# Random perturbations of critical equilibria

## application to hysteresis and conduction

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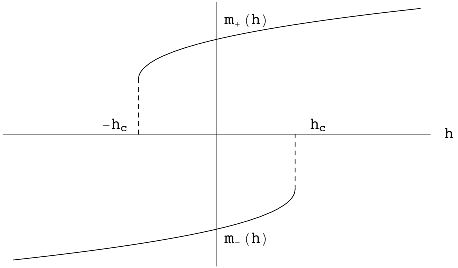
# Random Hysteresis Loops

Dynamical hysteresis appears when a time dependent magnetic field $h$ is applied to a ferromagnet whose temperature is kept fixed below the critical value



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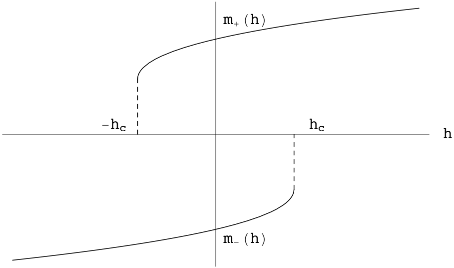


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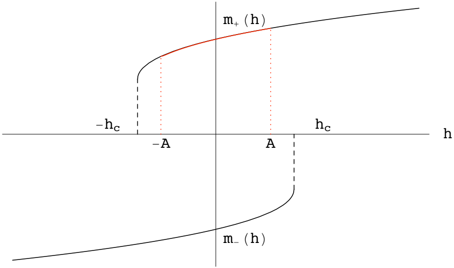


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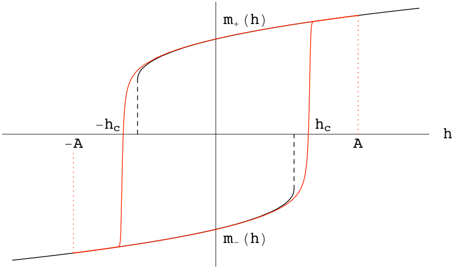


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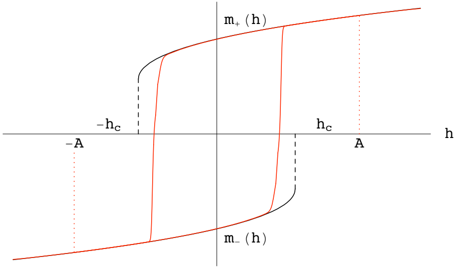


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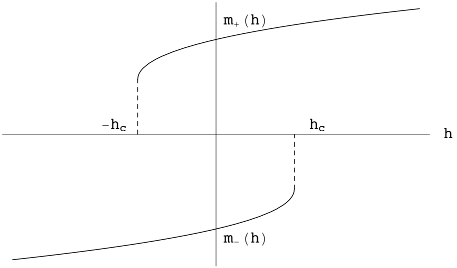


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- [mean field Ising model with Glauber dynamics]
  - critical $\omega$ and critical $A$ $\rightarrow$ random hysteresis cycles

# The Mean Field Ising Model

Let $\Lambda$ be a bounded region of $\mathbb{Z}^d$, we denote by $\sigma$ an Ising spin configuration in $\Lambda$,

$$\sigma = \{\sigma(i),\, i \in \Lambda\}, \quad \sigma : \Lambda \to \{-1, +1\}$$

and by $\mathcal{X} = \{-1, +1\}^N$ the phase space.

Let $N = |\Lambda|$, the magnetization density of the configuration $\sigma$ is

$$m_N = m_N(\sigma) := \frac{1}{N} \sum_{i \in \Lambda} \sigma(i)$$

$m_N$ takes values in $\mathcal{M}_N := \frac{1}{N}\Big\{ -N, -N+2, ..., N-2, N \Big\}$.

Let $h$ be the external magnetic field, the mean field hamiltonian is

$$H_{h,N}(\sigma) := N\Big( -\frac{m_N(\sigma)^2}{2} - h m_N(\sigma) \Big)$$

Let $\beta > 0$ be the inverse temperature, at the equilibrium the system is described by the mean field Gibbs measure

$$G_{\beta,h,N}(\sigma) := \frac{e^{-\beta H_{h,N}(\sigma)}}{Z_{\beta,h,N}}, \qquad Z_{\beta,h,N} := \sum_{\sigma \in \mathcal{X}_{m,N}} e^{-\beta H_{h,N}(\sigma)}$$

$\mathcal{X}_{m,N}$ the canonical ensemble of magnetization $m$.

The canonical free energy density is

$$\mathcal{F}_{\beta,h,m,N} = -\frac{1}{\beta N} \log Z_{\beta,h,N}$$

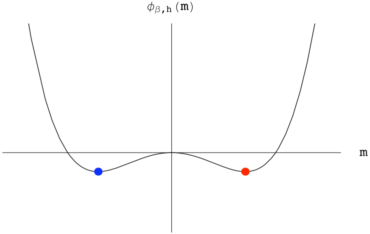
For any $m \in (-1,1)$,

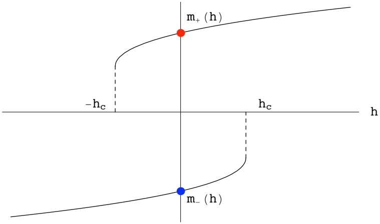
$$\lim_{N \to \infty} \mathcal{F}_{\beta,h,m,N} = \phi_{\beta,h}(m),$$

$$\phi_{\beta,h}(m) = \left\{ -\frac{m^2}{2} - hm \right\} - \frac{1}{\beta} I(m)$$

$$I(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2}$$

Let $\beta > 1$, then there exists $h_c > 0$ such that $\phi_{\beta,h}(m)$ is a double well for $|h| \leq h_c$.




$h = 0$

The critical points of $\phi_{\beta,h}(m)$, $m_\pm(h)$ and $m_0(h)$ satisfy the

mean field equation $$F(m,h) := -m + \tanh\{\beta(m+h)\} = 0$$

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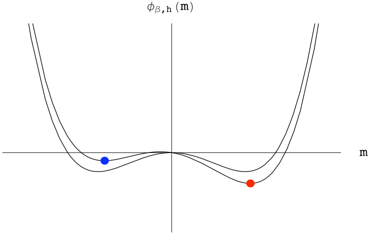


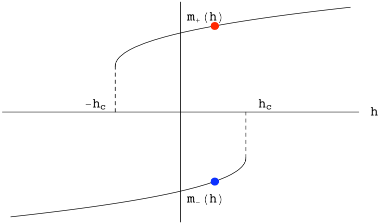


$h_c > h > 0$

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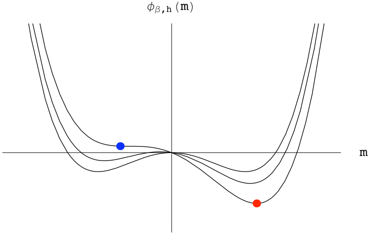


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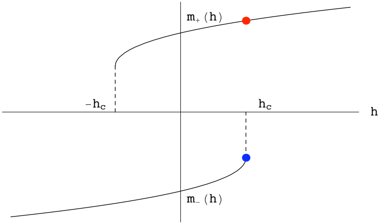


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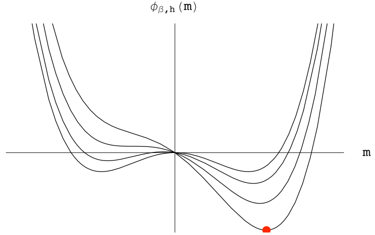


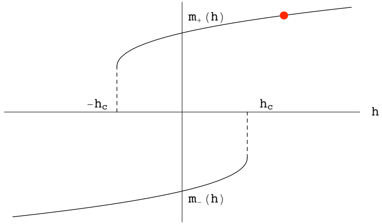


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# Glauber Dynamics

This is the Markov process $\sigma(t)$ on $\{-1,1\}^N$ with generator

$$Lf(\sigma) := \sum_{i=1}^{N} c(i,\sigma;h)\,(f(\sigma_i) - f(\sigma))$$

with $\sigma^{(i)}$ the configuration obtained from $\sigma$ by flipping the spin at $i$ and

$$c(i,\sigma;h) = \frac{e^{-\beta[H_{h,N}(\sigma^{(i)})-H_{h,N}(\sigma)]}}{e^{-\beta H_{h,N}(\sigma^{(i)})} + e^{-\beta H_{h,N}(\sigma)}}$$

the Glauber spin flip intensity at $i$ when the state is $\sigma$.

$c(i,\sigma;h)\,dt$ is the probability that the spin at $i$ flips in the time interval $[t,t+dt]$ knowing that at time $t$ the configuration is $\sigma$.

- the Gibbs measure is invariant for the Glauber dynamics

# The macroscopic Mean Field dynamics

The infinite volume dynamics is governed by the ODE

$$\frac{dm}{dt} = F(m,h), \qquad F(m,h) = -m + \tanh\{\beta(m+h)\} \tag{1}$$

Let $h(t)$ be a smooth function of $t$, $m_N(t) = m_N(\sigma(t))$ the markov process induced by $\sigma(t)$ which starts from $m_N \in \mathcal{M}_N$, $m_N \to m \in [-1,1]$ as $N \to \infty$.

**Theorem**

*For any $\delta > 0$ and any $T > 0$,*

$$\lim_{N\to\infty} \mathbf{P}_N \left\{ \sup_{t \le T} \left| m_N(t) - \bar{m}(t) \right| \ge \delta \right\} = 0$$

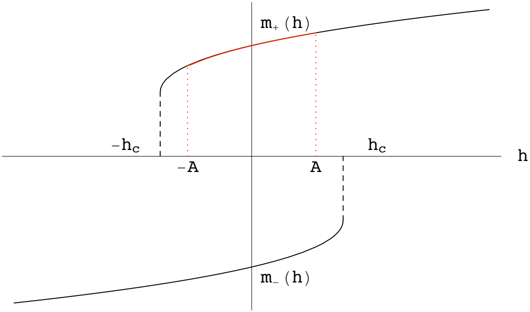
*where $\bar{m}(t)$ is the unique solution of* (1) *with $\bar{m}(0) = m$.*

# The adiabatic limit

Let the magnetic field oscillate with frequency $\omega$ and width $A$

$$h(t) := -A\cos t, \qquad h_\omega(t) = h(\omega t),$$

let $\bar{m}_\omega(t)$ be the solution of $\dot{m} = F(m, h_\omega)$ with $\bar{m}_\omega(0) = m_+(h_\omega(0))$.




In the adiabatic regime $\omega \simeq 0$

- $A < h_c + \mathcal{O}(\omega)$
  $\bar{m}_\omega(t)$ tracks $m_+(h_\omega(t))$
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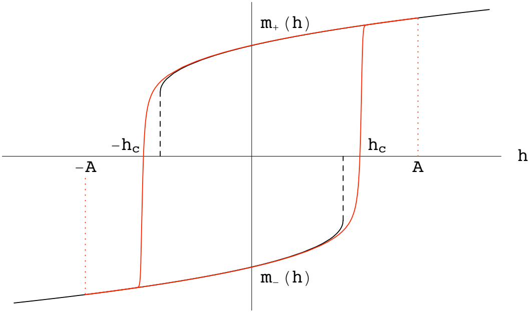


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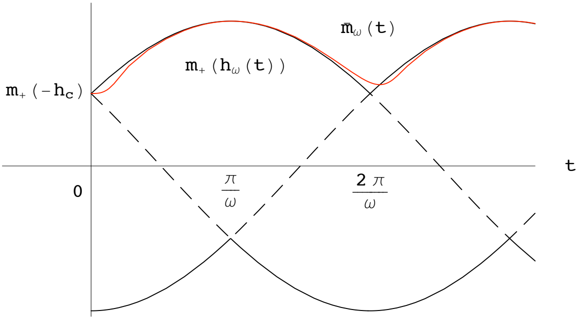
- $A < h_c + \mathcal{O}(\omega)$
  $\bar{m}_\omega(t)$ tracks $m_+(h_\omega(t))$
  at a distance $\mathcal{O}(\omega)$
- $A > h_c + \mathcal{O}(\omega)$
  $\bar{m}_\omega(t)$ tracks an hysteresis loop of area $\mathcal{A}_0 + \mathcal{O}(\omega^{2/3})$

# The adiabatic limit

If $A = h_c$ there is not hysteresis. Solutions stay $\sqrt{\omega}$ above the bifurcation point.




## Theorem

*Let* $A = h_c$, *then for any* $\tau > 0$, $$\lim_{\omega\to 0} \sup_{t \le \omega^{-1}\tau} \left|\bar{m}_\omega(t) - m_+(h_\omega(t))\right| = 0$$

.

# Slower oscillations

What happens for $A = h_c$ when the frequency $\omega$ depends on $N$?

- the relevant order of times is $\omega^{-1}$
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- the relevant order of times is $\omega^{-1}$
- for large but finite $N$ stochastic fluctuations of intensity $N^{-1/2}$ appear
- let $\mathcal{L}_{h_\omega}$ be the generator of $m_N(t)$, the dynamics is governed by

$$m_N(t) = m_N(0) + \int_0^t \mathcal{L}_{h_\omega} m_N(s)ds + M_N(t)$$

where $M_N(t)$ is a martingale

# Previous results

The issue has been modeled in

- Berglund N., Gentz B. 2002, *Ann. Appl. Prob.* 4 **12**
- Berglund N., Gentz B. 2002, *Nonlinearity* **15**

by the stochastic ODE

$$dm = f(m, h_\omega)dt + \sigma dw(t)$$

where $f$ derives from a periodically forced double well, e.g. $f(m, h) = m - m^3 + h$.

If $A = h_c$ and $\sigma = N^{-\frac{1}{2}}$ then

- for $\omega >> N^{-\frac{2}{3}}$ $\rightarrow$ no transition during one cycle
- for $N^{-\frac{2}{3}} >> \omega >> e^{-N^{-\frac{2}{3}}}$ $\rightarrow$ hysteresis cycle
- for $\omega << e^{-N^{-\frac{2}{3}}}$ $\rightarrow$ poorly localized paths

in the infinite volume limit.

# The result

$A = h_c$ and $\omega_N = \mathcal{O}(N^{-\frac{2}{3}})$ $\rightarrow$ the dynamics remains stochastic in the hydrodynamic limit

hysteresis loops become random

There exists $p \in (0, 1)$ such that, at each cycle

- with probability $p$ there is transition
- with probability $1 - p$ there is no transition

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Moreover $p = \mathbf{P}\,\{\text{there is } t \,:\, Y(t) = -\infty\}$

with $Y(t)$ solution of the problem

$$dY = (t^2 - Y^2)dt + \xi_\beta dw_t, \qquad \lim_{t\to-\infty}(Y(t)+t) = 0$$

for a suitable $\xi_\beta > 0$.

# Langevin equation in an oscillating potential

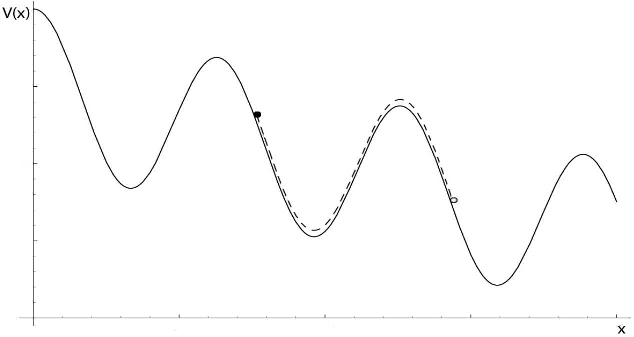
We deal with a particle moving in a periodic potential in the presence of viscosity subject to a stochastic noise and to an additional constant external force.

The equation of motion for the coordinate $x(t) \in \mathbb{R}$ of the particle is the Langevin equation:

$$\ddot{x} + \gamma\,\dot{x} + V_0'(x) = \alpha + \epsilon\,\dot{w}(t)$$

- $V_0(x)$ is a periodic potential
- $\gamma > 0$ is the viscosity coefficient
- $\alpha > 0$ is the external force
- $\epsilon$ is the noise intensity

The total potential is $V(x) = V_0(x) - \alpha x$




The equation of motion is $\ddot{x} + \gamma\,\dot{x} + V'(x) = \epsilon\,\dot{w}(t)$

equivalent to the first order equations system

$$\begin{cases} \dot{x} = p \\ \dot{p} = -\gamma p - V'(x) + \epsilon \dot{w} \end{cases}$$

# Deterministic orbits

Consider the deterministic system $(\epsilon = 0)$

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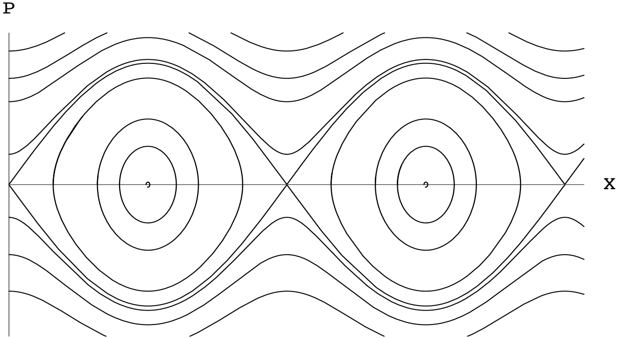
for $\gamma = \alpha = 0$ solutions are periodic



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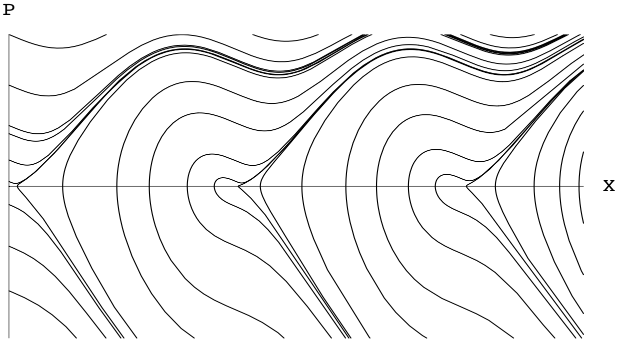
- for $\alpha > 1$ there are only running solutions



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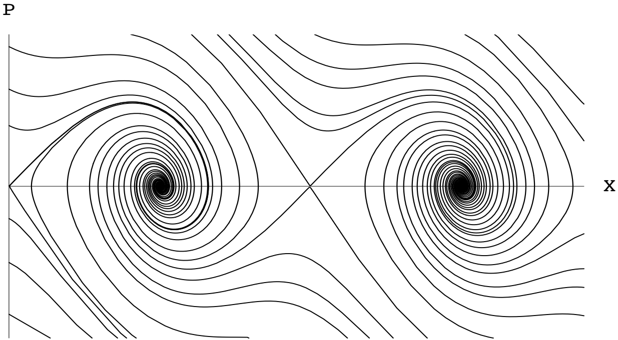

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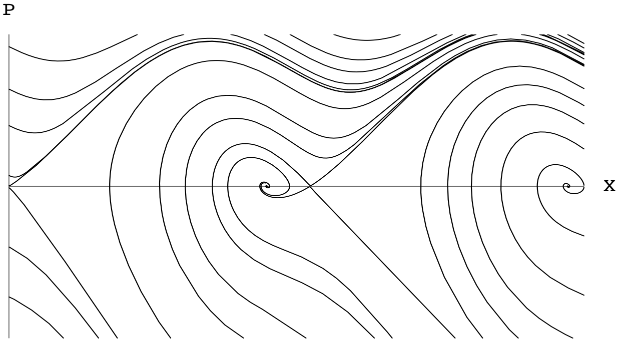

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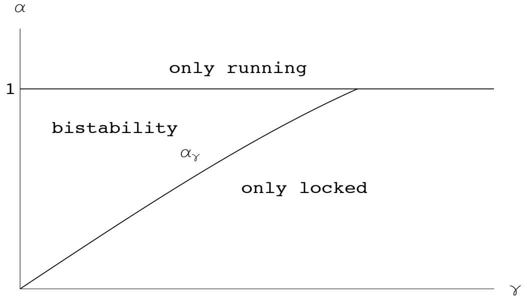
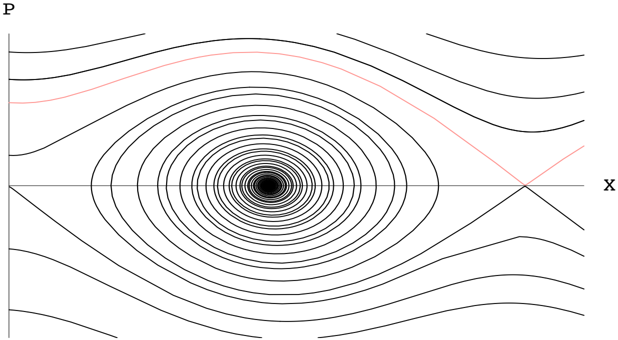



$$\alpha_\gamma = \mathcal{O}(\gamma) \qquad \text{as} \qquad \gamma \to 0$$

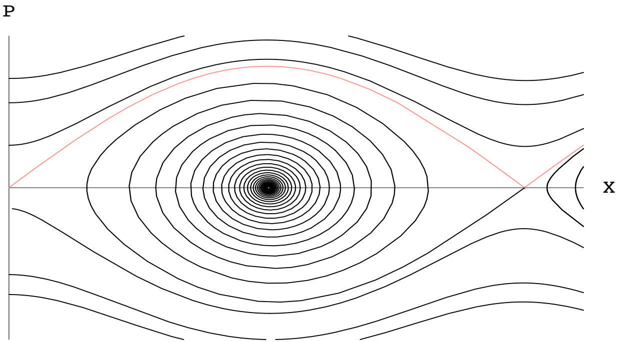
For $\alpha \leq \alpha_\gamma$




for any $k$ there exists a critical solution $(X_k^*(t), P_k^*(t))$ such that

$$\lim_{t\to\infty} X_k^*(t) = 2k\pi \qquad \text{and} \qquad \lim_{t\to\infty} P_k^*(t) = 0$$

For $\alpha = \alpha_\gamma$




the critical solution is heteroclinic, i.e.

$$\lim_{t\to\infty} X_k^*(t) = 2k\pi, \qquad \lim_{t\to\infty} P_k^*(t) = 0$$

$$\lim_{t\to 0} X_k^*(t) = 2(k-1)\pi \quad \text{and} \quad \lim_{t\to 0} P_k^*(t) = 0$$

# The problem

Let $\gamma$ small enough and $\alpha = \alpha_\gamma$, and denote by $\wp_k^*(x)$ the $k$-th heteroclinic orbit in the phase space, $\wp_k^*(X_k^*(t)) = P_k(t)$.

Consider the problem

$$\begin{cases} \dot{x} = p & x(0) = -\pi \\ \dot{p} = -\gamma p - V'(x) + \epsilon \dot{w} & p(0) = p_0 \end{cases} \tag{3}$$

with $\quad |p_0 - \wp_0^*(-\pi)| \leq \epsilon^{1+\delta} \quad$ for some $\quad \delta > 0$

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with $\quad |p_0 - \wp_0^*(-\pi)| \leq \epsilon^{1+\delta} \quad$ for some $\quad \delta > 0$

then, in the limit as $\epsilon \to 0$,

- at each time the probability for the particle to get across the next well is $1/2$
- the random variable associated to the number of wells crossed by the particle has a geometric distribution of parameter $1/2$
- the particle will finally be trapped in one of the wells

# Dynamics around the criticalities

A convenient choice of variables in a neighborhood of criticalities is given by

$$z_k(t) := p(t) - \lambda^-(x(t) - 2k\pi) \qquad v_k(t) := p(t) - \lambda^+(x(t) - 2k\pi)$$

with

$$\lambda^- := \frac{d}{dx}\wp_k^*(2k\pi^-) \qquad \text{and} \qquad \lambda^+ := \frac{d}{dx}\wp_{k+1}^*(2k\pi^+)$$

- at the beginning of the $k$-th critical interval $z_k(t)$ approximates the deviation in the phase plane from the $k$-th heteroclinic orbit $\wp_k^*(x)$
- at the end of the $k$-th critical interval $v_k(t)$ approximates the deviation in the phase plane from the $k+1$-th heteroclinic orbit $\wp_{k+1}^*(x)$

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The events "the $k$-th criticality has been/not been crossed" can be expressed by

$$\{z_k(T_k) \gtrless 0\}$$

$T_k$ the first exit time from the $k$-th critical interval

The dynamics in a neighborhood of criticalities is approximated by the linear system

$$\begin{cases} \dot{z} = \lambda^{+} z + \epsilon \dot{w} \\ \dot{v} = \lambda^{-} v + \epsilon \dot{w} \end{cases} \tag{4}$$

let $S_k$ be the first hitting time in the $k$-th critical interval, then

$$z_k(t) \simeq z_k(S_k)\, e^{\lambda^{+}(t-S_k)} + \epsilon\, e^{\lambda^{+} t} \int_{S_k}^{t} e^{-\lambda^{+} s} dw_s$$

and

$$v_k(t) \simeq v_k(S_k)\, e^{\lambda^{-}(t-S_k)} + \epsilon\, e^{\lambda^{-} t} \int_{S_k}^{t} e^{-\lambda^{-} s} dw_s$$

where

$$\lambda^{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\sqrt{1-\alpha_\gamma^2}}}{2}, \qquad \lambda^{\pm} = \pm 1 + \mathcal{O}(\gamma) \quad \text{as} \quad \gamma \to 0$$

# The main result

### Theorem

*There exists* $c > 0$ *such that, for any* $\epsilon > 0$ *small enough,*

$$\left| \mathbf{P}_\epsilon \left\{ z_k(T_k) \gtrless 0 \,\middle|\, z_{k-1}(T_{k-1}) > 0 \right\} - \frac{1}{2} \right| \leq c\epsilon^{\theta_\gamma}$$

*with*

$$\theta_\gamma = \frac{|\lambda^-|}{\lambda^+} - 1 = \mathcal{O}(\gamma) \qquad \textit{as} \qquad \gamma \to 0$$

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Let $\mathcal{N}$ be the r.v. associated to the number of wells crossed by $(x(t), p(t))$

$$\mathcal{N} := \inf\{k \ge 0 \;:\; z_k(T_k) < 0\} \in \mathbb{N} \cup \{0\}$$

### Theorem

*For any fixed* $k \in \mathbb{N} \cup \{0\}$

$$\lim_{\epsilon \to 0} \mathbf{P}_\epsilon \{\mathcal{N} = k\} = \frac{1}{2^{k+1}} \tag{5}$$