Sharp interface limit of the Stochastic Allen-Cahn equation in one space-dimension

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Introduction

Allen Cahn equation in one space-dimension:

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + f(u)$$

- f is negative derivative of symmetric double well potential, for example $f(u) = u - u^3$
- Used in a phenomenological model of phase separation

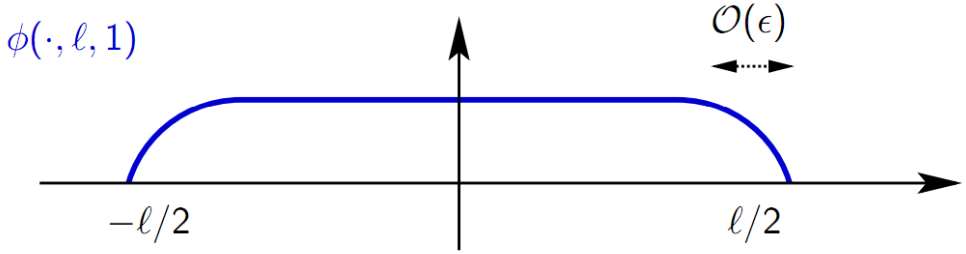
- $\varepsilon \to 0$ is the sharp interface limit, corresponding to cooling down a material to absolute zero

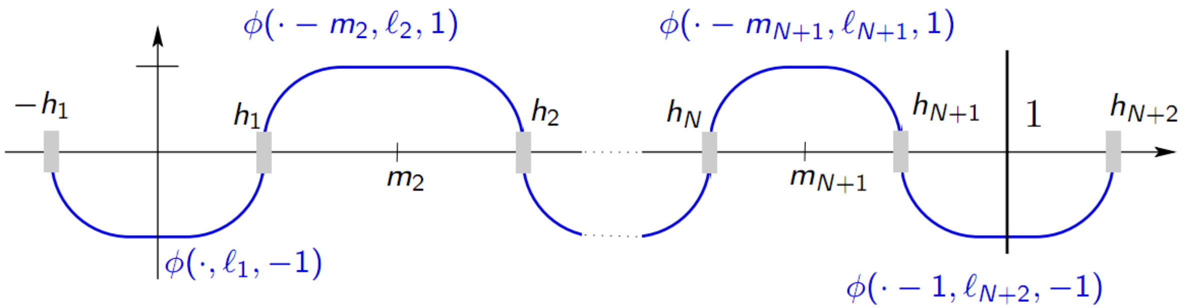
- Intuitively, we know that “most” intial conditions quickly lead to a solution which is 1 and -1 except for its boundaries of order ε

Approximate Slow Manifold

-Originally suggested by Fusco and Hale, used by Carr and Pego to characterise slow motion

-Construction: "gluing" together of time-invariant solutions with cutoff functions





Approximate Slow Manifold ctd

-Slow manifold is indexed by h (position of interfaces)

-Carr and Pego derived by orthogonal projection of solutions near slow manifold a system of ODE's for the front motion

-persists at least for times of order $O(e^{l/\varepsilon})$ where l is the minimum distance between the interfaces in the initial configuration

-Example of Metastability

Behaviour besides slow motion

-Phase separation (Chen 2004)

-Initially finitely many zeroes

-After $t = O(|\log \varepsilon|)$ solution bounded below in modulus by ½ except in $O(\varepsilon)$ neighbourhoods of its interfaces

-Laplacian almost negligible at this stage

Behaviour besides slow motion ctd

-Generation of metastable matterns (Chen 2004, Otto-Reznikoff(Westdickenberg 2006):

-After a further time of order $O\left(\frac{1}{\varepsilon}\right)$ solution is at an $O\left(e^{-C/\varepsilon}\right)$ distance from the slow manifold

-Afterwards, we see slow motion of each interface to its nearest neighbouring interface

-Annihilation (C1):

-At a certain $O(\varepsilon)$ distance we lose the ability to map orthogonally onto the manifold and the interfaces annihilate each other, converging to a new slow manifold within $O(|\log\varepsilon|)$ time, after which we see slow motion again

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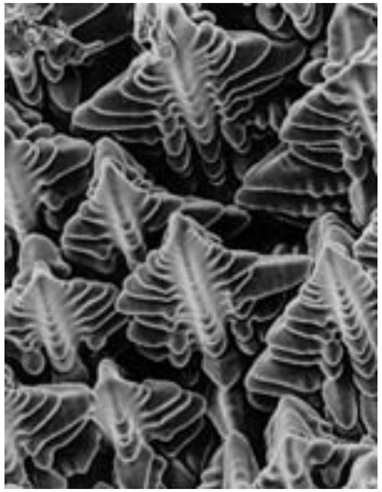
Behaviour besides slow motion ctd

-Clearly, as $t \to \infty$ the solution either becomes a time-invariant solution of one interface or attains one of the constant profiles +1 or -1

Why perturb it with noise?

- Answer comes from Physics:



(real-life lab picture)

- Numerically, a toy model of dendrites is much better approximated by the stochastic Allen-Cahn equation than by the deterministic case (Nestler et al):

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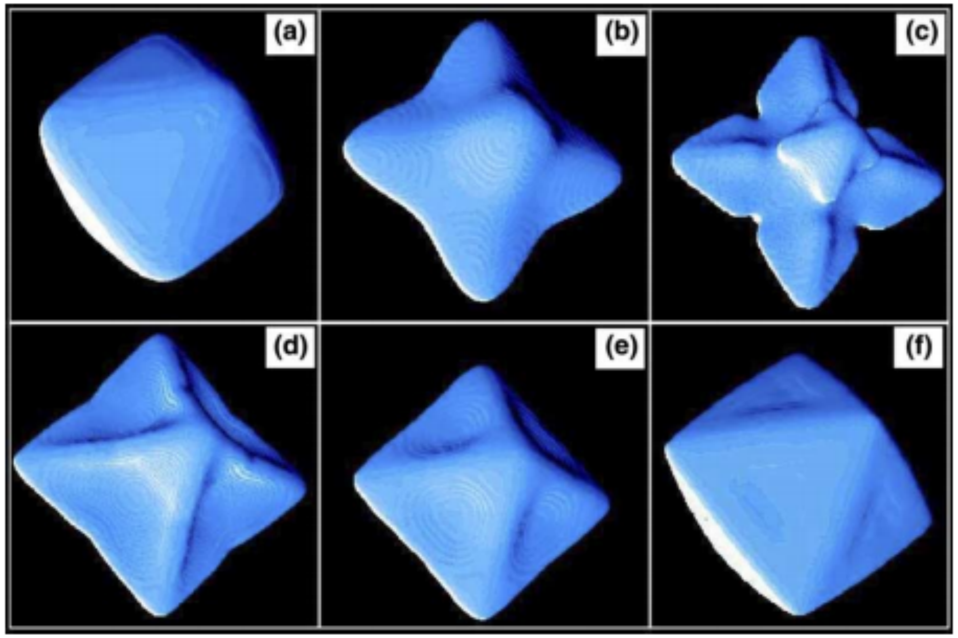
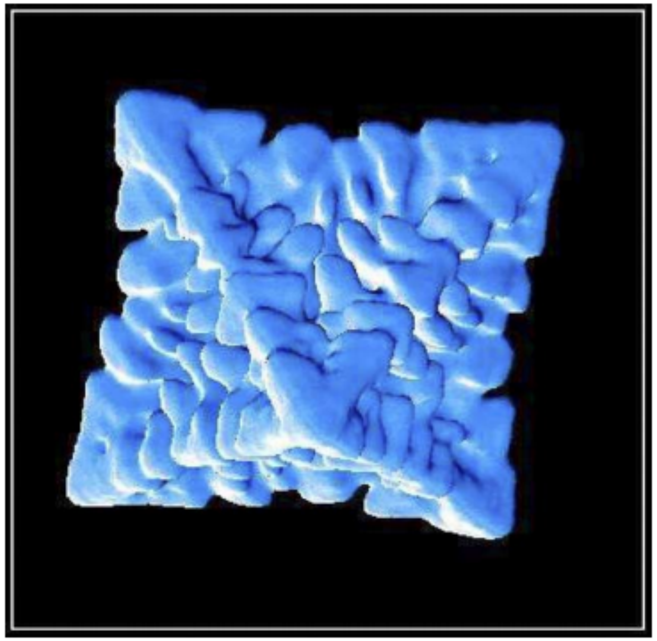
Why perturb with noise (ctd)?





Computation without thermal noise

Computation with thermal noise

Stochastic Allen-Cahn equation

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + f(u) + \varepsilon^\gamma \dot{W} \quad \text{on} \quad (0,1)$$

- The noise is infinitedimensional in order to be microscopic
- We restrict ourselves to small noise $\gamma > 2$

- The results should also hold for $\gamma > 3/2$

- W is a Q-Wiener process denoted as $W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k(\cdot)$ where $\{e_k(\cdot)\}$ is an orthonormal basis of $L^2(0,1)$ and $\{\beta_k(t)\}$ is a set of independent Brownian motions

Results in relation to PDE

- Phase separation, generation of metastable patterns and annihilation take $O(|\log \varepsilon|)$ time and are very similar to deterministic case
- Stochastic flow dominates over slow motion
- Based on an idea of Antonopoulou, Blömker, Karali applying the Ito formula yields a stable system of SDEs for motion of interfaces:

$$dh_k = O(\varepsilon^{2\gamma+1+\delta})dt + \varepsilon^{\gamma}\left\langle \frac{\varepsilon}{S_\infty} u_r^h + O(\varepsilon^{1+\delta}), dW \right\rangle \quad u_k^h = \frac{\partial u^h}{\partial h_k} \approx \frac{du^h}{dx}$$

around the interface and 0 beyond the midpoints between interfaces

- $\varepsilon^{1/2} \frac{u_k^h}{\sqrt{S_\infty}}$ converges to the square root of the Dirac Delta function as $\varepsilon \to 0$

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Results in relation to PDE

- Thus, after a timechange $t' = S_\infty \varepsilon^{2\gamma+1} t$ we see Brownian motions in the sharp interface limit
- The time taken by phase separation, generation of metastable patterns and annihilation converges to 0
- Therefore on the timescale t' the interfaces perform annihilating Brownian motions in the sharp interface limit

Ideas of the proofs

-Phase separation, generation of metastable patterns and annihilation:

-Can bound SPDE linearised at the stable points for times of polynomial order in $\frac{1}{\varepsilon}$

-Difference of this linearisation and Stochastic Allen-Cahn equation is the Allen-Cahn PDE with the linearised SPDE as a perturbation

-Phase Separation: up to logarithmic times the error between perturbed and non-perturbed equation is $o(1)$

-Pattern Generation: Use iterative argument that within an order 1 time we can halve

-Time of logarithmic order obtained from exponential rate in time

Ideas of the proofs

- Stochastic motion:
- Apply Itô formula to obtain equations
- Obtain random PDE perturbed by finite dimensional Wiener process using linearisation
- PDE techniques and Itô formula yield stability on timescales polynomial in $\frac{1}{\varepsilon}$
- Finally, one easily observes that the coupled system for manifold configuration and distance has a solution; showing their orthogonality completes the proof

Ideas of the proofs

-Sharp interface limit:

-Duration of convergence towards slow manifold and annihilation converges to 0

-Intuitively, the stopped SDE's converge to the square root of a Dirac Delta function integrated against space-time white noise

-Actual proof makes use of martingales and a stopped generalisation of Levy's characterisation of Brownian motion

Thank you for your attention!