Harris recurrence for strongly degenerate stochastic systems, with application to stochastic Hodgkin-Huxley models

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talk based on

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- Strongly degenerate time inhomogeneous SDEs: densities and support properties. Application to a Hodgkin-Huxley system with periodic input. `arXiv:1410.0341`
- Ergodicity for a stochastic Hodgkin-Huxley model driven by Ornstein-Uhlenbeck type input. `arXiv:1311.3458v3`, AIHP
- A general scheme for ergodicity in strongly degenerate stochastic systems. Ongoing work.

I: strongly degenerate stochastic systems – main result

for $m < d$, consider d-dim diffusion driven by m-dim Brownian motion

$$dX_t \;=\; b(t,X_t)\,dt \;+\; \sigma(X_t)\,dW_t \quad , \quad t \geq 0$$

with coefficients

$$b(t,x) = \begin{pmatrix} b^1(t,x) \\ \vdots \\ b^d(t,x) \end{pmatrix} \quad , \quad \sigma(x) = \begin{pmatrix} \sigma^{1,1}(x) & \dots & \sigma^{1,m}(x) \\ \vdots & & \vdots \\ \sigma^{d,1}(x) & \dots & \sigma^{d,m}(x) \end{pmatrix}$$

for $t \geq 0$, $x \in E$: state space $(E, \mathcal{E})$ Borel subset of $\mathbb{R}^d$ (with some properties)

coefficient smooth, but neither bounded nor globally Lipschitz

<u>assume:</u> unique strong solution exists, has infinite life time in $\mathrm{int}(E)$

<u>aim:</u> ask for Harris properties of $(X_t)_{t\geq 0}$ (non homogeneous in time)

when drift is time-periodic and when some Lyapunov function is at hand:

write $P_{s,t}(x,dy)$ $(0 \le s < t < \infty,\ x,y \in E)$ for the semigroup of $(X_t)_{t\ge 0}$

assumption A: i) the drift is T-periodic in the time argument

$$b(t,x) = b\left(i_T(t),x\right) \quad , \quad i_T(t) := t \text{ modulo } T$$

ii) we have a Lyapunov function:

$$\begin{cases} V : E \to [1,\infty) \ \ \mathcal{E}\text{-measurable, and for some compact } K: \\ P_{0,T}V \ \text{ bounded on } K \ , \ P_{0,T}V \le V - \varepsilon \ \text{ on } E \setminus K \end{cases}$$

T-periodicity of the drift implies that the semigroup is T-periodic

$$P_{s,t}(x,dy) \ = \ P_{s+kT,t+kT}(x,dy) \quad , \quad k \in \mathbb{N}_0 \ , \ x,y \in E$$

thus the T-skeleton chain $(X_{kT})_{k\in\mathbb{N}_0}$ is a time homogeneous Markov chain

Lyapunov condition grants that skeleton chain will visit K infinitely often alternative under assumption A: define torus $\mathbb{T} := [0, T]$, define $\overline{E} := \mathbb{T} \times E$,

add time as 0-component to the process X:

$$\overline{X}_t := (i_T(t)\,,\, X_t) \quad , \quad t \geq s \quad , \quad \overline{X}_0 = (s, x)$$

$\overline{X}$ is time homogeneous, $(1{+}d)$-dim, state space $(\overline{E}, \overline{\mathcal{E}})$

assumption B: i) for some $U \subset \mathbb{R}^d$ open and containing E, coefficients

$$(t, x) \to b^i(t, x)\ ,\ x \to \sigma^{i,j}(x)\ ,\ 1 \leq i \leq d\ ,\ 1 \leq j \leq m$$

of SDE are real analytic functions on $\overline{T} := \mathbb{T} \times U$

ii) there exists some $x^* \in \mathrm{int}(E)$ with the following two properties:

- x^* is of full weak Hoermander dimension (cf. section IV)
- x^* is attainable in a sense of deterministic control (cf. next slide)

'attainable in a sense of deterministic control': in view of control arguments, put SDE in Stratonovich form

$$dX_t = \widetilde{b}(t, X_t)\, dt + \sigma(X_t) \circ dW_t$$

with Stratonovich drift

$$\widetilde{b}^i(t,x) = b^i(t,x) - \frac{1}{2} \sum_{\ell=1}^{m} \sum_{j=1}^{d} \sigma^{j,\ell}(x)\, \frac{\partial \sigma^{i,\ell}}{\partial x^j}(x) \quad , \quad 1 \le i \le d$$

definition 1: call $x^* \in \mathrm{int}(E)$ attainable in a sense of deterministic control

if for every starting point $x \in E$ we can find some function $\dot{\mathrm{h}} : [0, \infty) \to \mathbb{R}^m$ depending on x and x^*, all components $\dot{\mathrm{h}}^\ell(\cdot)$ in L^2_{loc}, $1 \le \ell \le m$,

which drives a deterministic control system

$$\varphi = \varphi^{h,x,x^*} \quad \text{solution to} \quad d\varphi_t = \widetilde{b}(t, \varphi_t)dt + \sigma(\varphi_t)\, \dot{\mathrm{h}}(t) dt$$

from $x = \varphi_0$ towards $x^* = \lim_{t\to\infty} \varphi_t$

(control theorem: Strook and Varadhan 1972, see Millet and Sanz-Sole 1994) theorem 1: under assumptions A + B:

i) (d-dim:) the T-skeleton $(X_{kT})_{k\in\mathbb{N}_0}$ is a positive Harris recurrent chain with invariant probability μ on $(E,\mathcal{E})$

ii) ($1{+}d$-dim:) the process $\overline{X} := (\, i_T(t)\,,\, X_t\,)_{t\geq 0}$ is positive Harris recurrent with invariant probability $\overline{\mu}$ on $(\overline{E},\overline{\mathcal{E}})$

and both invariant measures are related by

$$\overline{\mu} \;=\; \frac{1}{T}\int_0^T ds\,(\epsilon_s \otimes \mu P_{0,s}) \quad \text{on} \quad \overline{E} = \mathbb{T}\times E$$

corollary 1: (SLLN) for functions $G : E \to \mathbb{R}$ in $L^1(\mu)$ and $F : \overline{E} \to \mathbb{R}$ in $L^1(\overline{\mu})$

$$\frac{1}{n}\sum_{k=1}^{n} G(X_{kT}) \quad\longrightarrow\quad \int \mu(dy)\,G(y)$$

$$\frac{1}{t}\int_0^t F(i_T(s),X_s)\,\Lambda(ds) \quad\longrightarrow\quad \frac{1}{T}\int_0^T \Lambda(ds)\int_E (\mu P_{0,s})(dy)\,F(s,y)$$

Q_x-almost surely as $n\to\infty$ or $t\to\infty$, for every starting point $x\in E$
on the torus $\mathbb{T}$, we may consider many finite measures $\Lambda(ds)$, not only uniform

II: example, a stochastic Hodgkin-Huxley system

V membran potential in a neuron, n, m, h gating variables, ξ dendritic input

autonomous diffusion $(\xi_t)_{t\geq 0}$ modelling dendritic input, analytic coefficients, carrying T-periodic deterministic signal $t \to S(t)$ encoded in its semigroup

describe temporal dynamics of the neuron by a 5d stochastic system (ξHH):

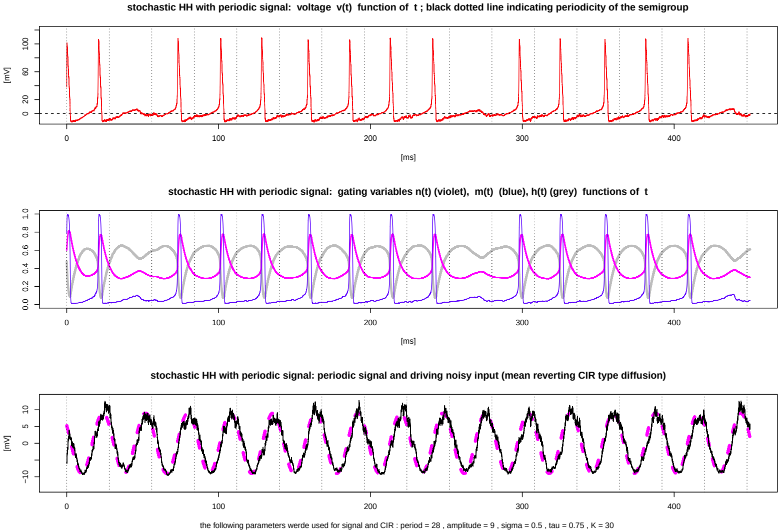
$$t \longrightarrow (V_t, n_t, m_t, h_t, \xi_t) =: X_t$$

5d SDE driven by 1d BM with state space $E = \mathbb{R} \times [0,1]^3 \times \mathbb{R}$ defined by

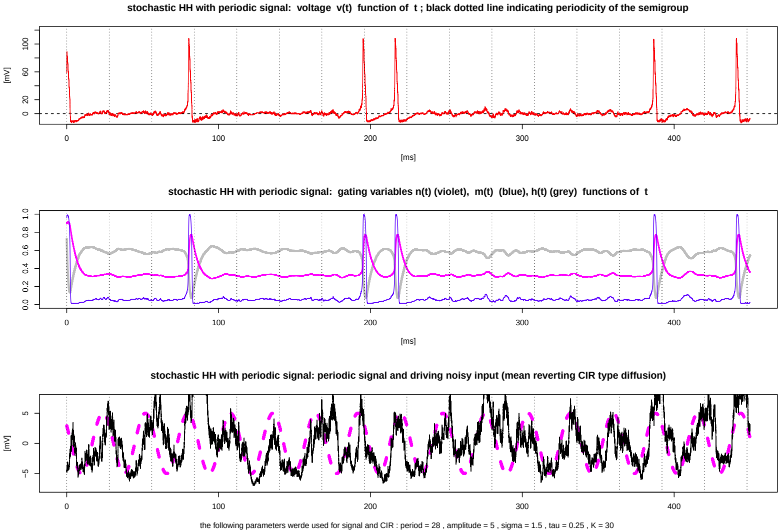
$$\begin{aligned}
dV_t &= d\xi_t - F(V_t, n_t, m_t, h_t)\, dt \\
dn_t &= [\alpha_n(V_t)(1-n_t) - \beta_n(V_t)n_t]\, dt \\
dm_t &= [\alpha_m(V_t)(1-m_t) - \beta_m(V_t)m_t]\, dt \\
dh_t &= [\alpha_h(V_t)(1-h_t) - \beta_h(V_t)h_t]\, dt \\
d\xi_t &= (S(t) - \xi_t)\, dt + dW_t
\end{aligned}$$

specific power series $F(V, n, m, h)$, strictly positive analytic fcts $\alpha_j(V)$, $\beta_j(V)$, $j = n, m, h$, see Izhikevich (2007), or Hodgkin and Huxley (1951)

trajectories may look like this (except that simulation here uses CIR type input)



or like this (depending on signal and choice of parameters for process $(\xi_t)_{t\geq 0}$)



classical deterministic HH systems with periodic deterministic signal $t \to \widetilde{S}(t)$:

$$\begin{aligned} dV_t &= \widetilde{S}(t)dt - F(V_t, n_t, m_t, h_t)\, dt \\ dn_t &= [\alpha_n(V_t)(1-n_t) - \beta_n(V_t)n_t]\, dt \\ dm_t &= [\alpha_m(V_t)(1-m_t) - \beta_m(V_t)m_t]\, dt \\ dh_t &= [\alpha_h(V_t)(1-h_t) - \beta_h(V_t)h_t]\, dt \end{aligned}$$

may show – depending on $\widetilde{S}(\cdot)$ – qualitatively quite different behaviour (spiking or non-spiking; single spikes or spike bursts, periodic or chaotic solutions; if periodic, periodicity of output may equal $\ell \geq 1$ periods of input; see interesting tableau based on numerical solutions in Endler 2012)

proposition 1: 'chamaeleon property' of (ξHH):

stochastic ξHH system $(X_t)_{0\leq t\leq T}$ coding deterministic signal $t \to S(t)$

imitates with positive probability over arbitrarily long (but fixed) time intervals

any deterministic HH with smooth and T-periodic signal $\widetilde{S}(\cdot) \neq S(\cdot)$

(the proof is a consequence of the control theorem) the stochastic Hodgkin-Huxley neuron (ξHH)

$$X_t = (V_t, n_t, m_t, h_t, \xi_t) \quad , \quad t \geq 0$$

is a strongly degenerate diffusion with state space E, and we can show

- assumptions A + B made above do hold, thus
- skeleton chain $(X_{kT})_k$ is positive Harris, invariant probability μ on $(E, \mathcal{E})$
- process $\overline{X} = (i_T(t), X_t)_t$ positive Harris, invariant probability $\overline{\mu}$ on $(\overline{E}, \overline{\mathcal{E}})$

Harris recurrence allows to analyze spiking patterns in the neuron via SLLN's:

$$\begin{aligned} A &:= \{x = (v, n, m, h, \zeta) : m > h\} && (\text{'active', during a spike}) \\ Q &:= \{x = (v, n, m, h, \zeta) : m < h\} && (\text{'quiet', or: between spikes}) \end{aligned}$$

events in $\mathcal{E}$, count spikes as follows: $\sigma_0 \equiv 0$, then for $n = 1, 2, \ldots$

$$\begin{aligned} \tau_n &:= \inf\{t > \sigma_{n-1} : X_t \in A\} && (n\text{-th spike beginning}) \\ \sigma_n &:= \inf\{t > \tau_n : X_t \in Q\} && (n\text{-th spike ending}) \end{aligned}$$

using decompositions into iid life periods and SLLN's (here we use Nummelin splitting in a sequence of 'accompanying' Harris processes with artificial atoms)

we can determine asymptotically a 'typical interspike time (ISI)' for the neuron

in the sense of a distribution function which depends on the signal $t \to S(t)$ and on the parameters of the SDE governing stochastic input $d\xi_t$

proposition 2: (Glivenko-Cantelli) define empirical distribution functions

$$\widehat{F}_n(t) = \frac{1}{n} \sum_{j=1}^{n} 1_{[0,t]}(\tau_{j+1} - \tau_j) \quad , \quad t \geq 0$$

then there is a honest distribution function F such that

$$\lim_{n\to\infty} \sup_{t\geq 0} \left| \widehat{F}_n(t) - F(t) \right| \quad = \quad 0$$

view F as the distribution function of 'the typical ISI' of the ξHH model neuron

(note: there may be single spikes or spike bursts, successive interspike times have no reason to be independent, geometric spike packets as in Berglund and Landon 2012 can easily be identified from such a limit distribution function F)

another (direct) application of cor 1: if for the ξHH model neuron

$$P_\mu \,(\text{more than one spike in } [0, T]) \;=\; 0$$

we can define a 'typical occurrence time for spikes' relatively to the periodicity interval via SLLN: define

$$\widehat{F}_n(r) := \frac{1}{n} \sum_{j=1}^{n} 1_{[0,rT]}(i_T(\tau_j)) \quad , \quad 0 \le r \le 1$$

then for arbitrary choice of a starting point $x \in E$ for the process X

$$F(r) \;:=\; \lim_{n \to \infty} \widehat{F}_n(r) \quad , \quad 0 \le r \le 1$$

exists a.s. and defines a honest distribution function on $[0, 1]$:

$$F(r) \;=\; P_\mu \,(\text{spike occurs before time } rT \mid \text{there is a spike in } [0, T])$$

many other applications in this spirit ...

III: proof of theorem 1, sketch of main arguments

back to setting of section I: d-dim SDE driven by m-dim BM, $m < d$,

$$dX_t \;=\; b(t, X_t)\, dt \;+\; \sigma(X_t)\, dW_t \quad , \quad t \geq 0$$

under assumptions A + B: drift T-periodic in time, existence of a Lyapunov function, analytic coefficients, existence of a point x^* which is of full weak Hoermander dimension and attainable in a sense of deterministic control

proof of theorem 1 consists of 3 main steps valid under assumptions A + B:

- control paths do transport weak Hoermander dimension
- all points in the state space are of full weak Hoermander dimension
- transition probabilities $P_{0,T}(\cdot, \cdot)$ locally admit continuous densities

then continue:

- rewrite this into a Nummelin minorization condition
  for the T-skeleton chain, with 'small set' some neighbourhood of x^*
- do Nummelin splitting (Nummelin 1978) in the skeleton chain $(X_{kT})_k$

IV: Lie brackets and weak Hoermander dimension

put SDE for process $X = (X_t)_{t \geq 0}$ in Stratonovich form

$$dX_t \;=\; \widetilde{b}(t, X_t)\, dt \;+\; \sigma(X_t) \circ dW_t$$

represent the time homogeneous process $\overline{X} = (i_T(t), X_t)_{t \geq 0}$ as

$$d\overline{X}_t \;=\; V_0(\overline{X}_t)\, dt \;+\; \sum_{\ell=1}^{m} V_\ell(\overline{X}_t) \circ dW_t^\ell$$

with W^ℓ the components of driving Brownian motion, $1 \leq \ell \leq m$, and with vector fields $V_0\,,\; V_1, \ldots, V_m : \overline{E} \to \mathbb{R}^{1+d}$

$$V_0(t,x) := \begin{pmatrix} 1 \\ \widetilde{b}^1(t,x) \\ \vdots \\ \widetilde{b}^d(t,x) \end{pmatrix} \;,\quad V_\ell(t,x) := \begin{pmatrix} 0 \\ \sigma^{1,\ell}(x) \\ \vdots \\ \sigma^{d,\ell}(x) \end{pmatrix} \;,\; 1 \leq \ell \leq m$$

with 0-component for 'time'; view $V_0,\; V_1, \ldots, V_m$ as differential operators

$$V_0 = \frac{\partial}{\partial t} + \sum_{j=1}^{d} \widetilde{b}^j(t,x) \frac{\partial}{\partial x^j} \;,\quad V_\ell = \sum_{j=1}^{d} \sigma^{j,\ell}(x) \frac{\partial}{\partial x^j} \;,\; 1 \leq \ell \leq m$$

for a vector field $L : \overline{E} \to \mathbb{R}^{1+d}$ whose 0-component already equals zero, Lie brackets $[V_0, L]$ and $[V_\ell, L]$, $1 \le \ell \le m$, take the form

$$[V_0, L]^0 = 0 \quad , \quad [V_0, L]^i = \frac{\partial L^i}{\partial t} + \sum_{j=1}^{d} \left(V_0^j \frac{\partial L^i}{\partial x^j} - L^j \frac{\partial V_0^i}{\partial x^j} \right) \ , \ i = 1, \dots, d$$

$$[V_\ell, L]^0 = 0 \quad , \quad [V_\ell, L]^i = \sum_{j=1}^{d} \left(V_\ell^j \frac{\partial L^i}{\partial x^j} - L^j \frac{\partial V_\ell^i}{\partial x^j} \right) \ , \ i = 1, \dots, d$$

(with superscript 'i' for i-th component): here 0-component is always zero

definition 2: for $N \ge 1$, define a set $\mathcal{L} := \mathcal{L}_N$ of vector fields $\overline{E} \to \mathbb{R}^{1+d}$ by

$$V_1, \dots, V_m \ \in \ \mathcal{L}$$

and at most N iteration steps

$$L \in \mathcal{L} \implies [L, V_0], [L, V_1], \dots, [L, V_m] \in \mathcal{L}$$

take $\mathcal{L}_N^* :=$ closure of $\mathcal{L}_N$ under Lie brackets, and $\Delta_{\mathcal{L}_N^*} := \mathrm{LA}(\mathcal{L}_N)$

note that all elements of $\mathcal{L}_N$, $\mathcal{L}_N^*$, $\Delta_{\mathcal{L}_N^*}$ have 0-component 'time' equal to zero thus d is a trivial upper bound for $\dim(\Delta_{\mathcal{L}_N^*})(s,x)$ for all $(s,x) \in \overline{E}$

definition 3: $x^* \in E$ is of full weak Hoermander dimension if for some N

$$\dim(\Delta_{\mathcal{L}_N^*})(s,x^*) \; = \; d \quad \text{independently of } s \in \mathbb{T}$$

remark 1: if $x^* \in \mathrm{int}(E)$ is of full weak Hoermander dimension, with N as above, then there exists an open neighbourhood U^* of x^* such that

$$\dim(\Delta_{\mathcal{L}_N^*})(s,x) \; = \; d \quad \text{for all } s \in \mathbb{T} \text{ and all } x \in U^*$$

theorem 2: for $(s,x) \in \overline{E}$, $\dim(\Delta_{\mathcal{L}_N^*})(s,x)$ does not depend on $N \geq 1$, and

$$\dim(\mathrm{LA}(V_0, V_1, \ldots, V_m))(s,x) \; = \; \dim(\Delta_{\mathcal{L}_N^*})(s,x) \; + \; 1$$

remark 2: i) the Lie algebra $\mathrm{LA}(V_0, V_1, \ldots, V_m)$ is the relevant one in view of control arguments: Sussmann (1973), Arnold and Kliemann (1987), cf. sect. V

ii) the Lie algebra $\Delta_{\mathcal{L}_N^*} = \Delta_{\mathcal{L}_1^*}$ is the relevant one for existence of continuous densities (Kusuoka and Strook 1985; existence locally: de Marco 2011) proof of theorem 2: starts from a construction of $\mathrm{LA}(V_0, V_1, \ldots, V_m)$ in analogy to the definition of sets $\mathcal{L}_N$, $\mathcal{L}_N^*$, $\Delta_{\mathcal{L}_N^*}$ of vector fields above:

for $N \geq 1$, define a set $\mathcal{D} := \mathcal{D}_N$ of vector fields by

$$V_0, V_1, \ldots, V_m \in \mathcal{D}$$

and at most N iteration steps

$$(*) \qquad L \in \mathcal{D} \implies [L, V_0], [L, V_1], \ldots, [L, V_m] \in \mathcal{D}$$

take $\mathcal{D}_N^* :=$ closure of $\mathcal{D}_N$ under Lie brackets, $\Delta_{\mathcal{D}_N^*}$ the linear hull: then

$$\Delta_{\mathcal{D}_N^*} = \mathrm{LA}(V_0, V_1, \ldots, V_m) \quad \text{does not dependend on } N$$

compare $\mathcal{L}_N$ to $\mathcal{D}_N$: $\mathcal{D}_N \setminus \mathcal{L}_N$ consists of V_0 and the 'descendence' of V_0 in the sense of iterated Lie brackets $(*)$; for these,

$$[\ldots[[V_0, V_\ell], L], \ldots] = -[\ldots[[V_\ell, V_0], L], \ldots]$$

belongs to $\mathcal{L}_N$ up to minus sign: $\mathcal{L}_N \subset \mathcal{D}_N \subset \{V_0, \pm L : L \in \mathcal{L}_N\}$

so V_0 is the only element of $\mathcal{D}_N$ which is linearly independent of $\mathcal{L}_N$, and the only element of $\mathcal{D}_N^*$ which is linearly independent of $\mathcal{L}_N^*$ $\square$

V: control arguments, support theorem

$\mathtt{H} :=$ Cameron-Martin space of functions $\mathtt{h} : [0, t_0] \to \mathbb{R}^m$ having absolutely continuous components $\mathtt{h}^\ell(t) = \int_0^t \dot{\mathtt{h}}^\ell(s) ds$ with $\dot{\mathtt{h}}^\ell \in L^2(0, t_0)$, $1 \le \ell \le m$

for $x \in E$ and $\mathtt{h} \in \mathtt{H}$ consider the deterministic system $\varphi = \varphi^{(\mathtt{h},x)}$ solution to

$$d\varphi_t \;=\; \widetilde{b}(t, \varphi_t)\, dt \;+\; \sigma(\varphi_t)\, \dot{\mathtt{h}}(t)\, dt \;\;,\;\; \varphi_0 = x$$

as a control path for X; call control $\mathtt{h}$ <u>admissible</u> if $\dot{\mathtt{h}}$ is piecewise constant (Arnold and Kliemann 1987)

<u>support theorem for diffusions</u> (Strook and Varadhan 1972, Millet and Sanz-Sole 1994): for bounded and smooth coefficents, for $0 < t_0 < \infty$:

$$\operatorname{supp}\left(\mathcal{L}(\,(X_t)_{0\le t\le t_0} | X_0 = x\,)\right) \;=\; \mathrm{cl}\left(\{\,(\varphi_t^{(\mathtt{h},x)})_{0\le t\le t_0} : \mathtt{h} \in \mathtt{H}\,\}\right)$$

with closure in $C([0, t_0], \mathbb{R}^d)$

under our assumptions A + B, we prove a <u>localized version</u>: control paths starting from $x \in E$ have some strictly positive lifetime in $\mathrm{int}(E)$, 'good' control paths for X have 'enough' lifetime (i.e. $\ge t_0$)

$\varphi = \varphi^{(\mathtt{h},x)}$ a control path for $X \quad \Longrightarrow \quad \overline{\varphi} := (i_T(\cdot), \varphi)$ control path for $\overline{X}$:

$$d\overline{\varphi}_t \;=\; \widetilde{b}(\overline{\varphi}_t)\,dt \;+\; \sigma(\overline{\varphi}_t)\,\dot{\mathtt{h}}(t)\,dt \quad , \quad \varphi_0 = x$$

admissible controls $\mathtt{h}$: there is a partition $\ldots < s_{r-1} < s_r < \ldots$ such that $\dot{\mathtt{h}}^\ell \equiv \gamma_\ell$ remains constant between times s_{r-1} and s_r: thus between times s_{r-1} and s_r, the control path $\overline{\varphi}^{(\mathtt{h},x)}$ for $\overline{X}$ moves along the vector field

$$V_0 \;+\; \gamma_1 V_1 + \ldots + \gamma_m V_m$$

where V_0, $V_1, \ldots, V_m$ have been defined in section IV; such control paths are 'piecewise integral curves of LA$(V_0, V_1, \ldots, V_m)$' in the sense of Sussmann 1973

thanks to assumption B (analytic coefficients) and to theorem 2:

lemma 1: along control paths $\varphi^{(\mathtt{h},x)}$ where $\mathtt{h}$ is admissible

i) (Sussmann 1973) $s \rightarrow \dim(\mathrm{LA}(V_0, V_1, \ldots, V_m))(s, \varphi^{(\mathtt{h},x)})$ remains constant

ii) hence also $s \rightarrow \dim \Delta_{\mathcal{L}_N^*}(s, \varphi^{(\mathtt{h},x)})$ remains constant

which proves that 'control paths do transport weak Hoermander dimension'

VI: finishing the proof of theorem 1

by assumption B, there exists $x^* \in \mathrm{int}(E)$ which

- is attainable in a sense of deterministic control
- is of full weak Hoermander dimension

lemma 2: i) all points $x \in E$ are of full weak Hoermander dimension

ii) for small neighbourhoods U^* of x^* and all $x \in E$:

$$Q_x(\, X_{jT} \in U^* \,) \; > \; 0 \quad \text{for } j \text{ large enough}$$

from now on, the steps are more and more classical:

lemma 3: trans. prob. $P_{0,T}(x, dy)$ admit Lebesgue densities $p_{0,T}(x, y)$ s. t.

i) $y \;\to\; p_{0,T}(x, y)$ is continuous when $x \in E$ is fixed

ii) $x \;\to\; p_{0,T}(x, y)$ is lower semicontinuous when $y \in E$ is fixed

lemma 4: for every $x \in E$ there is some $y = y(x) \in E$ and $\varepsilon > 0$ such that

$$(+) \qquad \inf_{x' \in B_\varepsilon(x)\,,\, y' \in B_\varepsilon(y)} p_{0,T}(x', y') \;\; > \;\; 0$$

for the T-skeleton chain $(X_{jT})_j$:

- condition A (Lyapunov fct): compact K is visited i.o.
- cover K with finitely many balls $B_{\varepsilon_k}(x_k)$ for which $(+)$ holds; then at least one of these $B_{\varepsilon_k}(x_k)$ is visited i.o., then $(+)$ is a Nummelin minorization with 'small set' $B_{\varepsilon_k}(x_k)$
- including x^* in $\mathrm{int}(K)$, there is $\varepsilon^* > 0$ such that $B_{\varepsilon^*}(x^*)$ satisfies $(+)$
- x^* is attainable in a sense of deterministic control: for every k, $B_{\varepsilon_k}(x_k)$ leads to $B_{\varepsilon^*}(x^*)$, thus $B_{\varepsilon^*}(x^*)$ is visited i.o.

thus $(+)$ for $B_{\varepsilon^*}(x^*)$ is a Nummelin minorization with 'small set' $B_{\varepsilon^*}(x^*)$:

lemma 5: for x^* of assumption B there is some $y^* = y^*(x^*) \in E$ and $\varepsilon^* > 0$ such that Nummelin minorization

$$(++) \qquad P_{0,T}(x, dy) \;\geq\; \alpha\, 1_{C^*}(x)\, \nu^*(dy)$$

holds with 'small set' $C^* := B_{\varepsilon^*}(x^*)$ and $\nu^* :=$ the uniform law on $B_{\varepsilon^*}(y^*)$

and Nummelin splitting with (artificial) atom $B_{\varepsilon^*}(y^*)$ yields iid life cycles ...

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