Extreme-value theory and the stochastic exit problem

7th Workshop on Random Dynamical Systems, Bielefeld, 12/12/14

Refs: NB, B Gentz, SIAM J Math Anal 46: 310-352 (2014) [arXiv: 1208.2557]
NB [arXiv: 1403.7393]

1. Extreme-value theory

$X_1, X_2, \dots$ $\mathbb{R}$-valued i.i.d. r.v.

$S_n = \sum_{i=1}^n X_i$

limit thm: $\lim_{n\to\infty} \frac{S_n - b_n}{a_n} \stackrel{\mathcal{L}}{=} Y$ $\quad (a_n > 0, b_n \in \mathbb{R})$

ex: CLT: $X_1 \in L^2$ $\begin{cases} b_n = n\,\mathbb{E}(X_1) \\ a_n = \sqrt{n \operatorname{Var}(X_1)} \\ Y \sim \mathcal{N}(0,1) \end{cases}$

Cauchy: $\frac{S_n}{n} \stackrel{\mathcal{L}}{=} X_1$

$\frac{\sum Y_i - b_n}{a_n} \stackrel{\mathcal{L}}{=} Y \Rightarrow Y$ stable

ex: $\mathbb{E}(e^{itY}) = e^{-|t|^\alpha}$

$\left.\begin{array}{l} F(t) = \mathbb{P}\{X_1 \leqslant t\} \\ M_n = \max\{X_1, \dots, X_n\} \end{array}\right\} \Rightarrow \mathbb{P}\{M_n \leqslant t\} = F(t)^n$

$F \in \mathcal{D}(\Phi) \stackrel{\text{def}}{\Leftrightarrow} \exists a_n > 0, b_n : \lim_{n\to\infty} \frac{M_n - b_n}{a_n} \stackrel{\mathcal{L}}{=} Y$

$\mathbb{P}\{Y \leqslant t\} = \Phi(t)$

$\Leftrightarrow \lim_{n\to\infty} F(a_n t + b_n)^n = \Phi(t) \; \forall t$

Rem: 1) $F \in \mathcal{D}(\Phi) \Rightarrow F \in \mathcal{D}(\Phi(a \cdot + b))$ $\forall a > 0, \forall b \in \mathbb{R}$

2) $F \in \mathcal{D}(\Phi) \Rightarrow \exists a > 0, b \in \mathbb{R}$
$\Phi(ax + b)^2 = \Phi(x) \; \forall x$

$$\boxed{\begin{array}{l}\text{Thm: [Fréchet '27, Fisher \& Tippett '28, Gnedenko '43]} \\ F(t) \neq 1_{\{t \geqslant c\}},\ F \in \mathcal{D}(\Phi) \\ \Rightarrow \Phi \in \{\Phi_\alpha, \Psi_\alpha, \Lambda\} \text{ where } \begin{cases} \Phi_\alpha(t) = e^{-t^{-\alpha}} 1_{\{t>0\}} \quad (\alpha > 0) & \text{Fréchet} \\ \Psi_\alpha(t) = e^{-(-t)^\alpha} 1_{\{t\leqslant 0\}} + 1_{\{t>0\}} & \text{Weibull} \\ \Lambda(t) = e^{-e^{-t}} & \text{Gumbel} \end{cases}\end{array}}$$

Let $R(t) := 1 - F(t) = \mathbb{P}\{X_1 > t\}$

Lemma: $F \in \mathcal{D}(\Phi) \Leftrightarrow \exists a_n > 0, b_n : \lim_{n\to\infty} n R(a_n t + b_n) = -\log \Phi(t)$ $\quad \forall t : \Phi(t) > 0$

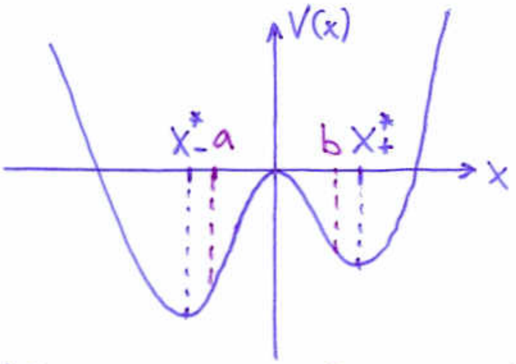
Gumbel: possible choice $\underbrace{b_n = \inf\{t : F(t) > 1 - \tfrac{1}{n}\}}_{\hookrightarrow R(b_n) = n}$ $\quad a_n = \inf\{t : F(t + b_n) > 1 - \tfrac{1}{ne}\}$

Ex: $\mathcal{N} \in \mathcal{D}(\Lambda)$

$$\boxed{\begin{array}{l}\text{Thm: } t_0 := \inf\{t : F(t) = 1\} \in \mathbb{R} \cup \{\infty\} \\ F \in \mathcal{D}(\Lambda) \Leftrightarrow \exists A(z), \lim_{z\to t_0^-} A(z) = 0 : \lim_{z \to t_0^-} \underbrace{\frac{R(z[1 + A(z)t])}{R(z)}}_{= \mathbb{P}\{X_1 > z[1+A(z)t] \,|\, X_1 > z\}} = -\log(\Lambda(t)) = e^{-t} \quad \forall t\end{array}}$$

possible choice: $A(b_n) = a_n / b_n \; \forall n$ $\quad (z = b_n)$ $\quad$ residual lifetime

2. Stochastic exit problem in dim 1



$$dx_t = -V'(x_t)\,dt + \sigma\,dW_t$$

$$\lambda = |V''(0)|$$

$$\tau_x = \inf\{t>0 : x_t = x\}$$

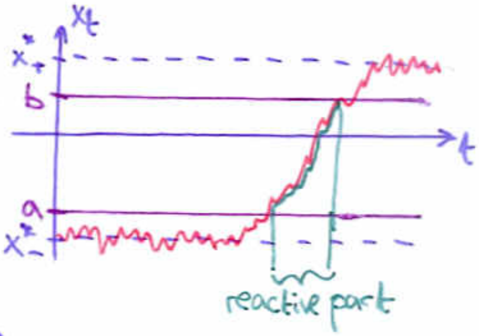
Well known: $\lim\limits_{\sigma\to 0} \mathbb{P}^a\{\tau_b > t\,\mathbb{E}^a[\tau_b]\} = e^{-t}$

$$\mathbb{E}^a[\tau_b] \sim C\, e^{2[V(0)-V(x_-^*)]/\sigma^2}$$

$$x_-^* < a < 0 < b \leq x_+^*$$

Thm A: [Cérou, Guyader, Lelièvre, Malrieu, ALEA 2013]

$$\lim_{\sigma\to 0} \text{Law}(\lambda\tau_b - 2|\log\sigma| \,|\, \tau_b<\tau_a) = \text{Law}(\underbrace{Z}_{\text{Gumbel}} + \underbrace{T(x_0,b)}_{\text{deterministic}})$$



Proof based on Doob's h-transform + exact computation

Yuri Bakhtin's approach [Stoch Dyn 2014 & arXiv: 1307.7060]

$$dx_t = \lambda x_t\,dt + \sigma\,dW_t \quad\Rightarrow\quad x_t = e^{\lambda t}\tilde{x}_t$$

$\lambda>0 \qquad x_0<0$

$$\tilde{x}_t = x_0 + \sigma\int_0^t e^{-\lambda s}\,dW_s \overset{\mathcal{L}}{=} x_0 + \sigma \underbrace{\tilde{W}_{(1-e^{-2\lambda t})/2\lambda}}_{\sim\sqrt{\frac{1-e^{-2\lambda t}}{2\lambda}}\,N}$$

Reflection principle: $\mathbb{P}\{\tau_0<t\} = 2\,\mathbb{P}\{\tilde{x}_t>0\}$

$$\mathbb{P}\{\tau_0<t \,|\, \tau_0<\infty\} = \frac{\mathbb{P}\{\tau_0<t\}}{\mathbb{P}\{\tau_0<\infty\}} = \frac{2\,\mathbb{P}\{\tilde{x}_t>0\}}{2\,\mathbb{P}\{\tilde{x}_\infty>0\}} = \mathbb{P}\{\tilde{x}_t>0 \,|\, \tilde{x}_\infty>0\}$$

$$\mathbb{P}\{\tau_0<t+\tfrac{1}{\lambda}|\log\sigma| \,|\, \tau_0<\infty\} = \mathbb{P}\{\tilde{x}_{t+\frac{1}{\lambda}|\log\sigma|}>0 \,|\, \tilde{x}_\infty>0\}$$

$$= \mathbb{P}\Big\{N > \frac{|x_0|}{\sigma}\sqrt{\frac{2\lambda}{1-\sigma^2 e^{-2\lambda t}}} \,\Big|\, N > \frac{|x_0|}{\sigma}\sqrt{2\lambda}\Big\}$$

$[-\log\Lambda(e^{-x}) = \Lambda(x)]$

$$\xrightarrow[\sigma\to 0]{} \exp\{-x_0^2\,\lambda\,e^{-2\lambda t}\} = \mathbb{P}\Big\{\frac{Z}{2} + \frac{\log(x_0^2\lambda)}{2\lambda} < t\Big\}$$

Thm B: [Day, Bakhtin] $\lim\limits_{\sigma\to 0} \text{Law}(\lambda\tau_0 - |\log\sigma| \,|\, \tau_0<\tau_a) = \text{Law}\Big(\frac{Z}{2} + \frac{\log(x_0^2\lambda)}{2\lambda}\Big)$

Thm C: [Day] $x_0=0,\ a<0<b$: $\lim\limits_{\sigma\to 0} \text{Law}(\lambda\tau_b - |\log\sigma| \,|\, \tau_b<\tau_a) = \text{Law}\Big(\Theta + \frac{\log(2b^2\lambda)}{2}\Big)$

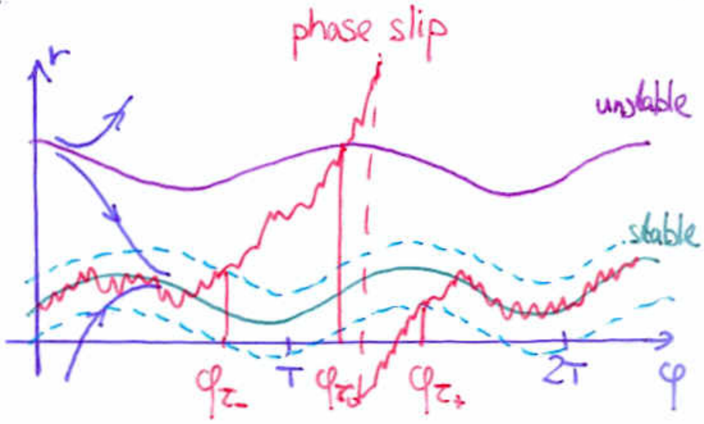
$$\Theta = -\log|N|$$

idea: $|x_t| \overset{\mathcal{L}}{=} \sigma\sqrt{\frac{1-e^{-2\lambda t}}{2\lambda}}\,|N|\,e^{\lambda t} \quad\Rightarrow\quad b \cong \sigma\frac{1}{\sqrt{2\lambda}}|N|\,e^{\lambda\tau}$

Thm B & Thm C $\Rightarrow$ Thm A because $\frac{1}{2}Z + \Theta \overset{\mathcal{L}}{=} Z + \frac{1}{2}\log(2)$

3. Stochastic exit problem in dim 2

$$\begin{cases} dr_t = f_r(r_t,\varphi_t)\,dt + \sigma g_r(r_t,\varphi_t)\,dW_t \\ d\varphi_t = f_\varphi(r_t,\varphi_t)\,dt + \sigma g_\varphi(r_t,\varphi_t)\,dW_t \end{cases}$$



ex: r = phase diff between 2 oscillators
φ = average phase

Q: Law of $\varphi_{\tau_0}, \varphi_{\tau_-}, \varphi_{\tau_+}$?

⚠ quasipotential const on unst. orbit

Thm: [B & Gentz, SIAM J Math Anal 2014]

$$\lim_{m\to\infty}\left(\lim_{\sigma\to 0} \text{Law}\left(\theta(\varphi_{\tau_0}) - |\log\sigma| - \lambda T\, Y_m^\sigma\right)\right) = \text{Law}\left(\frac{Z}{2} - \frac{\log(2)}{2}\right)$$

* λT: instability of orbit (Lyapunov exponent × period)
* θ: explicit parametrisation of orbit, $\theta(\varphi+1) = \theta(\varphi) + \lambda T$
* Y_m^σ: $\lim_{n\to\infty} \mathbb{P}\{Y_m^\sigma = n+1 \mid Y_m^\sigma = n\} = e^{-I_m/\sigma^2}$ $\quad I_m = I_\infty + \mathcal{O}(e^{-2m\lambda T})$

"asymptotically geometric" $\quad Y_\infty^\sigma$ = # of optimal path

Thm: [NB 2014]

For appropriate def of $\tau_\pm$

$$\begin{cases} \lim_{\sigma\to 0} \text{Law}\left(\theta(\varphi_{\tau_0}) - \theta(\varphi_{\tau_-}) - |\log\sigma|\right) = \text{Law}\left(\frac{Z}{2} - \frac{\log 2}{2} + c_1\right) \\ \lim_{\sigma\to 0} \text{Law}\left(\theta(\varphi_{\tau_+}) - \theta(\varphi_{\tau_0}) - |\log\sigma|\right) = \text{Law}\left(\Theta + c_2\right) \\ \lim_{\sigma\to 0} \text{Law}\left(\theta(\varphi_{\tau_+}) - \theta(\varphi_{\tau_-}) - 2|\log\sigma|\right) = \text{Law}\left(Z + c_1 + c_2\right) \end{cases}$$

and $\text{Law}\left(\theta(\varphi_{\tau_0^{n+1}}) - \theta(\varphi_{\tau_0^n}) - Y_m^\sigma\right) \to$ Logistic (density $\sim \frac{1}{\cosh^2(t)}$)