Universität Augsburg University

Stochastic Front Motion & Slow Manifolds

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joint work with :

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Contents

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Setting:

∞-dimensional stochastic system in $\mathcal{H}$

$$du = \mathcal{L}(u)dt + dW \qquad \text{(SPDE)}$$

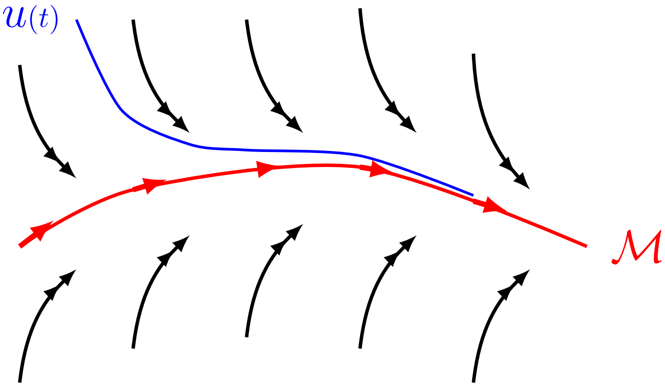
- $\mathcal{L}$ nonlinear operator (e.g. $\Delta u + f(u)$)
- small additive noise
- W- Wiener process in $\mathcal{H}$, covariance operator Q

$$\big(\mathbb{E}\langle u, W(t)\rangle\langle v, W(t)\rangle = tQ\langle u, v\rangle\big)$$

Deterministic slow manifold $\mathcal{M} \subset \mathcal{H}$

(Dynamics for $W = 0$)

For example, parametrized by position of interfaces



!! NOT NECESSARILY INVARIANT !!

Some Related Results

- Otto, Reznikoff [07] – Slow manifold - gradient systems
- Beck, Wayne [09] – Slow manifold – Burgers on $\mathbb{R}$
- Brassesco, Butta,et.al.[98,02,...] – single interface motion (stochastic Ginzburg-Landau, Phase Field Equation)
- Fatkullin, Kovacic, Vanden Eijnden [10] gradient systems
- Statistics, numerics, asymptotic expansions
  vanden Eijnden, Fatkullin[03], Fatkullin[10], Lythe[00,..], ...
- S. Weber [14], Shardlow [00] multi-kink stoch. Allen-Cahn
- Stannat et al [14,.....] travelling waves
- For SDE see Gentz & Berglund . ...

Stochastic Cahn-Hilliard – Brassesco [14]

The motion of a single interface in Cahn-Hilliard on $\mathbb{R}$ is non-Markovian (fractional Brownian motion)

General Idea of the Approach

(usually hidden in technical details)

Theorems (for stochastic dynamics)

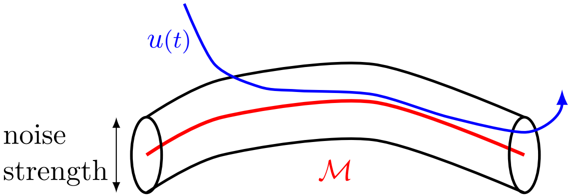
- Attractivity of the manifold $\mathcal{M}$
  - only for a neighboorhood
  - $\mathcal{M}$ not necessarily invariant for deterministic dynamics
- Stability of $\mathcal{M}$
  - exit from a neighboorhood of $\mathcal{M}$
  - here: weaker results than Large Deviation (not optimal)
- Motion along the manifold
  - Deterministic dynamics
  - Wiener process projected to $\mathcal{M}$

Theorem: Stability

Metatheorem (Stability)

With high probability, a solution

1. stays on the "order of noise-strength" close to $\mathcal{M}$
2. until it exits "at the end" of $\mathcal{M}$



Motion along the manifold

For some parameter space $\mathcal{P} \subset \mathbb{R}^N$

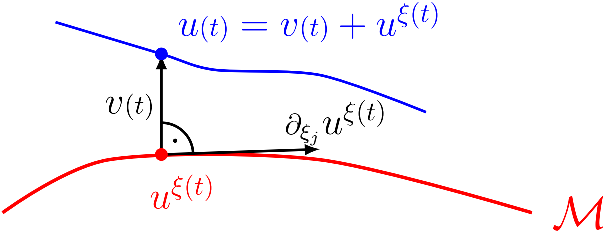
$$\mathcal{M} = \{u^\xi \in \mathcal{H} \ : \ \xi \in \mathcal{P}\} \, .$$

AIM:

Determine $b : \mathcal{P} \mapsto \mathbb{R}^N$ and $\sigma_j : \mathcal{P} \mapsto \mathcal{H}$ such that

$$u(t) \approx u^{\xi(t)} \quad \text{with} \quad d\xi_j = b_j(\xi)dt + \langle \sigma_j(\xi), dW \rangle \, .$$

$$u = u^\xi + v \quad \text{with} \quad v \perp \partial_{\xi_j} u^\xi \text{ and } \partial_{\xi_j} u^\xi \text{ is tangential}$$



Now:
Differentiate (Itô-formula)

$$u = u^\xi + v \qquad \text{and} \qquad \langle \partial_{\xi_j} u^\xi, v \rangle = 0\,,$$

eliminate dv and derive an equation for $d\xi_j$.

For Stability

$$\begin{aligned} dv &= du + du^\xi \\ &= \mathcal{L}(v + u^\xi)dt + \partial_\xi u^\xi \cdot d\xi + \text{Itō-Correction} \ + dW \\ &= D\mathcal{L}(u^\xi)v\ dt + N(u^\xi, v)dt + \partial_\xi u^\xi \cdot d\xi + \dots \end{aligned}$$

Now take scalar product with v

NEED:

- ... Bounds on linearized operator $\langle v, D\mathcal{L}(u^\xi)v\rangle$
- ... Control of Nonlinearity $N(u^\xi, v)$

To show bounds for v over large times with high probability

Stability

Show for $t \leq \tau^\star$

($\tau^\star$exit time from a neighboorhood of $\mathcal{M}$ - distance R_ϵ)

$$d\|v\|^2 = -a_\epsilon \|v\|^2 dt + \mathcal{O}(K_\epsilon)dt + \langle \mathcal{O}(\|v\|), dW \rangle$$

Inductively

$$\mathbb{E}\|v(\tau^\star)\|^{2p} \leq C_p \left(\frac{K_\epsilon + \|Q_\epsilon\|}{a_\epsilon} \right)^p \cdot \mathbb{E}\tau^\star$$

Theorem

If $K_\epsilon + \|Q_\epsilon\| \ll a_\epsilon R_\epsilon^2 \epsilon^\kappa$, then:

Exiting the neighboorhood of $\mathcal{M}$ before any time of order ϵ^{-q} due to v being large has small probability.

Stochastic Cahn-Hilliard

- Results of [Antonopoulu, Karali, DB, 12]
- Motion of interfaces betweeen several "pure" phases
- Approximate slow manifold
  - parametrized by the interface positions
  - Based on Bates & Xun [94,95] + Carr & Pego [89,90]

Phenomenological model for phase separation

For binary alloy, fluid,....
starting from an initially homogeneous mixture

$u(t, x)$ – concentration of one component

($t > 0$ time, x space)

$u = \pm 1$ (almost) “*pure*“ phases.

Stochastic Cahn-Hilliard Equation

$$\partial_t u = -\partial_x^2[\epsilon^2 \partial_x^2 u - f(u)] + \epsilon^\delta \partial_t \partial_x W, \qquad \text{(CH)}$$

Neumann-type (no flow) boundary conditions

$$\partial_x u(t,0) = \partial_x u(t,1) = 0 \qquad \partial_x^3 u(t,0) = \partial_x^3 u(t,1) = 0$$

- F smooth double well, $F' = f$ (E.g. $F(u) = \frac{1}{4}(1-u^2)^2$)
- Noise strength: ϵ^δ, $\delta > 9/2$ (technical reason)
- Interaction length/interface width: $0 < \epsilon \ll 1$

Noise

$$\partial_t u = -\partial_x^2[\epsilon^2 \partial_x^2 u - f(u)] + \epsilon^\delta \partial_t \partial_x W$$

$$W(t,x) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k(x)$$

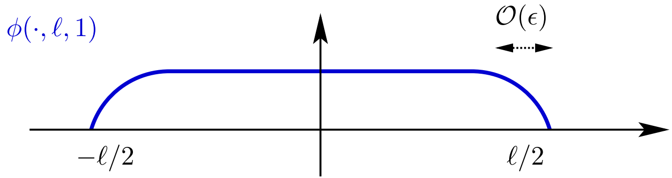
- e_k eigenfunctions of Dirichlet Laplacian (i.e., $\sin(\pi k x)$)
- $\alpha_k \to 0$ for $k \to \infty$, sufficiently fast (for Itō-formula)
- $\{\beta_k\}_{k \in \mathbb{N}}$ i.i.d. standard Brownian motion

Physics: Mass conservative space-time white noise, $\alpha_k = 1$

Slow Manifold Construction

Carr & Pego [89,90]

Stationary solutions of deterministic (CH)!
(upto boundary conditions)



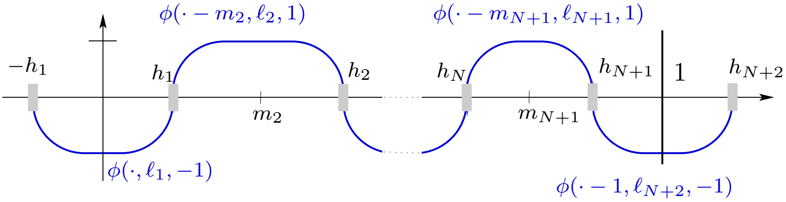
$\Phi(x, \ell, \pm 1)$ solves

$$\epsilon^2 \partial_x^2 \Phi = f(\Phi) \,, \qquad \Phi(-\ell/2) = 0 = \Phi(\ell/2)$$

Approximatimation u^h

u^h indexed by interfaces $h \in \Omega_\rho \subset \mathbb{R}^{N+1}$

Distance: $\ell_j = h_{j+1} - h_j > \epsilon/\rho$ (later $\rho = \epsilon^\kappa$, $0 < \kappa \ll 1$)



Near h_j use cut-off to glue the ϕ's smoothly.

(CH) is mass conservative. (i.e. $\int_0^1 u(t)dx = \int_0^1 u(0)dx$)

Fix mass $M = \int_0^1 u(t,x)\, dx$.

Definition (Slow Manifold)

$$\mathcal{M} = \left\{ u^h \ : \ h \in \Omega_\rho, \int_0^1 u^h dx = M \right\}$$

Interface h_{N+1} is a function of $\xi = (h_1, \ldots, h_N)$.

Define $u^\xi := u^h$

Technical Point:

If $u(t,x)$ solves (CH), then

$$\tilde{u}(t,x) = \int_0^x u(t,z)dz$$

solves the Integrated Cahn-Hilliard equation (ICH).

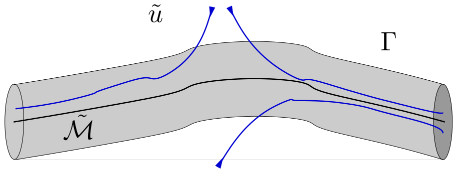
Advantage:

The linearized operator (around $\tilde{u}^\xi$)

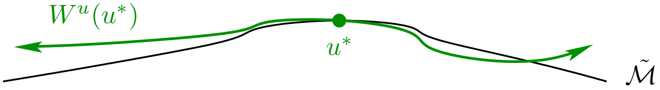
$$L^c = -\epsilon^2 \partial_x^4 + \partial_x f'(u^\xi)\partial_x$$

is self-adjoint. (It depends on ξ !)

$\exists$ exponentially (in ϵ) small neighboorhood Γ of $\tilde{\mathcal{M}}$ that is locally exponentially attracting with rate $\mathcal{O}(1)$



On $\tilde{\mathcal{M}}$ lies a stationary solution u^* (equidistant interfaces) with its N-dim. unstable manifold $W^u(u^*)$ exponentially close to $\tilde{\mathcal{M}}$.



Full SDE for ξ and $\tilde{v}$

Theorem

$\tilde{u}(0) = \tilde{u}^{\xi(0)} + \tilde{v}(0)$.

As long as $B_\epsilon(\tilde{v}) = \epsilon^2\|\partial_x^2\tilde{v}\|^2 + \|\partial_x\tilde{v}\|^2 = \mathcal{O}(\epsilon^3)$:

$$\tilde{u} = \tilde{u}^\xi + \tilde{v} \text{ is a solution of (ICH)}$$
$$\Leftrightarrow$$
$$\xi \text{ and } \tilde{v} \text{ solve (SDE) \& (SPDE)}$$

$$d\xi_k = b_k(\xi, \tilde{v})dt + \epsilon^\delta \langle \sigma_k(\xi\tilde{v}), dW\rangle \qquad \text{(SDE)}$$

with b and σ stated later.

$$d\tilde{v} = \mathcal{L}(\tilde{u}^\xi + \tilde{v})dt + \epsilon^\delta dW - d\tilde{u}^\xi\,, \qquad \text{(SPDE)}$$

where $\mathcal{L}(\tilde{u}) = -\epsilon^2\partial_x^4\tilde{u} + \partial_x f(\partial_x\tilde{u})$.

$$\sigma_r(\xi) = \sum_i A^{-1}_{ri}(\xi) E_i^\xi \,. \qquad (E_i^\xi \approx \partial_{\xi_i} \tilde{u}^\xi)$$

$$b_r(\xi) = \sum_i A^{-1}_{ri}(\xi) \Big\{ \langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi \rangle + \sum_j \langle \mathcal{Q}_\epsilon E_{ij}^\xi, \sigma_j(\xi) \rangle + \sum_{l,k} \frac{1}{2} \Big[\langle \tilde{v}, E_{ilk}^\xi \rangle - \langle \tilde{u}_{kl}^\xi, E_i^\xi \rangle - 2 \langle \tilde{u}_k^\xi, E_{il}^\xi \rangle \Big] \langle \mathcal{Q}_\epsilon \sigma_k(\xi), \sigma_l(\xi) \rangle \Big\}$$

$$A_{ri}(\xi, \tilde{v}) \approx 4\ell_{i+1}(\delta_{i,r} + \delta_{r,i-1}) + C_{ri}\epsilon$$

Recall:

Covariance operator of W is $\mathcal{Q}_\epsilon e_k = \epsilon^{2\delta} \alpha_k^2 e_k$

Indices are derivatives, $E_{ij}^\xi = \partial_{\xi_j} E_i^\xi$ and $\tilde{u}_{ij}^\xi = \partial_{\xi_j} \partial_{\xi_i} \tilde{u}^\xi$

Most of the terms in b are Itō-Stratonovic correction.

Theorem – Stability

Consider a solution $\tilde{u} = \tilde{u}^{\xi} + \tilde{v}$ of (ICH).
Fix $\kappa > 0$.
If $\mathcal{A}_{\epsilon}(\tilde{v}(0)) < \epsilon^{2\delta}$, then with high probability

$$\mathcal{A}_{\epsilon}(\tilde{v}(t)) = \mathcal{O}(\epsilon^{2\delta-\kappa})$$

for all $t \leq \epsilon^{-q}$ for any $q > 0$ or until interface breaks down.

Def. $\mathcal{A}_{\epsilon}(\tilde{v}) = \langle \tilde{v}, L^{c}\tilde{v} \rangle$ with $c\epsilon^{3}\|\tilde{v}\|^{2}_{H^1} \leq \mathcal{A}_{\epsilon}(\tilde{v}) \leq C\|\tilde{v}\|^{2}_{H^2}$.

Attraction

Exponential attraction holds for a larger neighbourhood of order $B_{\epsilon}(\tilde{v}) = \mathcal{O}(\epsilon^{6})$.

Is the (SDE) for ξ usefull?

Two interfaces ($N = 1$)

Second interface determined by first (mass conservation).

$$d\xi \approx \text{exp. small} + \epsilon^\delta \|\partial \tilde{u}^\xi\|^{-2} \langle \partial \tilde{u}^\xi, \circ dW \rangle \ ,$$

Stratonovic equation for $\epsilon^\delta W$ projected to $\tilde{\mathcal{M}}$

Space-time white noise ($\mathcal{Q}_\epsilon = \epsilon^{2\delta} \cdot Id$) – (too rough)

ξ is close to a Brownian motion with variance $\epsilon^{2\delta}/4\ell$.

[B-X-94] : $\|\tilde{u}_1^\xi\|^2 \approx 4\ell$ - distance between interfaces.

1D-Allen-Cahn

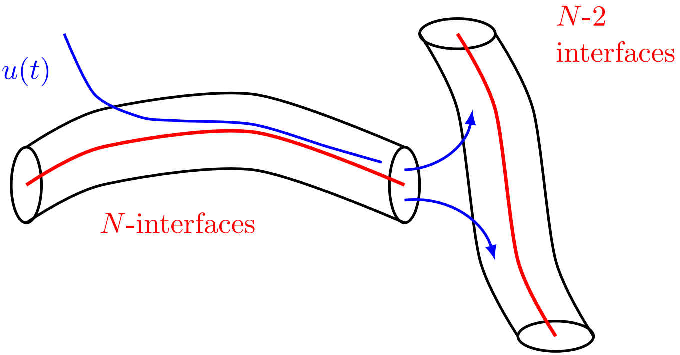
1D-Allen-Cahn

$$du = [\epsilon^2 \partial_x^2 u + u - u^3]dt + dW$$

& Neumann b.c

- similar dynamics than Cahn-Hilliard
- no mass conservation
- Interface motion similar [S. Weber 14, PhD]

- based on [X. Chen 04]
- Using maximum principle
- Exit from manifold with N-layers into the domain of attraction for N-2-layers



2D-Allen-Cahn

Motion of Droplets

On some sufficiently large domain $D \in \mathbb{R}^2$

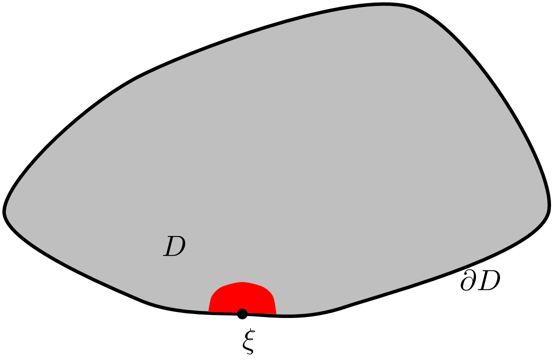
$$du = \left[\epsilon^2 \Delta u + u - u^3 - \frac{1}{|D|}\int_D (u - u^3)dx\right]dt + \epsilon^\delta dW$$

& Neumann b.c

Mass conservative Allen-Cahn

$$u^\xi \approx \begin{cases} 1 & : \quad \text{on ball of radius } 1-\epsilon \text{ around } \xi \in \partial D \\ -1 & : \quad \text{outside ball of radius } 1+\epsilon \text{ around } \xi \in \partial D \end{cases}$$

$$\mathcal{M} = \{u^\xi : \xi \in \partial D\} \simeq S^1$$



Theorem

Assume $\delta < 4$ and W sufficiently smooth in space.

Then:
Solutions stay close to $\mathcal{M}$ up to times of order ϵ^{-q}.

$$d\xi \approx \epsilon^2 c(\xi)dt + \epsilon^\delta \|\partial_\xi u^\xi\|^{-2} \langle \partial_\xi u^\xi, \circ dW \rangle$$

with $c(\epsilon) \approx \mathcal{K}'(\xi)$, where $\mathcal{K}'(\xi)$ is the curvature of ∂D at ξ

2D Case – Cahn-Hilliard

Results of [Antonopoulou, Karali, DB 15(?)]

2D-Case – Setting

Rescale time, such that:

$$du_\epsilon = \Delta(-\epsilon\Delta u_\epsilon + \epsilon^{-1} f'(u_\epsilon))dt + \epsilon^\sigma d\mathcal{W} \qquad \text{(CH)}$$

with Neumann boundary conditions on domain $\mathcal{D}$.

Chemical potential v_ϵ such that

$$\begin{cases} du_\epsilon &= -\Delta v_\epsilon dt + \epsilon^\sigma d\mathcal{W} \\ v_\epsilon &= -\frac{1}{\epsilon} f'(u_\epsilon) + \epsilon \Delta u_\epsilon \end{cases} \qquad \text{(SYS)}$$

2D-Case – very small noise
(based on Alikakos, Bates, Chen 94)

If $\sigma < \frac{14}{3}$, then the limit (u, v) satisfies free boundary problem

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathcal{D} \backslash \Gamma, \\ \partial_n v = 0 \quad \text{on } \partial \mathcal{D} \\ v = \lambda H \quad \text{on } \Gamma \\ 2V = \partial_n v^+ - \partial_n v^- \quad \text{on } \Gamma \end{cases}$$

NEED:

$\Gamma = \Gamma(t)$ is closed hypersurface dividing $u = 1$ from $u = -1$

with mean curvature H, velocity V,

unit outward normal n, constant $\lambda > 0$

Front motion

Dirk Blömker

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Conjecture for $\sigma = 1$

The limit (u, v) solves the stochastic Hele-Shaw problem.

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathcal{D} \backslash \Gamma, \\ \qquad \partial_n v = 0 \quad \text{on } \partial\mathcal{D} \\ \qquad v = \lambda H + \mathcal{W} \quad \text{on } \Gamma \\ \\ 2V = \partial_n v^+ - \partial_n v^- \quad \text{on } \Gamma \end{cases}$$

Thank you very much

for your attention