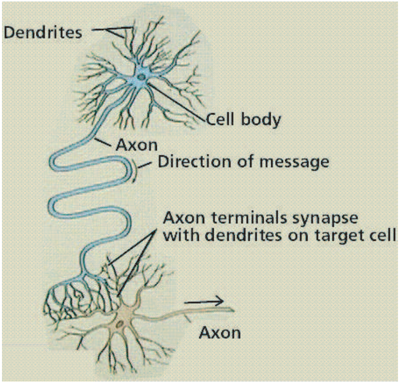
Gabriel Lord
Heriot Watt University, Edinburgh

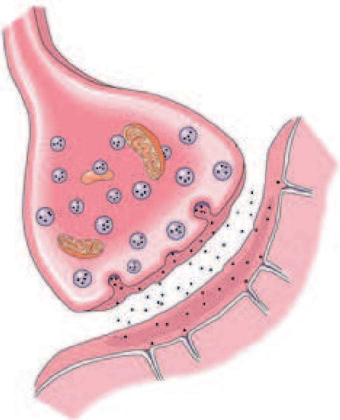
gabriel@ma.hw.ac.uk, http://www.ma.hw.ac.uk/∼gabriel

With Vera Thümmler - Bielefeld thuemmler@math.uni-bielefeld.de
With Emma Coutts - Heriot-Watt emmac@ma.hw.ac.uk

- Nagumo equation
  - Deterministic and SPDE
  - Stochastic Travelling Wave
  - Computations
- Spike Diffuse Spike : cable and spines
  - Stochsatic Travelling wave (not frozen)
  - Noise induced propagation
  - Wavespeeds

Neurons and Synapses





... interested in travelling wave propagation

Deterministic Nagumo - axon propagation

$$u_t = \Big[u_{xx} + u(1-u)(u-\alpha)\Big] \quad u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t > 0$$

where $\alpha \in (0, \frac{1}{2})$.

- Explicit TW solution connecting $u \equiv 1$ and $u \equiv 0$

$$u_{\text{det}}(x - \lambda t) = \left(1 + e^{\frac{\lambda t - x}{\sqrt{2}}}\right)^{-1}, \quad \text{wavespeed} \quad \lambda = \sqrt{2}\,(\tfrac{1}{2} - \alpha)$$

- Suppose we have a TW with wavespeed λ for

$$u_t = \Big[u_{xx} + f(u)\Big].$$

Into co-moving frame $u(x,t) = u(x - \lambda t, t)$

$$u_t = u_{xx} + \lambda u_x + f(u), \quad x \in \mathbb{R}, \quad t \geq 0 \qquad (1)$$

of which the travelling wave u is a stationary solution $(u_t = 0)$.

Deterministic case II

$$u_t = \left[u_{xx} + f(u) \right] \quad u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t > 0.$$

- What if we do not know wavespeed or wavespeed a func. of t?

Co-moving frame : unknown position $\gamma(t)$ and wavespeed $\lambda(t)$
$u(x,t) = u(x - \gamma(t), t)$

$$u_t = u_{\xi\xi} + \lambda(t) u_\xi + f(u)$$

Position of wave $\gamma(t) = \int_0^t \lambda(s) ds$.

Have an extra variable $\lambda(t)$ – add a phase condition $0 = \psi(u, \lambda)$.

Example phase condition :
Given a reference function $\hat{u}$, $\min \|u - \hat{u}\|_2^2$.

Stochastic Nagumo

$$du = \Big[u_{xx} + u(1-u)(u-\alpha)\Big]\,dt + (\nu + \mu u(1-u))dW(t)$$

Multiplicative noise : $\nu = 0$, $\mu \neq 0$ – parameter α (wave speed).

$$du = \Big[u_{xx}+f(u)\Big]\,dt+g(u,t)dW(t), \qquad \text{given} \quad u(0) = u_0, \qquad x \in \mathbb{R}.$$

- Assume Wiener processes of the form

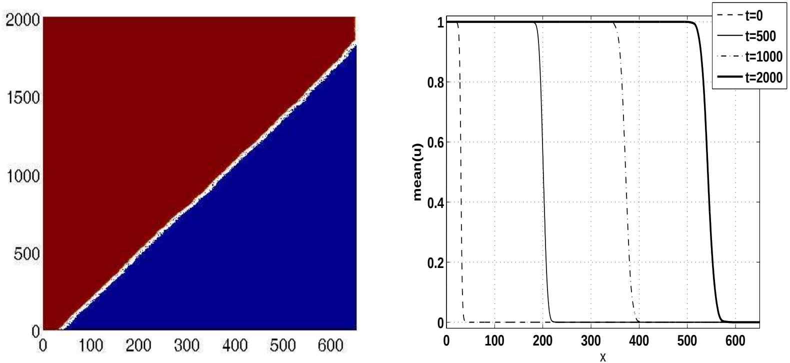
$$W(x,t) \ = \sum_{n\in\mathbb{Z}} b_n\phi_n(x)\beta_n(t) \ , \qquad \beta_n \text{ iid Brownian motions.}$$

Take space-time white noise and noise with exponential decay in correlation length.

- Results :

– equation for mean profile of front, small noise expansion
– wavespeed increases with noise intensity (Stratonovich).
[Armero, Sancho, Lacasta, Ramirez-Piscina, Sagues, 1996]

Stochastic Travelling wave



Typically :

- Reference to deterministic wave (eg small noise) Mikhailov, Schimansky–Geier & Ebeling '83
- Evolution of a level set : eg sFKPP Tribe, Elworthy& Zhao, Mueller & Sowers & Doering,

For reviews see for example : Garcia-Ojalvo & Sancho or Panja

Freezing a Stochastic Travelling wave

$$du = \Big[u_{xx} + f(u)\Big]\, dt + g(u,t)dW(t)$$

1) Add convection term to freeze wave

$$du = \Big[u_{xx} + f(u) + \lambda u_x\Big]\, dt + g(u,t)dW(t),$$

2) For some reference function $\hat{u}$ want :

$$\min \|u - \hat{u}\|_2^2$$

- SPDE has a travelling wave u if there exists a rv λ
s.t. $\|u - \hat{u}\|$ is minimized and v satisfies SPDAE

$$\begin{aligned} du &= [u_{xx} + \lambda(t)u_x + f(u)]\, dt + g(u)dW, \quad u(0) = u^0 \\ 0 &= \langle \hat{u}_x, u - \hat{u}\rangle \end{aligned} \qquad (2)$$

- $\lambda(t)$ "instantaneous" wave speed
- Wavespeed : $\Lambda(t) = \frac{1}{t}\int_0^t \lambda(s)ds$

Implementation : SPDAE

SPDAE : (on finite domain)

$$\begin{aligned} du &= [u_{xx} + \lambda(t)u_x + f(u)]\,dt + g(u)dW, \quad u(0) = u^0 \\ 0 &= \langle \hat{u}_x, u - \hat{u} \rangle \end{aligned}$$

Discretize in space
Semi–implicit Euler–Maruyama scheme gives :

$$\begin{aligned} u^{n+1} &= u^n + \Delta t \left[A u^{n+1} + \lambda^{n+1} D_{\lambda^n} u^n + f(u^n) \right] + g(u^n)\Delta W_n \\ 0 &= \langle \hat{u}_x, u^{n+1} - \hat{u} \rangle \end{aligned} \tag{3}$$

where ΔW_n is our Brownian increment.

$$\begin{pmatrix} I - \Delta t A & -\Delta t D_{\lambda^n} v^n \\ \Delta x\, D\hat{u} & 0 \end{pmatrix} \begin{pmatrix} v^{n+1} \\ \lambda^{n+1} \end{pmatrix} = \begin{pmatrix} v^n + \Delta t f(v^n) + g(u^n)\Delta W_n \\ \langle D\hat{u}, \hat{u} \rangle \end{pmatrix}.$$

Other time discretizations possible.

Frozen Nagumo Multiplicative Noise, $\alpha = 0.25$

Wavespeeds for SPDAE are computed from random variable $\lambda(t)$

$$E\lambda(t), \quad \Lambda(t) = \frac{1}{t}\int_0^t \lambda(s)ds, \quad E\Lambda(t), \quad \Lambda = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} E\Lambda(t)dt.$$

Example : 1000 realizations , $T_1 = 100$, $T_2 = 200$

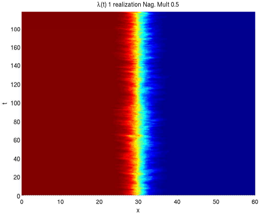
	$\mu = 0$	$\mu = 0.25$	$\mu = 0.5$	$\mu = 1$
Λ	0.354088	0.355231	0.359169	0.371950

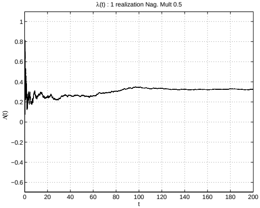
[$\hat{u} = u_{\text{det}}$ and $u^0 = u_{\text{det}}$]

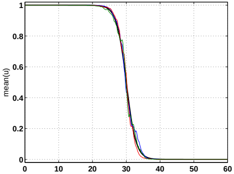
- Increasing in limiting wavespeed with noise intensity.

Frozen Nagumo Multiplicative, $\mu = 0.5$

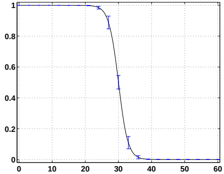
Single realization $\Lambda(t)$ 3 samples t=200.

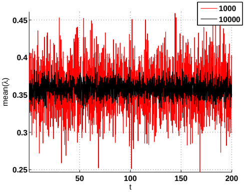


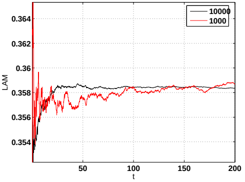




And means over 1000 and 10000 realizations:







Mean $E\lambda(t)$ $E\Lambda(t)$

Comparison with SPDE

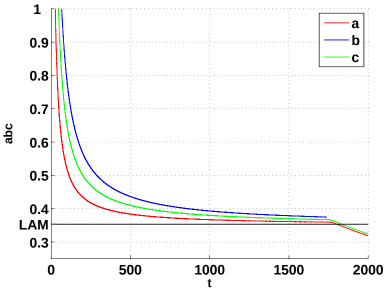
For SPDE wavespeeds compute $a(t)/t$, $b(t)/t$, $c(t)/t$

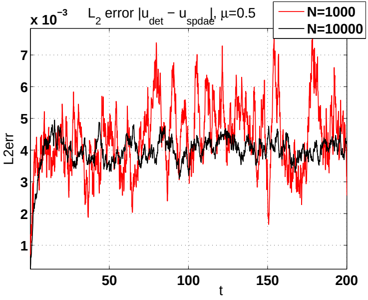
$$a(t) := \sup\{z : u(x,t) = u_-, x \leq z\}$$

$$b(t) := \sup\{z : u(x,t) = u_+, x \geq z\}$$

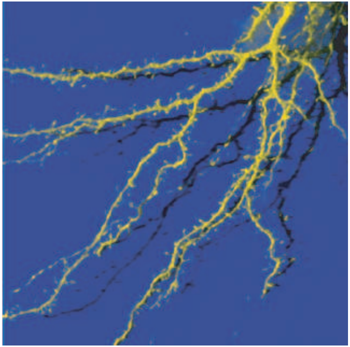
$$c(t) := \sup\{z : u(x,t) = (u_- + u_+)/2, x \leq z\}.$$

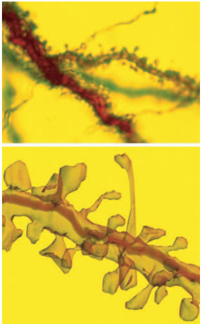
(also in literature $z(t) = \int_{x_0}^{L} u(x,t)dx$.)





Dendrites and spines

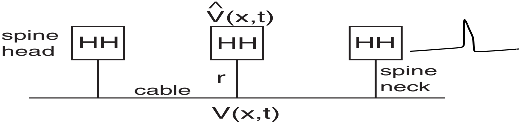




(synapses.mcg.edu)

- Spines provide surface area for synapses from other neurons
- $\approx$ 80% excitable synapses at dendritic spines in cortex
- Evidence for plasticity in spines
- Involved in : learning, memory, logic computation, pattern matching, temporal filtering

Baer-Rinzel Discrete Model



[J Neurophys, 65, 1991]

- Cable Equation :

$$\pi a C_m V_t = \frac{\pi a^2}{4R_a} V_{xx} - \frac{\pi a}{R_m} V + \rho \frac{\hat{V} - V}{r}$$

- Spine Dynamics :

$$\begin{aligned} \hat{V}_t &= -I(\hat{V}, m, n, h) - \frac{\hat{V} - V}{r} \\ \tau_X X_t &= X_\infty - X, \qquad X \in \{m, n, h\} \end{aligned}$$

- Discrete set of spines at $x = x_n$: $\rho(x) = \sum_n \delta(x - x_n)$

d := distance between spines

Spike Diffuse Spike Model

[with S. Coombes + Y. Timofeeva]

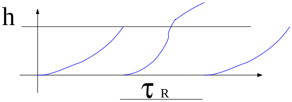
- Cable equation :

$$V_t = DV_{xx} - \tau V + Dr_a \sum_n \delta(x - x_n) \frac{\hat{V} - V}{r}$$

Action potentials $\hat{V}$ given by a function of a fixed form:

e.g. square function 

- Spine head dynamics : Integrate and fire process



$$\frac{dU_n}{dt} = -CU_n + \frac{V(x_n,t) - U_n}{r} - \underbrace{\widehat{C}h \sum_m \delta(t - T_n^m)}_{\text{RESET}},$$

- Refractory time τ_R.

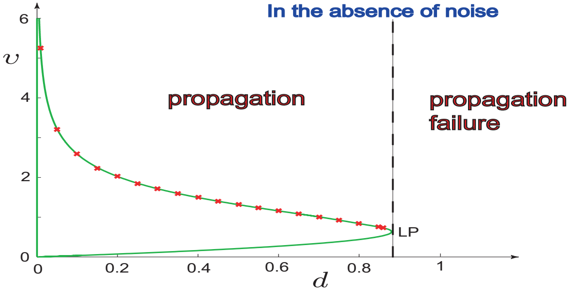
T_n^m : *mth* Firing time of *n*th spine + Refractory time.

SDS and Noise

- Stochastic gating of ion channels
- Voltage fluctuation in cable membrames

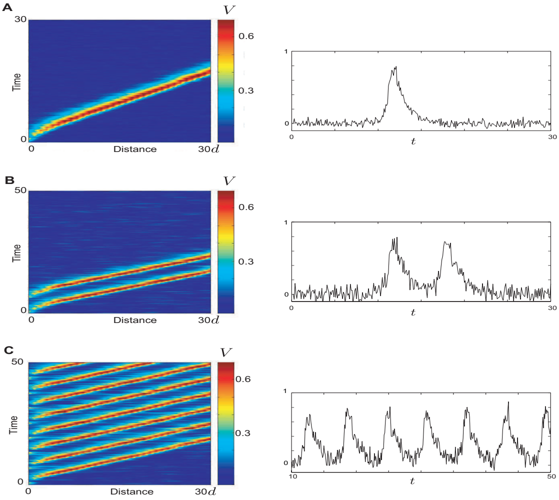
$$\mathrm{d}V = \left[D\Delta V - \frac{V}{\tau} + Dr_a\rho(x)\frac{\widehat{V} - V}{r} \right] \mathrm{d}t + \mu_V \mathrm{d}W_V(t,x),$$

$$\mathrm{d}U_n = \left[\frac{V_n}{\widehat{C}r} - \varepsilon_0 U_n - h\sum_m \delta(t - T_n^m) \right] \mathrm{d}t + \mu_U \mathrm{d}W_U(t,x).$$



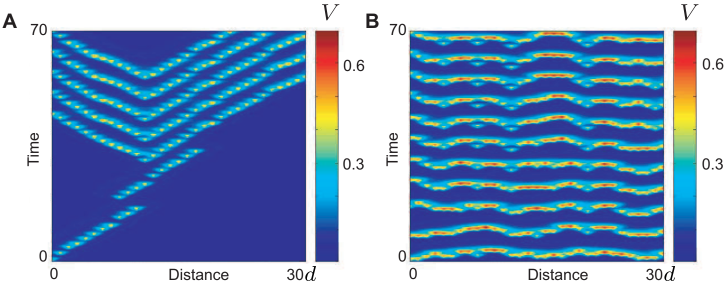
Noise induced propagation: $d = 1$, $\mu_U = 0$

Increasing μ_V: $\mu_V = 0.4, 0.8, 0.81$.



Noise induced propagation: $d = 1$, $\mu_V = 0$

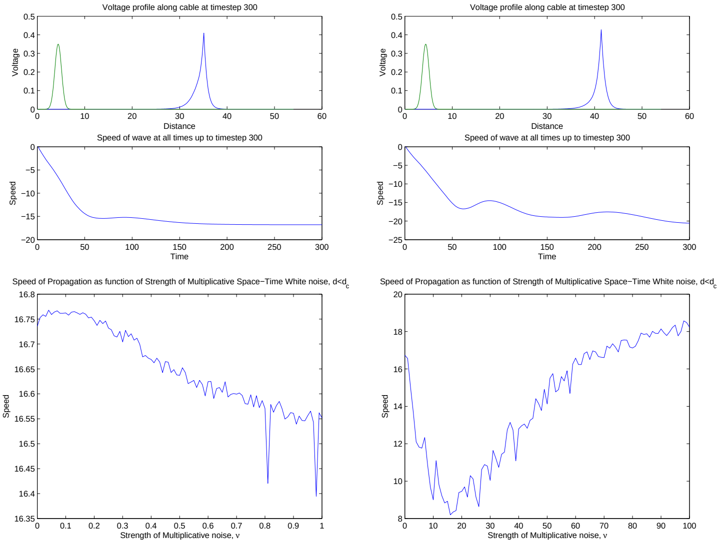
Increasing $\mu_U = 0.17, \mu_U = 0.4$.



More sensitive to noise in the spine heads.

Freezing the SDS

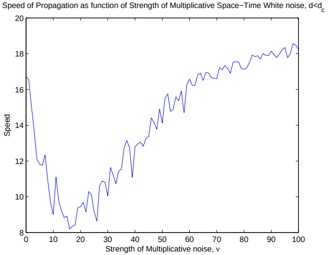
Small ($\mu = 0.6$) and large ($\mu = 80$) noise. Deterministic propagates.



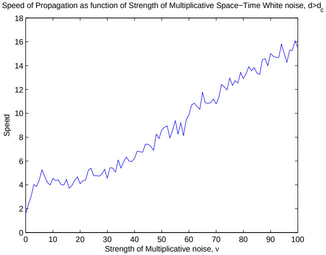
SDS speeds

Computed by freezing:

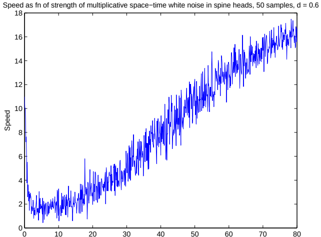
Det Propagates

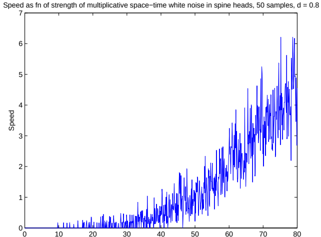


Noise Induced



Computed by level set :





Summary

- Definition of a stochastic travelling wrt reference $\hat{u}$.
- Results agree with those by direct simulation of SPDE
- Can apply to complicated solutions (eg SDS model)

Some adavantages of method

  - Well defined profile & wavespeed
  - Efficient : smaller domain
  - Does not rely on deterministic wave/ small noise
  - Faster convergence than via level sets a, b, c.

- Disadvantage of method
  - in transients numerical insability - large convection (u0)
  - need for a reference solution uref

Rome ...

Call for Mini-Symposia :

SIAM Conference on Nonlinear Waves and Coherent Structures

July 21-24, 2008
Università di Roma
“La Sapienza”, Rome, Italy

Also : sde-net https://list-serve.hw.ac.uk/mailman/listinfo/sde-net