Analysis and stochastic processes on metric measure spaces

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Abstract

This contribution deals with the properties of certain differential and nonlocal operators on various spaces, with the emphasis on the relationship between the analytic properties of the operators in question and the geometric properties of the underlying space. In most situations, these operators are Markov generators. In such cases, we are also concerned with probabilistic aspects, such as the path properties of the corresponding Markov process.

1 Analysis on manifolds

The main object of interest in this part of the project is the Laplace-Beltrami operator Δ on a Riemannian manifold M. In most cases M can be assumed geodesically complete and non-compact. Denote by $B(x,r)$ the geodesic ball on M of radius r centered at $x \in M$, and by $V(x,r)$ the Riemannian volume of $B(x,r)$.

Manifold M is called *parabolic* if any positive superharmonic function on M is const. It is known that the following properties are equivalent:

- M is parabolic;
- there is no positive Green function of Δ on M;
- Brownian motion on M is recurrent;

(see [16]).

1.1 Elliptic operators

1.1.1 Semi-linear elliptic inequalities

Consider on M the differential inequality

$$\Delta u + u^\sigma \leq 0 \tag{1}$$

where $\sigma > 1$ is a constant, and ask if it has a positive solution u on M. This question was initially motivated by certain problems in differential geometry, but after many years of research of many authors it has become a popular question in PDEs.

A classical result of Gidas and Spruck [15] says that the equation

$$\Delta u + u^\sigma = 0 \text{ in } \mathbb{R}^n$$

with $n > 2$ and $\sigma < \frac{n+2}{n-2}$ has no positive solution, whereas for any $\sigma \geq \frac{n+2}{n-2}$ this equation has a positive solution. The case of an inequality (1) has a different answer: if $\sigma \leq \frac{n}{n-2}$ then (1) has no positive solution, whereas for $\sigma > \frac{n}{n-2}$ there are positive solutions.

The existing methods of handling the differential inequality (1) and various generalizations use quite strongly specific properties of PDEs in $\mathbb{R}^n$ (see, for example, [47]). Here we are interested in understanding minimal geometric assumptions needed for non-existence of a positive solution of (1). Assume that M is geodesically complete. A famous theorem of Cheng and Yau [10] says that if, for some x and all $r \gg 1$,

$$V(x,r) \leq Cr^2, \tag{2}$$

then M is parabolic. Since a solution of (1) is superharmonic, we see that under (2) the inequality (1) has no positive solution.

The following is a combined result of [29] and [43].

Theorem 1 *Let M be a geodesically complete, non-compact manifold. If, for some $x \in M$ and all $r \gg 1$,*

$$V(x,r) \leq Cr^p \log^q r, \tag{3}$$

where

$$p = \frac{2\sigma}{\sigma - 1} \quad \textit{and} \quad q = \frac{1}{\sigma - 1}, \tag{4}$$

then the inequality (1) has no positive solution.

Note that $p > 2$ so that the assumption (3) is weaker than (2).

The conditions (3)-(4) are sharp in the following sense: if

$$p = \frac{2\sigma}{\sigma - 1} \quad \text{and} \quad q > \frac{1}{\sigma - 1}$$

then there is an example of M satisfying (3) and having a positive solution of (1).

1.1.2 Negative eigenvalues of Schrödinger operators

Let V be a non-negative function on $\mathbb{R}^n$. Denote by $Neg(V, \mathbb{R}^n)$ the number of negative eigenvalues of the Schrödinger operator $H = -\Delta - V(x)$ on $\mathbb{R}^n$, assuming that V is such that the operator H with domain $C_0^\infty(\mathbb{R}^n)$ is essentially self-adjoint in $L^2(\mathbb{R})$. In the case $n \geq 3$ it is known that

$$Neg(V, \mathbb{R}^n) \leq C_n \int_{\mathbb{R}^n} V^{n/2} dx, \tag{5}$$

which is the content of a celebrated theorem of Cwikel–Lieb–Rozenblum (see [11], [46], [48]). In the case $n = 2$ this estimate is not true, and an equally good upper bound for $Neg(V)$ is still unknown. However, $Neg(V, \mathbb{R}^2)$ admits a *lower bound*:

$$Neg(V, \mathbb{R}^2) \geq c \int_{\mathbb{R}^2} V dx,$$

where $c > 0$ is an absolute constant, which was proved in [36].

Obtaining good enough *upper bounds* for $Neg\left(V, \mathbb{R}^2\right)$ is unexpectedly difficult. A major contribution to this area was done by M. Solomyak [50], which was then improved by E.Shargorodsky [49]. In [35] we obtained a new type of upper bounds. Fix some $p > 1$ and define for any non-negative integer n the following quantities:

$$A_n(V) = \int_{\{e^{2^{n-1}} < |x| < e^{2^n}\}} V(x)(1 + |\ln|x||)dx,$$

$$B_n(V) = \left(\int_{\{e^n < |x| < e^{n+1}\}} V^p(x)|x|^{2(p-1)}dx \right)^{1/p}.$$

Similarly A_n and B_n are defined for $n < 0$. The main result of [35] is the following theorem.

Theorem 2 *We have*

$$Neg\left(V, \mathbb{R}^2\right) \leq 1 + C \sum_{\{n \in \mathbb{Z}: A_n > c\}} \sqrt{A_n} + C \sum_{\{n \in \mathbb{Z}: B_n > c\}} B_n, \tag{6}$$

where C, c are some positive constants depending only on p.

The complexity of this estimate is in striking contrast with (5); it reflects a more complicated mechanism of formation of negative eigenvalues in $\mathbb{R}^2$ which is related to the *parabolicity* of $\mathbb{R}^2$.

In [35] we introduced many new tools. In particular, we used the Green function $g_0(x, y)$ of the operator $-\Delta + V_0$ in $\mathbb{R}^2$ where $V_0 \geq 0$ is any non-zero function from $C_0^\infty\left(\mathbb{R}^2\right)$, and proved the following estimate of g_0:

$$g_0(x, y) \simeq \log\langle x\rangle \wedge \log\langle y\rangle + \log_+ \frac{1}{|x - y|},$$

that follows from the estimate (11) of the heat kernel of $-\Delta + V_0$ that is discussed below. The sign $\simeq$ means that the ratio of the both sides is between two positive constants.

Although (6) covers most previously known upper bounds of $Neg\left(V, \mathbb{R}^2\right)$, it still does not cover some interesting potentials in $\mathbb{R}^2$ where the finiteness of $Neg\left(V, \mathbb{R}^2\right)$ can be seen in ad hoc way.

1.1.3 Estimates of the Green function

Another question about Schrödinger operator is obtaining estimates of the Green function $g_V(x, y)$ of $-\Delta + V$ on an arbitrary manifold M via the Green function $g(x, y)$ of Δ. The following universal lower estimate was proved in [18].

Theorem 3 *On any nonparabolic Riemannian manifold M and for any $V \geq 0$, we have*

$$g_V(x, y) \geq g(x, y) \exp\left(-\frac{\int_M g(x, z)g(z, y)V(z)\,dz}{g(x, y)}\right). \tag{7}$$

A striking feature of this result is that it does not require any restriction on M. Moreover, the same result holds in a higher generality of abstract harmonic spaces.

1.2 The heat equation

A central object in Analysis on Manifold is the *heat kernel* $p_t(x, y)$ that is the fundamental solution of the heat equation

$$\partial_t u = \Delta u,$$

where $t > 0$ is a time and x, y are points of M. For example, if $M = \mathbb{R}^n$ then the heat kernel is given by the classical Gauss-Weierstrass formula

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right).$$

The question of obtaining heat kernel estimates under certain geometric assumptions on the underlying manifold M has been extensively studied a few decades (see [8], [12], [51]). For example, if the manifold M is geodesically complete and has non-negative Ricci curvature then, by a theorem of Li and Yau [45],

$$p_t(x,y) \asymp \frac{C}{V\left(x,\sqrt{t}\right)} \exp\left(-\frac{d^2(x,y)}{ct}\right), \tag{8}$$

where d is the geodesic distance on M, and C, c are positive constant. The sign $\asymp$ means that both $\leq$ and $\geq$ are true but with different values of C, c.

1.2.1 Heat kernels on connected sums

Here we consider heat kernel estimates on the *connected sum* $M_1 \# M_2$ of two manifolds M_1, M_2 of equal dimensions. By definition, $M_1 \# M_2$ denotes any manifold that is obtained by connecting exterior domains in M_1 and M_2 via a compact connected manifold. For example, even estimating the heat kernel on $\mathbb{R}^n \# \mathbb{R}^n$ is a highly non-trivial task. Although the first approach to the latter problem was initiated in [6] in 1996, the full answer was obtained in [41] in 2009.

Theorem 4 *If x, y are two points lying on different sheets of $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n \geq 3$, then, for large enough $t, |x|, |y|$,*

$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}}\right) e^{-\frac{d^2(x,y)}{ct}}.$$

More generally, consider a connected sum $M = M_1 \# ... \# M_k$ where we assume that, for each manifold M_i, the heat kernel satisfies the two-sided Li-Yau estimate (8). The question of estimating of the heat kernel on such a manifold M was largely solved in a series of papers of A. Grigor'yan and L. Saloff-Coste culminating in [41]. A remarkable observation of [41] is that one has to distinguish *parabolic* and *non-parabolic* ends M_i. The results of [41] are exhaustive when the manifold M is non-parabolic, that is, when at least one end M_i is non-parabolic. Assume also that, for some $o_i \in M_i$ and all large enough r,

$$V(o_i, r) \simeq r^{\alpha_i},$$

where $\alpha_i > 0$. Denote $|x| = d(x, o_i)$.

Theorem 5 *Assume that $\alpha_i \neq 2$ for all $i = 1, .., k$. Set*

$$\alpha_i^* = \begin{cases} \alpha_i, & \text{if } \alpha_i < 2, \\ 4 - \alpha_i, & \text{if } \alpha_i > 2 \end{cases}$$

and

$$\alpha = \min\{\alpha_i^* : i = 1, ..., k\}.$$

Then, for all $t \gg 1$, $x \in M_i$ and $y \in M_j$ with $i \neq j$ and large enough $|x|, |y|$,

$$\begin{aligned} p_t(x,y) \quad &\asymp \quad C\left(\frac{1}{t^{\alpha/2}|x|^{\alpha_i^*-2}|y|^{\alpha_j^*-2}} \frac{1}{t^{\alpha_j^*/2}|x|^{\alpha_i^*-2}} + \frac{1}{t^{\alpha_i^*/2}|y|^{\alpha_{j-2}^*}}\right) \\ &\times |x|^{(2-\alpha_i)_+} |y|^{(2-\alpha_j)_+} \exp\left(-\frac{d^2(x,y)}{ct}\right). \end{aligned}$$

In the case when $k = 2$ and $\alpha_1 = \alpha_2 = n > 2$, we obtain $a = \alpha_i^* = n$, and we obtain the estimate of Theorem 4.

Consider an example of a mixed case $M = M_1 \# M_2$ with

$$M_1 = \mathbb{R}_+^1 \times \mathbb{S}^2 \quad \text{and} \quad M_2 = \mathbb{R}^3.$$

In this case the manifold M_1 is parabolic with the volume growth exponent $\alpha_1 = 1$, and M_2 is non-parabolic with $\alpha_2 = 3$. It follows that

$$\alpha_1^* = 4 - \alpha_1 = 3, \qquad \alpha_2^* = \alpha_2 = 3, \quad \text{and } \alpha = \min(\alpha_1^*, \alpha_2^*) = 3.$$

Hence, if $x \in M_1$ and $y \in M_2$, then we obtain by Theorem 5

$$p_t(x,y) = \frac{C}{t^{3/2}} \left(1 + \frac{|x|}{|y|}\right) e^{-\frac{d^2(x,y)}{ct}}.$$

The parabolic case was treated in [41] only in a special case, while the general parabolic case still remains open. In particular, the following estimate was proved for $M = \mathbb{R}^2 \# \mathbb{R}^2$ (equivalently, for a catenoid) in [41].

Theorem 6 *If x and y are two points lying on different sheets of $M = \mathbb{R}^2 \# \mathbb{R}^2$ then, for $|x|, |y| \geq \sqrt{t} \gg 1$,*

$$p_t(x,y) \asymp \frac{C}{t} \left(\frac{1}{\log |x|} + \frac{1}{\log |y|}\right) e^{-\frac{d^2(x,y)}{ct}},$$

while for $|x|, |y| \leq \sqrt{t}$ we have

$$p_t(x,y) \asymp \frac{C}{t \log^2 \sqrt{t}} \left(\log \sqrt{t} + \log^2 \sqrt{t} - \log |x| \log |y|\right).$$

The proofs in [41] are based on [39], [38], [40], [42].

1.2.2 Heat kernels of Schrödinger operators

Consider in $\mathbb{R}^n$ the Schrödinger operator

$$H = -\Delta + \Phi$$

where $\Phi \geq 0$ is a smooth function, and let $p_t^\Phi(x,y)$ be the heat kernel of H. Here we describe some results about the estimates of p_t^Φ obtained in [17] using the method of h-transform from [40].

It is well-known that if $n > 2$ and, for some $\varepsilon > 0$,

$$\Phi(x) \leq C |x|^{-(2+\varepsilon)}, \quad \text{for all } |x| > 1, \tag{9}$$

then

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{n/2}} e^{-\frac{|x-y|^2}{ct}}. \tag{10}$$

This estimate reflects the fact that the potentials with the upper bound (9) are small perturbations of the Laplace operator (so called *short range* potentials), so that the estimate (10) is obtained by a perturbation argument.

The case $n = 2$ is quite different as it is stated below. Set

$$\langle x \rangle := 2 + |x|.$$

Theorem 7 *Let Φ be a non-zero function with a compact support in $\mathbb{R}^2$. Then the heat kernel of H satisfies*

$$p_t^\Phi(x,y) \asymp \frac{C \log\langle x \rangle \log\langle y \rangle}{t \log\left(\langle x \rangle + \sqrt{t}\right) \log\left(\langle y \rangle + \sqrt{t}\right)} e^{-\frac{|x-y|^2}{ct}}. \tag{11}$$

In particular, in the range $t \geq \langle x \rangle^2 + \langle y \rangle^2$, we have

$$p_t^\Phi(x,y) \simeq \frac{\log\langle x \rangle \log\langle y \rangle}{t \log^2 t}.$$

Now let us consider the most interesting potential

$$\Phi(x) = b |x|^{-2}, \quad \text{for all } |x| > 1. \tag{12}$$

that is on the borderline between the short and long range potentials.

Theorem 8 *Let Φ be a potential* (12) *in $\mathbb{R}^n$ with $n \geq 2$. Then the heat kernel of H satisfies the estimate for all $t > 0$ and $x, y \in \mathbb{R}^n$:*

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{n/2+\beta}} \left(\frac{1}{\sqrt{t}} + \frac{1}{\langle x\rangle}\right)^{-\beta} \left(\frac{1}{\sqrt{t}} + \frac{1}{\langle y\rangle}\right)^{-\beta} e^{-\frac{|x-y|^2}{ct}}, \tag{13}$$

where

$$\beta = -\frac{n}{2} + 1 + \sqrt{\left(\frac{n}{2} - 1\right)^2 + b}.$$

In particular, in the most interesting range $t \geq \langle x\rangle^2 + \langle y\rangle^2$, the estimate (13) becomes

$$p_t^\Phi(x,y) \simeq \frac{\langle x\rangle^\beta \langle y\rangle^\beta}{t^{n/2+\beta}}$$

Note that the value of the coefficient b in (12) determines the exponent $\frac{n}{2} + \beta$ of the power decay of the heat kernel as $t \to \infty$. Since b takes values in $(0, \infty)$, the exponent of t ranges in $(\frac{n}{2}, \infty)$.

For comparison let us mention that, for a long range potential

$$\Phi(x) = b\,|x|^{-(2-\alpha)}, \quad \text{for all } |x| > 1,$$

with $\alpha \in (0,2)$, the long time decay of the heat kernel is already superpolynomial as follows:

$$p_t^\Phi(0,0) \asymp C \exp\left(ct^{\frac{2-\alpha}{2+\alpha}}\right).$$

1.2.3 Heat kernels of operators with singular drift

Consider in $\mathbb{R}^n \setminus \{0\}$ the operator

$$Lu = \Delta u - \nabla\psi \cdot \nabla u$$

with a singular potential

$$\psi(x) = |x|^{-\alpha}$$

where $\alpha > 0$. We have proved in [37] the following estimates of the heat kernel of L.

Theorem 9 *For all $0 < t < 1$, we have*

$$\sup_{x,y} p_t(x,y) \leq \exp\left(Ct^{-\frac{\alpha}{\alpha+2}}\right)$$

and

$$\sup_x p_t(x,x) \geq \exp\left(ct^{-\frac{\alpha}{\alpha+2}}\right),$$

for some positive constants C, c.

The singularity of the drift term at the origin causes a higher rate of blow up of the heat kernel at $t \to 0$, and the fact that the latter should be given by the term $\exp\left(t^{-\frac{\alpha}{\alpha+2}}\right)$ is not obvious at all and was not predicted by any "physical" argument.

By a certain transformation we reduce the problem to estimating the heat kernel of a weighted Laplace operator, and the latter amounts to proving a certain isoperimetric inequality on a weighted manifold $(\mathbb{R}^n, \mu)$, where measure μ is given by

$$d\mu = \exp\left(-\frac{1}{|x|^\alpha}\right) dx.$$

Due to specific properties of this measure μ, the previously known methods for obtaining isoperimetric inequalities on warped products did not work, and we had to develop in [37] a new machinery for that.

1.3 Escape rate of Brownian motion

A manifold M is called *stochastically complete* if Brownian motion on M has lifetime ∞, which is equivalent to the condition

$$\int_M p_t(x,y)\,d\mu(y) = 1$$

for all $x \in M$ and $t > 0$.

It is known that a geodesically complete Riemannian manifolds is stochastically complete provided

$$\int^{\infty} \frac{rdr}{\log V(x,r)} = \infty \tag{14}$$

for some $x \in M$. It was proved in [19] that under the condition (14) one can obtain also quantitative estimate on how fast Brownian motion escapes to ∞. The following result was proved in [19].

Theorem 10 *Let M be a Cartan-Hadamard manifold, satisfying (14). Fix a point $x \in M$ and define a function $\varphi(t)$ for large t by the identity*

$$t = \int_1^{\varphi(t)} \frac{r\,dr}{\log V(x,r)}.$$

Then, Brownian motion on M at time t stays in the ball $B(x, \varphi(Ct))$ for large enough t with probability 1, where $C > 0$ is an absolute constant (for example, $C = 130$).

In other words, the function

$$R(t) = \varphi(Ct)$$

is an *upper rate function* of Brownian motion.

Examples of spherically symmetric manifolds show that this estimate of the escape rate in terms of V is essentially sharp.

For example, if

$$V(x,r) \simeq r^{\alpha}$$

then we obtain an upper rate function

$$R(t) = \text{const}\,\sqrt{t\log t}. \tag{15}$$

Note for comparison that by Khinchine's law of the iterated logarithm, an optimal upper rate function in $\mathbb{R}^n$ is

$$R(t) = \sqrt{(4+\varepsilon)\,t\log\log t}.$$

The function (15) is therefore not very sharp in $\mathbb{R}^n$ because of distinction between $\log\log t$ and $\log t$, but it is sharp in the class of all manifolds with polynomial volume growth (see [28])

Historically the upper rate function (15) was obtained by Hardy and Littlewood in 1914 for sums of independent Bernoulli random variables, which was superseded in ten years by Khinchine's law. From the modern point of view, the Hardy-Littlewood function (15) still make sense as an optimal upper rate function for Brownian motion on manifolds with polynomial volume growth.

2 Analysis on metric measure spaces

2.1 Heat kernels on fractal-like spaces

Let (M,d,μ) be a metric measure space, that is, (M,d) is a metric space and μ is a Radon measure on M with full support. We denote by $B(x,r)$ the metric balls in M and assume that all metric balls are precompact. Set $V(x,r) = \mu(B(x,r))$.

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(M,\mu)$ (see [14]). We investigate the properties of the Hunt process associated the Dirichlet form, and its heat kernel $p_t(x,y)$ that is defined as the integral kernel (should it exists) of the corresponding heat semigroup.

We distinguish two main cases: when the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *local*, that is, the associated process is a diffusion, and when the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of *jump type*, that is, it is given by

$$\mathcal{E}(f,g) = \int_M \int_M (f(x) - f(y))(g(x) - g(y)) J(x,y) \, d\mu(x) \, d\mu(y), \tag{16}$$

where J is a symmetric *jump kernel.*

Fix two positive parameters α, β. We look for conditions on the measure and energy that would ensure the following heat kernel bounds:

- sub-Gaussian bound in the local case:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right); \tag{17}$$

- stable-like bound in the jump case:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} = \frac{Ct}{\left(t^{1/\beta} + d(x,y)\right)^{\alpha+\beta}}. \tag{18}$$

It was proved in [23] that in the both cases α is the Hausdorff dimension of (M,d) and, moreover,

$$V(x,r) \simeq r^\alpha. \tag{19}$$

In the case of (17), the parameter β is called the *walk dimension*, which is an invariant of (M,d) as well. By [23], in this case $\beta \geq 2$ (in fact, $\beta > 2$ for most interesting fractals). In the case of (18), the parameter β is called the *index* of the associated jump process.

There are many reasons for considering these two types of estimates. Firstly, both are known to hold on various families of fractals, in particular, on the Sierpinski gasket and carpet (cf. [2]).

Secondly, the following dichotomy was proved in [30]: if $p_t(x,y)$ satisfies the estimate

$$p_t(x,y) \asymp Ct^{-\alpha/\beta} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

with some function Φ then this has to be either (17) or (18).

It was proved in [3] that the sub-Gaussian estimate (17) is equivalent to the parabolic Harnack inequality.

An important problem is to find some practical conditions on (M,d,μ) and $(\mathcal{E}, \mathcal{F})$ that should be equivalent to (17) resp. (18).

Some results about existence of the heat kernel and its upper bounds were obtained in [20], [22], [27].

In order to state the results about equivalent conditions for the estimates (17), let us define first the following notions.

Definition 11 A function $u \in \mathcal{F}$ is called harmonic in an open set $\Omega \subset M$ if

$$\mathcal{E}(u, \varphi) = 0$$

for all $\varphi \in \mathcal{F} \cap C_0(\Omega)$.

Definition 12 We say that the *uniform elliptic Harnack inequality* is satisfied if there is a constant C such that, for any function $u \in \mathcal{F}$ that is harmonic and non-negative in a ball $B(x,r) \subset M$,

$$\operatorname*{esssup}_{B(x,r/2)} u \leq C \operatorname*{essinf}_{B(x,r/2)} u.$$

Definition 13 For any compact set $K \subset M$ and open set $\Omega \supset K$, define the *capacity* of the capacitor (K, Ω) by

$$\operatorname{cap}(K,\Omega) = \inf\left\{\mathcal{E}(\varphi,\varphi) : \varphi \in \mathcal{F} \cap C_0(\Omega), \ \varphi|_K \equiv 1\right\}.$$

The series of works [22] [44], [21] leads to the following result.

Theorem 14 *Under certain connectivity property of* (M,d), *the sub-Gaussian estimate* (17) *is equivalent to the conjunction of the following three conditions:*

- *the volume regularity* (19)*;*
- *the uniform elliptic Harnack inequality;*
- *the capacity condition: for all balls* $B=B(x,r)$ *and* $2B=B(x,2r)$,

$$\operatorname{cap}(B,2B)\simeq r^{\alpha-\beta}. \tag{20}$$

Of course, the elliptic Harnack inequality is in general quite difficult to verify, so the search for better conditions goes on.

If M is a complete Riemannian manifold with the canonical Dirichlet form, then the Gaussian heat kernel bound (that is, the case $\beta=2$ in (17)) is known to be equivalent to the conjunction of the following two conditions:

- the volume regularity (19);
- the Poincaré inequality

$$\int_{B(x,2r)}|\nabla f|^2\,d\mu\geq\frac{c}{r^2}\int_{B(x,r)}\left(f-\overline{f}\right)^2d\mu, \tag{21}$$

  where $\overline{f}=\frac{1}{\mu(B(x,r))}\int_{B(x,r)}f\,d\mu$.

In the most interesting case $\beta>2$ that typically occurs in fractals, one replaces the Poincaré inequality (21) by the β-Poincaré inequality

$$\int_{B(x,r)}d\Gamma\langle f,f\rangle\geq\frac{c}{r^\beta}\int_{B(x,r)}\left(f-\overline{f}\right)^2d\mu, \tag{22}$$

where $\Gamma\langle f,f\rangle$ is the energy measure of f. Then both (19) and (22) are also necessary for (17), but *not* sufficient.

In order to state the next result, we need the notion of a generalized capacity.

Definition 15 Let $u\in\mathcal{F}\cap L^\infty(M)$. For any compact set $K\subset M$ and open set $\Omega\supset K$, define the *generalized capacity* of the capacitor (K,Ω) by

$$\operatorname{cap}_u(K,\Omega)=\inf\left\{\mathcal{E}\left(u^2\varphi,\varphi\right):\varphi\in\mathcal{F}\cap C_0(\Omega),\ \varphi|_K\equiv1\right\}.$$

The following theorem is a slightly reformulated result of [25].

Theorem 16 *The estimate* (17) *is equivalent to the conjunction of three properties:*

- *the volume regularity* (19)*;*
- *the* β*-Poincaré inequality* (22)*;*
- *the generalized capacity estimate: for any function* $u\in\mathcal{F}\cap L^\infty$ *and for any two concentric balls* $B_1:=B(x,R)$ *and* $B_2:=B(x,R+r)$,

$$\operatorname{cap}_u(B_1,B_2)\leq\frac{C}{r^\beta}\int_B u^2d\mu. \tag{23}$$

However, the latter condition is still difficult to check. Our conjecture is that it can be replaced by a simpler capacity condition (20). Note that (23) with $u = 1$ and $R = r$ is equivalent to (20).

A similar question is in place for the stable-like estimate (18). Some approach to upper bounds was developed in [24]. The equivalent conditions for the two-sided estimates (18) in the case $\beta < 2$ were obtained by Z.-Q. Chen and T. Kumagai [9], who proved that (18) is equivalent to the volume regularity (19) and the following estimate of the jump kernel J:

$$J(x, y) \simeq \frac{1}{d(x, y)^{\alpha+\beta}}. \tag{24}$$

The condition (24) replaces in this case the Poincaré inequality. The case $\beta \geq 2$ is still open.

There is one specific setting though where obtaining heat kernel bounds for the jump kernel $J(x, y) = d(x, y)^{-(\alpha+\beta)}$ is relatively easy for any $\beta > 0$: this is the case when (M, d) an *ultra-metric* space. The theory of Markov processes on ultra-metric spaces was developed in [5], using specific properties of ultra-metric. In particular, this theory applies when $M = \mathbb{Q}_p$ is the space of p-adic numbers with the p-adic distance, and yields the estimate (18) with $\alpha = 1$ (see the estimate (27) in Section 2.3 below).

2.2 Stochastic completeness of jump processes

In [26] we investigated the stochastic completeness of the jump process associated with the Dirichlet form (16). We say that the distance function $d(x, y)$ and the jump kernel $J(x, y)$ are *adapted* to each other, if there exists a constant C such that

$$\int_M (1 \wedge d(x, y)^2) J(x, y) d\mu(y) \leq C \quad \text{for all } x \in M. \tag{25}$$

For example, the following jump kernel in $\mathbb{R}^n$

$$J(x, y) = \frac{\text{const}}{|x - y|^{n+\alpha}},$$

is adapted to the Euclidean distance provided $\alpha \in (0, 2)$. Moreover, by Lévy-Khinchine theorem, the Lévy measure $W(dy)$ of any Lévy process in $\mathbb{R}^n$ satisfies the condition

$$\int_{\mathbb{R}^n \setminus \{0\}} \left(1 \wedge |y|^2\right) W(dy) < \infty.$$

Since $W(dy)$ corresponds in our notation to $J(x, y) d\mu(y)$, we see that the Euclidean distance in $\mathbb{R}^n$ is adapted to any Lévy process.

The main result of [26] is the following theorem.

Theorem 17 *If J and d are adapted and if, for some $x \in M$ and $c > 0$,*

$$V(x, r) \leq \exp(cr \log r) \quad \textit{for all large enough } r,$$

then the jump process with the jump kernel J is stochastically complete.

2.3 Jump processes on ultra metric spaces

An ultra-metric space is a metric space (M, d) where the distance function satisfies the ultra-metric inequality

$$d(x, y) \leq \max(d(x, z), d(z, y)),$$

that is obviously stronger than the usual triangle inequality. The ultra-metric inequality implies that any two metric balls $B(x, r)$, $B(y, r)$ of the same radius are either disjoint or identical. This in turn implies that, for any non-negative real r, the family of all distinct balls of radius r form a partition of M.

Let (M, d) be a locally compact ultra-metric space. A model example is the field $\mathbb{Q}_p$ of p-adic numbers with the p-adic distance or its straightforward generalization $\mathbb{Q}_p^n$. Fix a Radon measure μ

on M with full support, a probability distribution function $\sigma(r)$ on $[0,+\infty)$ and define the following operator P on functions on M:

$$Pf(x)=\int_0^\infty\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}f\,d\mu\right)d\sigma(r) \tag{26}$$

(cf. [4] and [5]). This operator is clearly a Markov operator. As it follows from the aforementioned property of ultra-metric balls, P is a bounded non-negative definite self-adjoint operator in the Hilbert space $L^2(M,\mu)$. The latter allows us to define the heat semigroup $\{P_t\}_{t\geq0}$ simply by $P_t=P^t$ and, hence, the associated continuous time random walk $\{X_t\}_{t\geq0}$ on M (note that typically Markov operators are not positive definite, so that the operator P^t cannot be defined in general).

The spectral decomposition for P_t follows easily from the representation (26), which leads to explicit expression for the heat kernel $p_t(x,y)$ of P_t and then also to simple estimates of $p_t(x,y)$ (see [5]).

For example, let $M=\mathbb{Q}_p$ with the p-adic distance $d(x,y)=\|x-y\|_p$ and the Haar measure μ. Then $\mu(B(x,r))\simeq r$. Choose

$$\sigma(r)=\exp(-(c/r)^\alpha),$$

where $\alpha,c>0$.

Theorem 18 *In $\mathbb{Q}_p$ the heat kernel of the heat semigroup $\{P_t\}$ with the above probability distribution function $\sigma(r)$ satisfies the estimate*

$$p_t(x,y)\simeq\frac{t}{(t^{1/\alpha}+\|x-y\|_p)^{1+\alpha}}, \tag{27}$$

for all $t>0$ and $x,y\in\mathbb{Q}_p$. Consequently, the Green function $g(x,y)$ of $\{P_t\}$ is finite if and only if $\alpha<1$, and in this case

$$g(x,y)\simeq\|x-y\|_p^{\alpha-1}.$$

As a locally compact abelian group, $\mathbb{Q}_p$ has the dual group that is again $\mathbb{Q}_p$. Hence, the Fourier transfer is defined as a unitary operator in $L^2(\mathbb{Q}_p,\mu)$. Using the Fourier transfer, Vladimirov and Volovich [52], [53] introduced a class $\mathfrak{D}^\alpha$ of *fractional derivatives* on functions on $\mathbb{Q}_p$. This operator acts as follows:

$$\mathfrak{D}^\alpha f(x)=\frac{p^\alpha-1}{1-p^{-\alpha-1}}\int_{\mathbb{Q}_p}\frac{f(x)-f(y)}{\|x-y\|_p^{1+\alpha}}d\mu(y).$$

The following theorem was proved in [5].

Theorem 19 *Operator $\mathfrak{D}^\alpha$ coincides with the generator of the semigroup $\{P_t\}$ with the following probability distribution function*

$$\sigma(r)=\exp(-(p/r)^\alpha).$$

Consequently, the heat kernel of $\mathfrak{D}^\alpha$ satisfies (27).

It does not seem possible to obtain this estimate of the heat kernel of $\mathfrak{D}^\alpha$ by using the Fourier Analysis approach.

3 Homology theory on graphs

In a series of papers [31], [32], [33], [34], we introduced the notion of a differential form on a digraph (=directed graph) with the exterior derivative d, as well as the dual object − a ∂-invariant path with the boundary operator ∂, which leads to the dual notions of cohomology and homology of graphs.

Let V be a finite set. An *elementary p-path* on V is any sequence $i_0...i_p$ of $(p+1)$ vertices of V, which is also denoted by $e_{i_0...i_p}$. The formal linear combinations of all $e_{i_0...i_p}$ with coefficients from a field $\mathbb{K}$ form a linear space Λ_p. Define a linear *boundary operator* $\partial:\Lambda_p\to\Lambda_{p-1}$ by

$$\partial e_{i_0...i_p}=\sum_{k=0}^{p}(-1)^k e_{i_0...\widehat{i_k}...i_p},$$

where $\widehat{i_k}$ means omission of i_k.

Let $G = (V, E)$ be a digraph, where E is the set of directed edges (=arrows) on V. A p-path $e_{i_0 \ldots i_p}$ is called *allowed* if all the pairs $i_k i_{k+1}$ are arrows. Denote by $\mathcal{A}_p$ the subspace of Λ_p generated by all allowed p-paths. In general, if $v \in \mathcal{A}_p$ then ∂v does not have to be in $\mathcal{A}_{p-1}$. For example, on the digraph

$$^0\bullet \to \bullet^1 \to \bullet^2$$

the 2-path e_{012} is allowed and, hence, lies in $\mathcal{A}_2$ while its boundary

$$\partial e_{012} = e_{12} - e_{02} + e_{01} \tag{28}$$

is not in $\mathcal{A}_1$ because e_{02} is not allowed.

This observation motivates the following definition.

Definition 20 Define the subspace Ω_p of $\mathcal{A}_p$ by

$$\Omega_p = \Omega_p(G) = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}.$$

The elements of Ω_p are called ∂-*invariant* p-paths.

For example, if G contains the following "triangle"

$$\begin{tikzcd} & \bullet^1 \arrow[dr] & \\ ^0\bullet \arrow[ur] \arrow[rr] & & \bullet^2 \end{tikzcd}$$

then the 2-path e_{012} is ∂-invariant by (28) . If G contains the following "square"

$$\begin{tikzcd} ^1\bullet \arrow[r] & \bullet^3 \\ ^0\bullet \arrow[u] \arrow[r] & \bullet^2 \arrow[u] \end{tikzcd}$$

then the 2-path $v = e_{013} - e_{023}$ is ∂-invariant, because $v \in \mathcal{A}_2$ and

$$\partial v = (e_{13} - e_{03} + e_{13}) - (e_{23} - e_{03} + e_{02}) = e_{13} + e_{13} - e_{23} - e_{02} \in \mathcal{A}_1.$$

It is easy to see that ∂ acts from Ω_p to Ω_{p-1} and that $\partial^2 = 0$. Hence, we obtain a chain complex $\Omega_*(G)$

$$\ldots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \Omega_{p+1} \leftarrow \ldots$$

where $p \geq 0$ and $\Omega_{-1} = \{0\}$. The homology groups $H_p = H_p(G)$ of this chain complex are called the *path homologies* of G.

There are also dual notions of d-invariant p-forms, cochain complex $\Omega^*(G)$ and path cohomologies $H^*(G)$ of G that we do not address here. If G is a (undirected) graph then G can always be considered as a digraph, by turning each edge of G into a double arrow.

There has been a number of attempts to define the notion of (co)homology for graphs. For example, one can consider a graph as an one-dimensional simplicial complex, or take into account all its cliques (=complete subgraphs) as simplexes of the corresponding dimensions. However, such homologies do not have usually necessary functorial properties.

Another approach to homologies of digraphs can be realized via Hochschild homologies, using a natural *path algebra* of a graph. However, it is known that in this case the Hochschild homologies of order ≥ 2 are trivial, which makes this approach useless. In singular homology theories of graphs, certain "small" graphs are predefined as basic cells. However, simple examples show that the singular homology groups do depend essentially on the choice of the basic cells.

Our notion of path homologies of digraphs has the following advantages.

- The path homologies of all dimensions can be non-trivial, even for planar graphs the path homologies can be non-trivial in dimension 2.

- The path homologies can be easily computed using any software package containing operations with matrices.

- The path homology theory is compatible with the homotopy theories of graphs [1] and digraphs [31].
- The path homologies have good functorial properties with respect to graph-theoretical operations, for example, the homologies of the Cartesian product of digraphs (as well as of the join) satisfy the Künneth formula.
- The path homology theory is dual to the cohomology theory of digraphs. The latter was introduced independently by A.Dimakis and F.Müller-Hoissen [13], using a classification of Bourbaki [7] of exterior derivations on algebras.

One of the most essential and technically difficult results of our work is the Künneth formula for products. For two digraphs X and Y, denote by $X\Box Y$ their *Cartesian product*, that is, the product based on the pattern $\Box$.

Theorem 21 *For any two finite digraphs X and Y we have*

$$\Omega_*\left(X\Box Y\right)\cong\Omega_*\left(X\right)\otimes\Omega_*\left(Y\right),$$

that is, for any integer $r\geq 0$,

$$\Omega_r\left(X\Box Y\right)\cong\bigoplus_{\{p,q\geq 0:p+q=r\}}\left(\Omega_p\left(X\right)\otimes\Omega_q\left(Y\right)\right)$$

Consequently, by the abstract theorem of Künneth, the same isomorphism holds for homologies:

$$H_*\left(X\Box Y\right)\cong H_*\left(X\right)\otimes H_*\left(Y\right).$$

The fact that the Künneth formula holds at the level of chain complexes is very surprising. It contrasts the classical algebraic topology, where the Künneth formula holds only in homologies. This result provides an indirect evidence that our notion of the chain complex $\Omega_*\left(G\right)$ for digraphs is very meaningful by itself.

Define now the *join* $X*Y$ of digraphs X,Y as a digraph whose set of vertices is a disjoint union of the sets of vertices of X and Y, and the set of arrows of $X*Y$ consists of all the arrows of X,Y as well as of new arrows from any vertex of X to any vertex of Y. In the next result we use the augmented chain complex $\widetilde{\Omega}_*$.

Theorem 22 *For any two finite digraphs X, Y and for any integer $r\geq -1$, we have*

$$\widetilde{\Omega}_r\left(X*Y\right)\cong\bigoplus_{\{p,q\geq -1:p+q=r-1\}}\left(\widetilde{\Omega}_p\left(X\right)\otimes\widetilde{\Omega}_q\left(Y\right)\right).$$

It follows that, for any $r\geq 0$,

$$\widetilde{H}_r\left(X*Y\right)\cong\bigoplus_{\{p,q\geq 0:p+q=r-1\}}\left(\widetilde{H}_p\left(X\right)\otimes\widetilde{H}_q\left(Y\right)\right).$$

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