Analysis on Manifolds

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Contents

Chapter 1

Riemannian manifolds and Laplace-Beltrami operator

08.04.24 **Lecture 1**

We introduce in this Chapter the notions of smooth and Riemannian manifolds, Riemannian measure, and the Riemannian Laplace operator.

1.1 Topological spaces and manifolds

Topological spaces. Recall that a *topological space* is a couple $(M, \mathcal{O})$ where M is any set and $\mathcal{O}$ is a collection of subsets of M that are called *open* and satisfy the following axioms:

- $\emptyset$ and M are open;
- union of any family of open sets is open;
- intersection of two open sets is open.

A subset F of M is called closed if its complement $F^c := M \setminus F$ is open. A subset K of M is called compact if any open covering $\{\Omega_\alpha\}$ of K contains a finite subcover. It is easy to prove that any closed subset of a compact set is also compact (Exercise 1).

Definition. A topological space M is called *Hausdorff* if, for any two disjoint points $x, y \in M$, there exist two disjoint open sets $U, V \subset M$ containing x and y, respectively. One says in this case that the sets U and V separate the points x, y.

In a Hausdorff space M, any compact subset K of M is closed (see Exercise 2).

Definition. We say that M has a *countable base* if there exists a countable family $\{B_j\}_{j=1}^{\infty}$ of open sets in M such that any other open set is a union of some sets B_j. The family $\{B_j\}$ is called a base of the topology of M.

Let M be a topological space and S be any subset of M. Then S itself is a topological space with the *induced* topology, that is, open sets in S are intersections of open sets in M with S. If M has a countable base, then S also has countable base; if M is Hausdorff, the same is true also for S.

Let X and Y be two topological spaces. A mapping $F : X \to Y$ is called continuous if for any open subset V of Y, the preimage $F^{-1}(Y)$ is an open subset of X. It is known that if F is continuous then, for any compact subset K of X, the image $F(K)$ is a compact subset of Y.

A mapping $F : X \to Y$ is called a homeomorphism if F is bijective, and both F and its inverse mapping are continuous.

Any metric space (M, d) is a topological space with the following standard topology: a subset $\Omega \subset M$ is called open if for any $x \in \Omega$ there is a metric ball

$$B(x, r) := \{y \in M : d(x, y) < r\}$$

with radius $r > 0$ that is a subset of Ω. It is easy to see that all metric balls are open sets. The topology of a metric space is automatically Hausdorff because for any two distinct points $x, y \in M$, the balls $B(x, r/2)$ and $B(y, r/2)$ with $r = d(x, y)$ separate the points x, y.

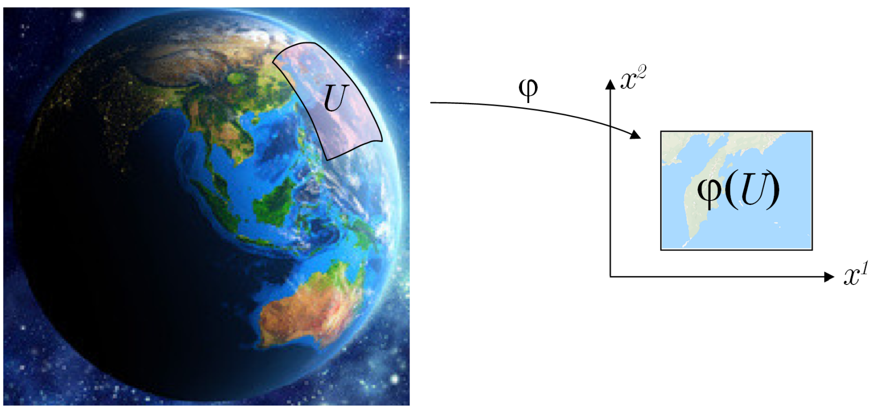
A metric space has a countable base if and only if it is separable, that is, if it contains a countable dense subset D. Indeed, if such a set exists then all balls of rational radii centered at the points of D form a countable base. Conversely, if $\{B_j\}$ is a countable base then choosing one point in each B_j, we obtain a countable dense subset D of M.

For example, $\mathbb{R}^n$ as a metric space with the Euclidean distance is an example of a Hausdorff topological space with a countable base.

C-manifolds. Let us define the notion of a manifold.

Definition. A n-dimensional *chart* on a topological space M is any couple (U, φ) where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$ (which is called the image of the chart).

Any chart (U, φ) on M gives rise to the *local coordinate system* $x^1, x^2, ..., x^n$ in U by taking the φ-pullback of the Cartesian coordinate system in $\mathbb{R}^n$. Hence, we can say that a chart is an open set $U \subset M$ with a local coordinate system. Normally, we will identify U with its image $\varphi(U)$ so that the coordinates $x^1, x^2, ..., x^n$ can be regarded as the Cartesian coordinates in a region in $\mathbb{R}^n$.



A chart on the surface of the earth

Definition. A *C-manifold* of dimension n is a Hausdorff topological space M with a countable base such that any point of M belongs to a n-dimensional chart. The collection of all n-dimensional charts on M is called an *atlas*.

This terminology originates from geography and refers to a geographical atlas of the Earth, where each sheet can be regarded as (the image of) a 2-dimensional chart on the Earth's surface.

For example, $\mathbb{R}^n$ is a C-manifold and $U = \mathbb{R}^n$ is a single n-dimensional chart that covers $\mathbb{R}^n$. Let us consider some subsets of $\mathbb{R}^n$ that are C-manifolds.

Example. Let V be an open subset of $\mathbb{R}^n$ and $f : V \to \mathbb{R}^m$ be a continuous mapping. Then its graph

$$\Gamma = \left\{(x, f(x)) \in \mathbb{R}^{n+m} : x \in V\right\}$$

is a C-manifold because it is covered by a single n-dimensional chart (Γ, φ) where

$$\begin{aligned} \varphi &: \Gamma \to V \\ \varphi\,(x, f(x)) &= x \end{aligned}$$

is a homeomorphism.

Example. A *hypersurface* M in $\mathbb{R}^{n+1}$ is a subset of $\mathbb{R}^{n+1}$ such that, for any point $x \in M$, there exists an open set $\Omega \subset \mathbb{R}^{n+1}$ containing x such that $\Omega \cap M$ is a graph with respect to one of the coordinates $x^1, ..., x^{n+1}$ of a continuous function $f : V \to \mathbb{R}$ defined on an open subset V of $\mathbb{R}^n$. Since $\Omega \cap M$ is a chart and M can be covered by such charts, we conclude that M is a C-manifold.

Example. Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^1-function. Consider the null set of F, that is, the set

$$M = \left\{x \in \mathbb{R}^{n+1} : F(x) = 0\right\},$$

and assume that $\nabla F(x) \neq 0$ for any point $x \in M$. Then M is a hypersurface and, hence, a C-manifold of dimension n (Exercise 7).

For example, the unit sphere

$$\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}$$

is a C-manifold of dimension n because it is the null set of the function $F(x) = \|x\|^2 - 1$, and $\nabla F = 2x \neq 0$ for all $x \in \mathbb{S}^n$.

12.04.24 Lecture 2

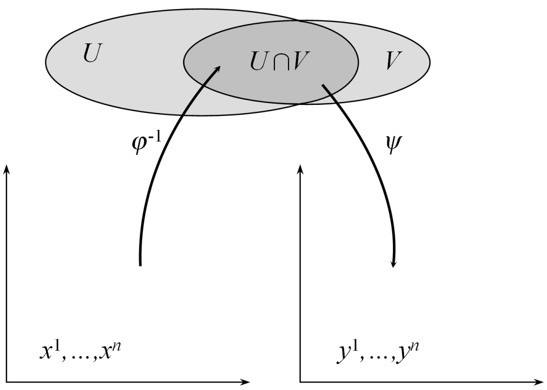
The hypothesis of a countable base will be mostly used via the next simple lemma. Let us first fix some notations. For any set $A \subset M$, define the closure $\overline{A}$ of A as the intersection of all closed sets containing A. In other words, $\overline{A}$ is the smallest closed set containing A. We will use the relation $A \Subset B$ (*compact inclusion*) between two subsets A and B of M, which means the following: the closure $\overline{A}$ of A is a compact set and $\overline{A} \subset B$. A set A whose closure of compact is called *precompact* (or relatively compact).

Lemma 1.1 *For any C-manifold M, there is a countable family $\{U_i\}_{i=1}^{\infty}$ of charts covering all M and such that $U_i \Subset V_i$ for some chart V_i.*

Proof. Any point $x \in M$ is contained in a chart, say V_x. Choose $U_x \Subset V_x$ to be a small open ball around x so that U_x is also a chart. Hence, we obtain a covering of M by charts $\{U_x\}_{x\in M}$ such that each of then is compactly included in another chart. It remains to choose a countable subcover. By definition, manifold M has a countable base. Choose from this base only those elements that are contained in one of the sets U_x; let it be a sequence $\{B_j\}_{j=1}^{\infty}$. Since U_x is open, it is a union of some sets B_j. It follows that $\{B_j\}$ is a covering of M. Select for each B_j exactly *one* chart U_x containing B_j, say U_{x_j}. Thus, we obtain a countable family of charts $\left\{U_{x_j}\right\}$ covering M, which finishes the proof. $\blacksquare$

In particular, we see that a C-manifold M is a *locally compact* topological space.

If (U, φ) and (V, ψ) are two charts on a C-manifold M then in the intersection $U \cap V$ two coordinate systems are defined, say $x^1, ..., x^n$ and $y^1, ..., y^n$. The change of the coordinates from $x^1, ..., x^n$ to $y^1, ..., y^n$ is given then by *continuous* functions $y^i = y^i(x^1, ..., x^n)$ because they are the components of the mapping $\psi \circ \varphi^{-1}$. Similarly, the change from $y^1, ..., y^n$ to $x^1, ..., x^n$ are given by continuous functions $x^i(y^1, ..., y^n)$ that are the components of the mapping $\varphi \circ \psi^{-1}$.



The mapping $\psi \circ \varphi^{-1}$

Smooth manifolds. Now we define the notion of a smooth manifold.

Definition. A family $\mathcal{A}$ of charts on a C-manifold is called a C^k-*atlas* (where k is a positive integer or $+\infty$) if the charts from $\mathcal{A}$ covers all M and the change of coordinates in the intersection of any two charts from $\mathcal{A}$ is given by C^k-functions. Two C^k-atlases are said to be compatible if their union is again a C^k-atlas. A family of all compatible C^k-atlases determines a C^k-*structure* on M.

Definition. A C^k-manifold is a C-manifold endowed with a C^k-structure. A smooth manifold is a C^∞-manifold.

Alternatively, one can say that a C^k-manifold is a couple $(M, \mathcal{A})$, where M is a C-manifold and $\mathcal{A}$ is a C^k-atlas on M. However, if the two C^k-atlases $\mathcal{A}$ and $\mathcal{A}'$ are compatible then $(M, \mathcal{A})$ and $(M, \mathcal{A}')$ determine the same C^k-manifold.

In this course we are going to consider mostly smooth manifolds. By default, the term "manifold" will be used as a synonymous of "smooth manifold". By a chart on a smooth manifold we will always mean a chart from its C^∞-structure, that is, any chart compatible with the defining atlas $\mathcal{A}$.

Here are some examples of smooth manifolds.

1. $\mathbb{R}^n$ with the atlas consisting of a single chart $(\mathbb{R}^n, \mathrm{id})$.
2. Any C^∞-hypersurface that is locally a graph of a C^∞-function, is a smooth manifold.
3. If $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞-function whose null set $M = \{F = 0\}$ is non-degenerate then M is a smooth manifolds. For example, the unit sphere $\mathbb{S}^n$ is a smooth manifold.

If Ω is an open subset of M then Ω naturally inherits all the above structures of M and becomes a smooth manifold if M is so. Indeed, the open sets in Ω are defined as the intersections of open sets in M with Ω, and in the same way one defines charts and atlases in Ω.

If f is a (real valued) function on a smooth manifold M and k is a non-negative integer or ∞ then we write $f \in C^k(M)$ (or $f \in C^k$) if the restriction of f to any chart is a C^k function of the local coordinates $x^1, ..., x^n$. The set $C^k(M)$ is a linear space over $\mathbb{R}$ with respect to the usual addition of functions and multiplication by constant.

1.2 Cutoff functions and partition of unity

For any function $f \in C(M)$, its support is defined by

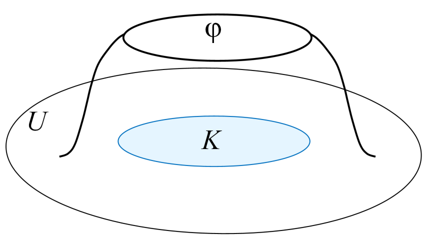
$$\operatorname{supp} f = \overline{\{x \in M : f(x) \neq 0\}},$$

where the bar stands for the closure. It follows from the definition of $\operatorname{supp} f$ that if $f \equiv 0$ outside a closed set $F \subset M$ then $\operatorname{supp} f \subset F$.

Denote by $C_0^k(M)$ the subspace of $C^k(M)$, which consists of functions with compact supports. If Ω is an open subset of M then $C_0^k(\Omega)$ denotes the set of all functions $f \in C_0^k(M)$ such that $\operatorname{supp} f \subset \Omega$.

Definition. Let M be a smooth manifold, U be an open subset of M and K be a compact subset of U. We say that a function function φ on M is a *cutoff function* of K in U if

- $\varphi \in C_0^\infty(U)$
- $\varphi \equiv 1$ in a neighborhood of K
- $0 \leq \varphi \leq 1$ on M.



A cutoff function φ of K in U

Lemma 1.2 *For any open subset U of $\mathbb{R}^n$ and any compact set $K \subset U$, there exists a cutoff function of K in U.*

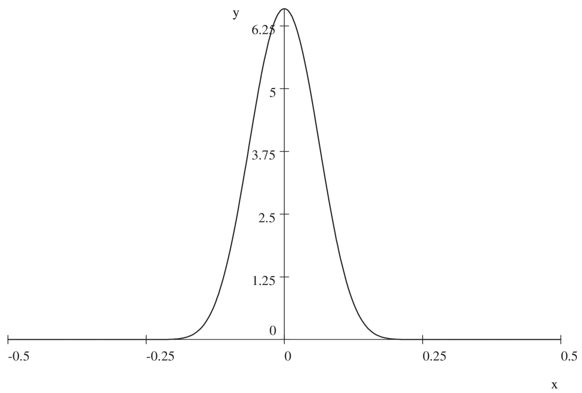
In the proof we use the notion of a mollifier. We say that a function $\psi \in C_0^\infty(\mathbb{R}^n)$ is a *mollifier* if $\operatorname{supp}\psi \subset B_1(0)$, $\psi \geq 0$, and

$$\int_{\mathbb{R}^n} \psi d\mu = 1. \tag{1.1}$$

For example, the following function

$$\psi(x) = \begin{cases} c \exp\left(-\frac{1}{\left(\frac{1}{4}-|x|^2\right)^2}\right), & |x| < 1/2 \\ 0, & |x| \geq 1/2 \end{cases} \tag{1.2}$$

is a mollifier, for a suitable normalizing constant $c > 0$.



The mollifier (1.2) in $\mathbb{R}$.

If ψ is a mollifier then, for any $0 < \varepsilon < 1$, the function

$$\psi_\varepsilon := \varepsilon^{-n}\psi\left(\frac{x}{\varepsilon}\right)$$

is also a mollifier, and $\operatorname{supp}\psi_\varepsilon \subset B_\varepsilon(0)$.

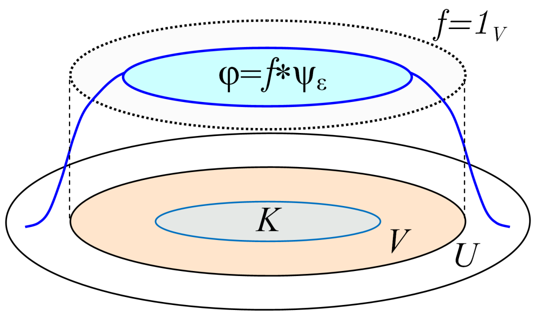
Proof of Lemma 1.2. Let V be an open neighborhood of K such that $V \Subset U$, and set $f = 1_V$. Fix a mollifier ψ, $\varepsilon > 0$ and consider the convolution

$$f * \psi_\varepsilon(x) = \int_{\mathbb{R}^n} f(x-y)\psi_\varepsilon(y)\,dy = \int_{B_\varepsilon(x)} f(z)\psi_\varepsilon(x-z)\,dz.$$

Since $f \in L^1(\mathbb{R}^n)$, we have $f * \psi_\varepsilon \in C^\infty(\mathbb{R}^n)$. Clearly,

$$0 \leq f * \psi_\varepsilon(x) \leq \sup|f| \int_{\mathbb{R}^n} \psi_\varepsilon(y)\,dy = \sup|f| = 1.$$

If ε is small enough then $f * \psi_\varepsilon$ is supported in U so that $f * \psi_\varepsilon \in C_0^\infty(U)$.



Construction of a cutoff function

Besides, for small enough ε and for any $x \in K$, we have $B_\varepsilon(x) \subset V$, whence $f|_{B_\varepsilon(x)} = 1$ and

$$f * \psi_\varepsilon(x) = \int_{B_\varepsilon(x)} f(z)\, \psi_\varepsilon(x - z)\, dz = \int_{B_\varepsilon(x)} \psi_\varepsilon(x - z)\, dz = 1.$$

Hence, the function $\varphi = f * \psi_\varepsilon$ is a cutoff function of K in U, provided ε is small enough. ■

The following statement provides a convenient vehicle for transporting the local properties of $\mathbb{R}^n$ to manifolds.

Proposition 1.3 *Let K be a compact subset of a smooth manifold M and $\{U_j\}_{j=1}^k$ be a finite family of charts covering K. Then there exist non-negative functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_{j=1}^k \varphi_j \equiv 1$ in an open neighbourhood of K and $\sum_{j=1}^k \varphi_j \leq 1$ in M.*

A sequence of functions $\{\varphi_j\}$ as in Proposition 1.3 is called a *partition of unity* at K subordinate to the cover $\{U_j\}$.

A particular case of Proposition 1.3 with $k = 1$ says that, for any compact K and any chart $U \supset K$, there exists a non-negative function $\varphi \in C_0^\infty(U)$ such that $\varphi \equiv 1$ in a neighborhood of K and $\varphi \leq 1$ on M; that is, φ is a cutoff function of K in U.

Corollary 1.4 *Let $\{U_\alpha\}$ be an arbitrary family of charts covering M. Then, for any function $f \in C_0^\infty(M)$, there exists a finite sequence $\{f_j\}_{j=1}^k$ of functions from $C_0^\infty(M)$ such that each f_j is supported in one of the charts U_α and*

$$f = f_1 + \ldots + f_k \quad \text{on } M. \tag{1.3}$$

Proof. Let $K = \operatorname{supp} f$ and let $U_1, \ldots, U_k$ be a finite subfamily of $\{U_\alpha\}$ that covers K. By Proposition 1.3, there exists a partition of unity $\{\varphi_j\}_{j=1}^k$ at K subordinate to $\{U_j\}_{j=1}^k$. Set $f_j = f\varphi_j$ so that $f_j \in C_0^\infty(U_j)$. Then we have

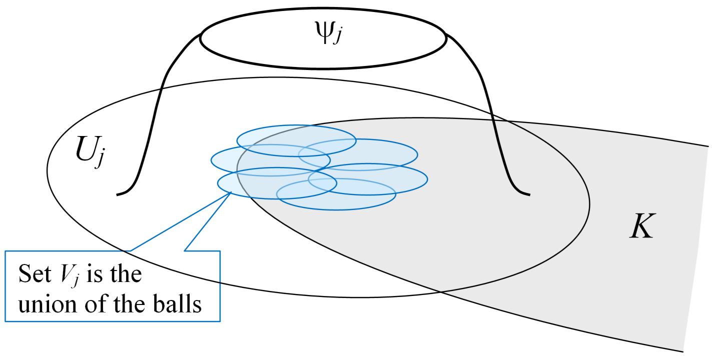
$$\sum_{j=1}^k f_j = f \quad \text{on } M,$$

because on K we have $\sum_j \varphi_j = 1$, while outside K all the functions f and f_j vanish. ■

Proof of Proposition 1.3. We claim that there exists open sets $V_j \Subset U_j$ such that $\{V_j\}_{k=1}^k$ is a covering of K. Indeed, since any point $x \in K$ belongs to a chart U_j, there is a ball B_x in this chart centered at x and such that $B_x \Subset U_j$. The family of balls $\{B_x\}_{x\in K}$ obviously covers K. Since K is compact, there is a finite subfamily $\{B_i\}_{i=1}^m$ covering K. For any $j = 1, ..., k$, consider the set

$$V_j := \bigcup_{\{i:B_i \Subset U_j\}} B_i.$$

By construction, the set V_j is open, $V_j \Subset U_j$, and the union of all sets V_j covers K.



Function ψ_j is a cutoff function of V_j in U_j.

15.04.24 Lecture 3

By Lemma 1.2 there exists a cutoff function $\psi_j \in C_0^\infty(U_j)$ of $\overline{V_j}$ in U_j considering U_j as a subset of $\mathbb{R}^n$. Now we consider U_j as a subset of M and extend ψ_j to M by setting $\psi_j = 0$ in $M \setminus U_i$, so that $\psi_j \in C_0^\infty(M)$.

Define now functions φ_j, $j = 1, ..., k$, by

$$\varphi_j = \psi_j (1 - \psi_1) ... (1 - \psi_{j-1}), \tag{1.4}$$

that is,

$$\varphi_1 = \psi_1, \quad \varphi_2 = \psi_2 (1 - \psi_1), \ ..., \quad \varphi_k = \psi_k (1 - \psi_1) ... (1 - \psi_{k-1}).$$

Obviously, $\varphi_j \in C_0^\infty (U_j)$ and $\varphi_j \geq 0$. It is easy to check by induction in l the following identity

$$1 - \sum_{j=1}^{l} \varphi_j = (1 - \psi_1) ... (1 - \psi_l). \tag{1.5}$$

Indeed, for $l = 1$ it is trivial. If it is true for some l, then

$$\begin{aligned} 1 - \sum_{j=1}^{l+1} \varphi_j &= (1 - \psi_1) ... (1 - \psi_l) - \varphi_{l+1} \\ &= (1 - \psi_1) ... (1 - \psi_l) - \psi_{l+1} (1 - \psi_1) ... (1 - \psi_l) \\ &= (1 - \psi_1) ... (1 - \psi_l) (1 - \psi_{l+1}), \end{aligned}$$

which proves the induction step.

It follows from (1.5) with $l = k$ that

$$\sum_{j=1}^{k} \varphi_j = 1 - (1 - \psi_1) \ldots (1 - \psi_k) \le 1. \tag{1.6}$$

Since $1 - \psi_j = 0$ on V_j, (1.6) implies that $\sum_{j=1}^{k} \varphi_j \equiv 1$ on the union $\bigcup_{j=1}^{k} V_j$ that is an open neighborhood of K, which was to be proved. ∎

1.3 Tangent space and tangent vectors

Let M be a smooth manifold and x_0 be a point on M.

Definition. A mapping $\xi : C^\infty(M) \to \mathbb{R}$ is called an $\mathbb{R}$-*differentiation* at $x_0 \in M$ if

- ξ is linear;
- ξ satisfies the *product rule* in the following form:

$$\xi(fg) = \xi(f)\, g(x_0) + \xi(g)\, f(x_0),$$

for all $f, g \in C^\infty$.

The set of all $\mathbb{R}$-differentiations at x_0 is denoted by $T_{x_0}M$. For any $\xi, \eta \in T_{x_0}M$ one defines the sum $\xi + \eta$ as the sum of two functions on C^∞, and similarly one defined $\lambda\xi$ for any $\lambda \in \mathbb{R}$. It is easy to check that both $\xi + \eta$ and $\lambda\xi$ are again $\mathbb{R}$-differentiations, so that $T_{x_0}M$ is a linear space over $\mathbb{R}$.

Definition. The linear space $T_{x_0}M$ is called the *tangent space* of M at x_0, and its elements (that is, $\mathbb{R}$-differentiations) are also called *tangent vectors* at x_0.

In $\mathbb{R}^n$ we have the following example of $\mathbb{R}$-differentiation:

$$\xi(f) = \frac{\partial f}{\partial x^i}(x_0),$$

that is clearly linear and satisfies the product rule. In particular, $T_{x_0}\mathbb{R}^n$ contains n $\mathbb{R}$-differentiations $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ that are clearly linearly independent.

Moreover, for any vector $v \in \mathbb{R}^n$, the directional derivative $\frac{\partial f}{\partial v}(x_0)$ is also a $\mathbb{R}$-differentiation, which allows us to identify $\mathbb{R}^n$ as a subspace of $T_{x_0}\mathbb{R}^n$. Since

$$\frac{\partial f}{\partial v} = v^i \frac{\partial f}{\partial x^i}$$

(where we assume the convention about *summation over repeated indices*; in this case, summation over i), it follows that

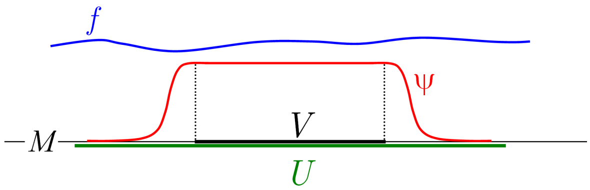
$$\frac{\partial}{\partial v} = v^i \frac{\partial}{\partial x^i}.$$

Theorem 1.5 *If M is a smooth manifold of dimension n then the tangent space $T_{x_0}M$ is a linear space of the same dimension n.*

Consequently, $\dim T_{x_0}\mathbb{R}^n = n$, which implies that every $\mathbb{R}$-differentiation in $\mathbb{R}^n$ has the form $\frac{\partial}{\partial v}$ for some $v \in \mathbb{R}^n$. We will prove Theorem 1.5 after a series of claims.

Claim 1. *Let $U \subset M$ be a chart and $V \Subset U$ be an open subset of U. Then, for any function $f \in C^\infty(U)$, there exists a function $F \in C_0^\infty(M)$ such that $f \equiv F$ on V.*

Proof. Indeed, let ψ be a cutoff function of $\overline{V}$ in U (see Lemma 1.2).



Functions f and ψ in Claim 1

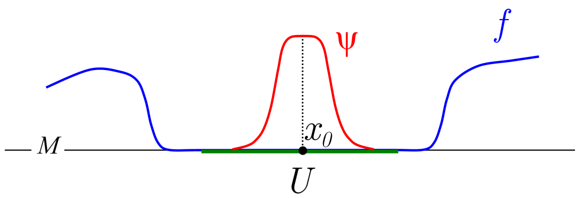
Then define function F by

$$\begin{cases} F = \psi f & \text{in } U, \\ F = 0 & \text{in } M \setminus U, \end{cases}$$

which clearly satisfies all the requirements. ∎

Claim 2. *Let $f \in C^\infty(M)$ and let $f \equiv 0$ in an open neighbourhood U of the point $x_0 \in M$. Then $\xi(f) = 0$ for any $\xi \in T_{x_0}M$. Consequently, if f_1 and f_2 are smooth functions on M such that $f_1 \equiv f_2$ in an open neighbourhood of a point $x_0 \in M$ then $\xi(f_1) = \xi(f_2)$ for any $\xi \in T_{x_0}M$.*

Proof. By reducing U we can assume that U is a chart. Let V be an open neighborhood of x_0 that is compactly included in U. Let ψ be a cutoff function of V in U so that $\psi(x_0) = 1$. Then we have $f\psi \equiv 0$ on M, which implies $\xi(f\psi) = 0$.



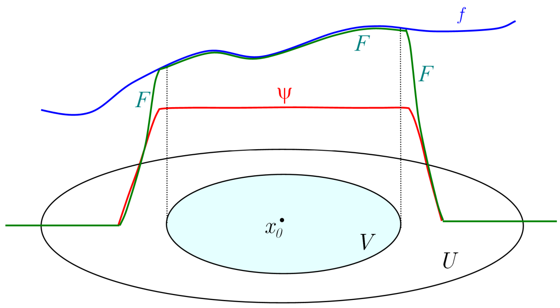
Functions f and ψ in Claim 2

On the other hand, we have by the product rule

$$\xi(f\psi) = \xi(f)\,\psi(x_0) + \xi(\psi)\,f(x_0) = \xi(f),$$

because $\psi(x_0) = 1$ and $f(x_0) = 0$. Hence, $\xi(f) = 0$. The second part follows from the first one applied to the function $f = f_1 - f_2$. ∎

Remark. Originally a tangent vector $\xi \in T_{x_0}M$ is defined as a functional on $C^\infty(M)$. The results of Claims 1 and 2 imply that ξ can be regarded as a functional on $C^\infty(U)$ where U is any open neighbourhood of x_0. Indeed, by Claim 1, for any $f \in C^\infty(U)$ there exists a function $F \in C^\infty(M)$ such that $f = F$ in a small open neighborhood V of x_0.



Functions $f \in C^\infty(U)$ and $F \in C_0^\infty(M)$

Hence, define $\xi(f)$ by $\xi(f) := \xi(F)$. By Claim 2, this definition of $\xi(f)$ does not depend on the choice of F.

Claim 3. *Let f be a smooth function in a ball $B = B_R(o)$ in $\mathbb{R}^n$ where o is the origin of $\mathbb{R}^n$. Then there exist smooth functions $h_1, h_2, ..., h_n$ in B such that, for any $x \in B$,*

$$f(x) = f(o) + x^i h_i(x), \tag{1.7}$$

where we assume summation over the repeated index i. Also, we have

$$h_i(o) = \frac{\partial f}{\partial x^i}(o). \tag{1.8}$$

Proof. By the fundamental theorem of calculus applied to the function $t \mapsto f(tx)$ on the interval $t \in [0,1]$, we have

$$f(x) = f(o) + \int_0^1 \frac{d}{dt} f(tx)\, dt, \tag{1.9}$$

whence it follows

$$f(x) = f(o) + \int_0^1 x^i \frac{\partial f}{\partial x^i}(tx) dt.$$

Setting

$$h_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx) dt$$

we obtain (1.7). Clearly, $h_i \in C^\infty(B)$. The identity (1.8) follows from the line above by substitution $x = o$. ■

Claim 4. *Under the hypothesis of* Claim 3, *there exist smooth functions h_{ij} in B (where $i, j = 1, 2, ..., n$) such that, for any $x \in B$,*

$$f(x) = f(o) + x^i \frac{\partial f}{\partial x^i}(o) + x^i x^j h_{ij}(x). \tag{1.10}$$

Proof. Applying (1.7) to the function h_i instead of f we obtain that there exist smooth functions h_{ij} in B, such that

$$h_i(x) = h_i(o) + x^j h_{ij}(x).$$

Substituting this into the representation (1.7) for f and using $h_i(o) = \frac{\partial f}{\partial x^i}(o)$ we obtain

$$f(x) = f(o) + x^i h_i(x) = f(o) + x^i \frac{\partial f}{\partial x^i}(o) + x^i x^j h_{ij}(x).$$

■

Now we can prove Theorem 1.5.

Proof of Theorem 1.5. Let $x^1, x^2, ..., x^n$ be local coordinates in a chart U containing x_0. All the partial derivatives $\frac{\partial}{\partial x^i}$ evaluated at x_0 are $\mathbb{R}$-differentiations at x_0, and they are clearly linearly independent. We will prove that any tangent vector $\xi \in T_{x_0}M$ can be represented in the form

$$\xi = \xi^i \frac{\partial}{\partial x^i} \quad \text{where} \quad \xi^i = \xi\left(x^i\right). \tag{1.11}$$

Note that, by the above Remark, the $\mathbb{R}$-differentiation ξ applies also to smooth functions defined in a neighborhood of x_0; in particular, $\xi\left(x^i\right)$ is well-defined. The identity (1.11) implies that $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ is a basis in the linear space $T_{x_0}M$ and, hence, $\dim T_{x_0}M = n$.

Without loss of generality, we can assume that x_0 is the origin o of the chart U. For any smooth function f on M, we have by (1.10) the following representation in a ball $B \subset U$ centred at o:

$$f(x) = f(o) + x^i \frac{\partial f}{\partial x^i}(o) + x^i x^j h_{ij}(x),$$

where h_{ij} are some smooth functions in B. Using the linearity of ξ, we obtain

$$\xi(f) = \xi(1) f(o) + \xi\left(x^i\right) \frac{\partial f}{\partial x^i}(o) + \xi\left(x^i x^j h_{ij}\right). \tag{1.12}$$

By the product rule, we have

$$\xi(1) = \xi(1 \cdot 1) = \xi(1)\, 1 + \xi(1)\, 1 = 2\xi(1),$$

whence $\xi(1) = 0$. Set $u_i = x^j h_{ij}$. By the linearity and the product rule, we have

$$\xi\left(x^i u_i\right) = \xi\left(x^i\right) u_i(o) + \xi(u_i)\, x^i(o) = 0,$$

because $x^i(o) = 0$ and $u_i(o) = x^j(o)\, h_{ij}(o) = 0$. Hence, in the right hand side of (1.12), the first and the third term vanish. Setting $\xi^i := \xi\left(x^i\right)$, we obtain

$$\xi(f) = \xi^i \frac{\partial f}{\partial x^i}, \tag{1.13}$$

which is equivalent to (1.11). ■

22.04.24 **Lecture 4**

The numbers ξ^i are referred to as the *components* of the vector ξ in the coordinate system $x^1, ..., x^n$. One often uses the following alternative notation for $\xi(f)$:

$$\xi(f) \equiv \frac{\partial f}{\partial \xi}.$$

Then the identity (1.13) takes the form

$$\boxed{\frac{\partial f}{\partial \xi} = \xi^i \frac{\partial f}{\partial x^i}}, \tag{1.14}$$

which allows to think of ξ as a *direction* at x_0 and to interpret $\frac{\partial f}{\partial \xi}$ as a directional derivative.

A *vector field* on a smooth manifold M is a family $\{\xi(x)\}_{x\in M}$ of tangent vectors such that $\xi(x) \in T_xM$ for any $x \in M$. In the local coordinates $x^1, ..., x^n$, it can be represented in the form

$$\xi(x) = \xi^i(x) \frac{\partial}{\partial x^i}.$$

The vector field $\xi(x)$ is called *smooth* if all the functions $\xi^i(x)$ are smooth in any chart.

1.4 Cotangent space

As any other finite dimensional linear space, T_xM possesses the *dual space* T_x^*M that consists of all linear functionals on T_xM:

$$\omega : T_xM \to \mathbb{R}.$$

Then T_x^*M is also a linear space over $\mathbb{R}$; moreover, it is known from linear algebra that

$$\dim T_x^*M = \dim T_xM = n.$$

Definition. The linear space T_x^*M is referred to as the *cotangent space* of M at x. The elements of T_x^*M are called *tangent covectors*.

For any $\omega \in T_x^*M$ and $\xi \in T_xM$, the value $\omega(\xi)$ will be also denoted by $\langle \omega, \xi \rangle$ and referred to as the *pairing* of ω and ξ. This notation reflects the fact that every vector $\xi \in T_xM$ can be regarded as a linear functional on T_x^*M given by $\xi(\omega) = \langle \omega, \xi \rangle$. Note that all linear functionals on T_x^*M have this form (that is, the second dual space $T_x^{**}M$ is identified with T_xM).

Fix a point $x \in M$ and let f be a smooth function in a neighborhood of x.

Definition. Define the *differential* df at x as a tangent covector as follows:

$$\langle df, \xi \rangle := \xi(f) \text{ for any } \xi \in T_xM, \tag{1.15}$$

where $\xi(f)$ is the value of the $\mathbb{R}$-differentiation ξ at the function f.

Given the local coordinates $x^1, ..., x^n$, we can consider each coordinate x^i as a function in the chart. In particular, dx^i is a tangent covector.

Lemma 1.6 *$\{dx^i\}_{i=1}^n$ is a basis in T_x^*M.*

Proof. Indeed, any basis $\{e_1, ..., e_n\}$ in a linear space has a dual basis $\{e^1, ..., e^n\}$ in the dual space that is defined by

$$\langle e^i, e_j \rangle = \delta_j^i := \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

Since $\left\{\frac{\partial}{\partial x^i}\right\}$ is a basis in T_xM, we obtain that $\{dx^i\}$ is the dual basis in T_x^*M because

$$\langle dx^i, \frac{\partial}{\partial x^j} \rangle = \frac{\partial}{\partial x^j} x^i = \delta_j^i.$$

■

Consequently, any tangent covector $\omega \in T_x^*M$ has an expansion in this basis:

$$\omega = \omega_i dx^i,$$

where the coefficients $\omega_i \in \mathbb{R}$ are referred to as the components of ω. Hence, for any tangent vector $\xi = \xi^i \frac{\partial}{\partial x^i}$, we obtain

$$\langle \omega, \xi \rangle = \langle \omega_i dx^i, \xi^j \frac{\partial}{\partial x^j} \rangle = \omega_i \xi^j \delta_j^i = \omega_i \xi^i.$$

In particular, for $\xi = \frac{\partial}{\partial x^i}$ we obtain

$$\omega_i = \langle \omega, \frac{\partial}{\partial x^i} \rangle.$$

For example, for the covector df we obtain from (1.15) that

$$(df)_i = \langle df, \frac{\partial}{\partial x^i} \rangle = \frac{\partial f}{\partial x^i}$$

and, hence,

$$\boxed{df = \frac{\partial f}{\partial x^i} dx^i.} \tag{1.16}$$

1.5 Riemannian metric

Let M be a smooth n-dimensional manifold.

Definition. A *Riemannian metric* (or a *metric tensor*) on M is a family $\mathbf{g} = \{\mathbf{g}(x)\}_{x \in M}$ such that, for any $x \in M$, $\mathbf{g}(x)$ is a symmetric, positive definite, bilinear form on the tangent space T_xM, *smoothly depending* on $x \in M$.

The metric tensor determines an inner product $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ in any tangent space T_xM by

$$\langle \xi, \eta \rangle_{\mathbf{g}} := \mathbf{g}(x)(\xi, \eta) \quad \text{for all } \xi, \eta \in T_xM$$

so that T_xM becomes a Euclidean (=inner product) space.

In the local coordinates $x^1, ..., x^n$, we have

$$\langle \xi, \eta \rangle_{\mathbf{g}} = \left\langle \xi^i \frac{\partial}{\partial x^i}, \eta^j \frac{\partial}{\partial x^j} \right\rangle_{\mathbf{g}} = g_{ij}(x)\, \xi^i \eta^j$$

where

$$\boxed{g_{ij}(x) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_{\mathbf{g}}}. \tag{1.17}$$

Clearly, $(g_{ij}(x))_{i,j=1}^n$ is a symmetric positive definite $n \times n$ matrix. The functions $g_{ij}(x)$ are called the *components* of the metric tensor $\mathbf{g}$ in the coordinates $x^1, ..., x^n$.

Definition. The condition that $\mathbf{g}(x)$ *smoothly depends* on x means that all the components $g_{ij}(x)$ are C^∞-functions in any chart.

The metric tensor can be represented in the local coordinates as follows:

$$\boxed{\mathbf{g} = g_{ij} dx^i dx^j}, \tag{1.18}$$

where $dx^i dx^j$ stands for the *tensor product* of the covectors dx^i and dx^j (sometimes also denoted by $dx^i \otimes dx^j$), that is a bilinear functional on $T_x M$ defined by

$$dx^i dx^j (\xi, \eta) = \langle dx^i, \xi \rangle \langle dx^j, \eta \rangle \quad \text{for all } \xi, \eta \in T_x M.$$

Indeed, since

$$\langle dx^i, \xi \rangle = \xi\left(x^i\right) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^i,$$

it follows that

$$g_{ij} dx^i dx^j (\xi, \eta) = g_{ij} \xi^i \eta^j = \langle \xi, \eta \rangle_{\mathbf{g}},$$

which proves (1.18).

Definition. A *Riemannian manifold* is a couple $(M, \mathbf{g})$ where M is a smooth manifold and $\mathbf{g}$ is a Riemannian metric on M.

A trivial example of a Riemannian manifold is $\mathbb{R}^n$ with the *canonical Euclidean metric* $\mathbf{g}_{\mathbb{R}^n}$ defined in the Cartesian coordinates $x^1, ..., x^n$ by

$$\mathbf{g}_{\mathbb{R}^n} = \left(dx^1\right)^2 + ... + (dx^n)^2.$$

For this metric, we have $(g_{ij}) = \mathrm{id}$.

Let $(M, \mathbf{g})$ be a Riemannian manifold. The metric tensor $\mathbf{g}$ can be regarded as a linear mapping

$$\mathbf{g}(x) : T_x M \to T_x^* M \tag{1.19}$$

as follows. For any vector $\xi \in T_x M$, define $\mathbf{g}(x)\, \xi \in T_x^* M$ by the identity

$$\langle \mathbf{g}(x)\, \xi, \eta \rangle = \langle \xi, \eta \rangle_{\mathbf{g}} \text{ for all } \eta \in T_x M, \tag{1.20}$$

Rewriting (1.20) in the local coordinates, we obtain

$$(\mathbf{g}(x)\, \xi)_j\, \eta^j = g_{ij} \xi^i \eta^j.$$

which implies

$$\boxed{\left(\mathbf{g}\left(x\right)\xi\right)_j = g_{ij}\xi^i}. \tag{1.21}$$

In particular, the components of the linear operator $\mathbf{g}\left(x\right)$ are g_{ij} – the same as the components of the metric tensor.

If the Riemannian metric $\mathbf{g}$ is fixed then it is customary to drop $\mathbf{g}$ from all the notations. For example, the notation of the inner product of two tangent vectors ξ, η becomes $\langle \xi, \eta \rangle$. Moreover, the notation for the covector $\mathbf{g}\left(x\right)\xi$ becomes just ξ; that is, the same as for the vector. However, the notation ξ^i is still used to denote the components of the vector ξ in the basis $\left\{\frac{\partial}{\partial x^i}\right\}$, while ξ_j will be used to denote the components of the covector ξ in the basis $\left\{dx^j\right\}$. The relation between the vector components ξ^i and the covector components ξ_j is given then by

$$\xi_j := \left(\mathbf{g}\left(x\right)\xi\right)_j = g_{ij}\xi^i.$$

The operation of passing from ξ^i to ξ_j is called *lowering the index.*

26.04.24 Lecture 5

Lemma 1.7 *The linear operator $\mathbf{g}(x) : T_xM \to T_x^*M$ is invertible. The inverse mapping*

$$\mathbf{g}^{-1}\left(x\right) : T_x^*M \to T_xM$$

has in the local coordinates the following form for any $u \in T_x^*M$:

$$\boxed{\left(\mathbf{g}^{-1}\left(x\right)u\right)^i = g^{ij}u_j}, \tag{1.22}$$

where the matrix $\left(g^{ij}\right)$ is the inverse of $\left(g_{ij}\right)$, that is,

$$\left(g^{ij}\right) = \left(g_{ij}\right)^{-1}.$$

Proof. The operator $\mathbf{g}(x)$ is injective: indeed, if $\xi \neq 0$ then also $\mathbf{g}\left(x\right)\xi \neq 0$ because

$$\langle \mathbf{g}\left(x\right)\xi, \xi \rangle = \left\{\xi, \xi\right\}_{\mathbf{g}} > 0.$$

Since the spaces T_xM and T_x^*M have the same dimensions, it follows that $\mathbf{g}(x)$ is bijective and, hence, invertible.

Fix $u \in T_x^*M$ and set $\xi = \mathbf{g}^{-1}(x)u$ so that $u = \mathbf{g}(x)\xi$. By (1.21) we have

$$u_j = g_{kj}\xi^k.$$

Using the fact that $\left(g^{ij}\right)$ is the inverse matrix of $\left(g_{ij}\right)$, we obtain

$$g^{ij}u_j = g^{ij}g_{kj}\xi^k = g^{ij}g_{jk}\xi^k = \delta_k^i\xi^k = \xi^i,$$

which is equivalent to (1.22). ∎

Denoting the vector $\mathbf{g}^{-1}(x)\,u$ also by u, we obtain the following relation between the vector and covector components of u:

$$u^i := \left(\mathbf{g}^{-1}(x)\,u\right)^i = g^{ij}u_j.$$

The operation of passing from u_j to u^i is called *raising the index*. Clearly, this is the inverse operation to lowering the index.

Definition. The operator $\mathbf{g}^{-1}(x)$ determines an inner product in T_x^*M as follows: for all $u, v \in T_x^*M$, set

$$\langle u, v\rangle_{\mathbf{g}^{-1}} := \langle \mathbf{g}^{-1}(x)\,u, \mathbf{g}^{-1}(x)\,v\rangle_{\mathbf{g}}. \tag{1.23}$$

In the local coordinates we have

$$\langle u, v\rangle_{\mathbf{g}^{-1}} = g^{ij}u_iv_j$$

because by (1.23), (1.20) and (1.22)

$$\langle u, v\rangle_{\mathbf{g}^{-1}} = \langle \mathbf{g}^{-1}(x)\,u, \mathbf{g}^{-1}(x)\,v\rangle_{\mathbf{g}} = \langle u, \mathbf{g}^{-1}(x)\,v\rangle = u_i\left(\mathbf{g}^{-1}(x)\,v\right)^i = g^{ij}u_iv_j.$$

By elimination $\mathbf{g}$ from all the notations, we see that the expression $\langle u, v\rangle$ has the same value in the following four possible cases:

- u and v are covectors, and $\langle u, v\rangle$ is their inner product in T_x^*M;
- u and v are vectors, and $\langle u, v\rangle$ is their inner product in T_xM;
- u is a covector, v is a vector, and $\langle u, v\rangle$ is their pairing;
- u is a vector, v is a covector, and $\langle u, v\rangle$ is their pairing.

Definition. For any $f \in C^\infty(M)$ define its *gradient* $\nabla f(x)$ at any point $x \in M$ by

$$\nabla f(x) = \mathbf{g}^{-1}(x)\,df(x), \tag{1.24}$$

that is, $\nabla f(x)$ is a vector that is obtained from the covector $df(x)$ by raising the index.

Applying (1.20) with $\xi = \nabla f(x)$, we obtain, for any $\eta \in T_xM$,

$$\boxed{\langle \nabla f, \eta\rangle_{\mathbf{g}} = \langle df, \eta\rangle = \frac{\partial f}{\partial \eta}}, \tag{1.25}$$

which can be considered as an alternative definition of the gradient. In the local coordinates $x^1, ..., x^n$, we obtain by (1.22) and (1.24)

$$\boxed{(\nabla f)^i = g^{ij}\frac{\partial f}{\partial x^j}}. \tag{1.26}$$

If h is another smooth function on M then we obtain from (1.23)

$$\langle \nabla f, \nabla h\rangle_{\mathbf{g}} = \langle df, dh\rangle_{\mathbf{g}^{-1}} = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial h}{\partial x^j}. \tag{1.27}$$

1.6 Submanifolds

The notion of a submanifold. If M is a smooth manifold then any open subset $\Omega \subset M$ trivially becomes a smooth manifold by restricting all charts to Ω. Also, if $\mathbf{g}$ is a Riemannian metric on M then $\mathbf{g}|_\Omega$ is a Riemannian metric on Ω. Hence, any open subset Ω of M can be considered as a (Riemannian) submanifold of a (Riemannian) manifold M of the same dimension.

Consider a more interesting notion of a submanifold of smaller dimension. Any subset S of a smooth manifold M can be regarded as a topological space with induced topology. It is easy to see that S inherits from M the properties of being Hausdorff and having a countable base.

Definition. A set $S \subset M$ is called a (embedded) *submanifold* of dimension m if, for any point $x_0 \in S$, there is a chart $U \ni x_0$ in M with local coordinates $x^1, ..., x^n$ such that $S \cap U$ is given in the local coordinates by the equations

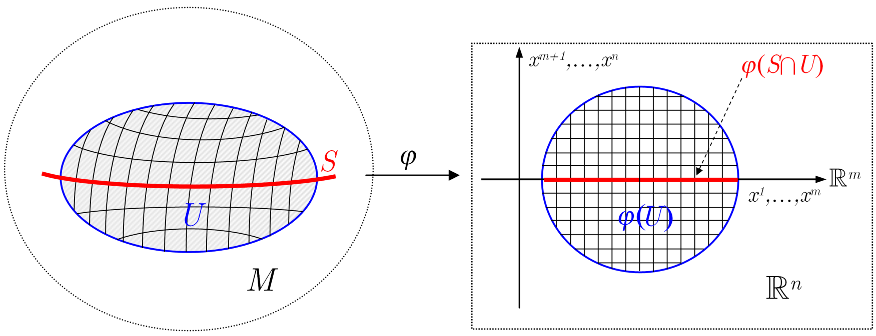
$$x^{m+1} = x^{m+2} = ... = x^n = 0. \tag{1.28}$$

The condition (1.28) implies that $S \cap U$ is a chart on S with coordinates $x^1, ..., x^m$ and, consequently, *S is a smooth manifold of dimension m.*

More precisely this can be justified as follows. Let the coordinates in U be given by a homeomorphism φ of U onto an open subset of $\mathbb{R}^n$. Then the condition that S in U is given by the equations (1.28) means that

$$\varphi(S \cap U) = \left\{x \in \varphi(U) : x^{m+1} = ... = x^n = 0\right\} = \varphi(U) \cap \mathbb{R}^m,$$

where we identify $\mathbb{R}^m$ with a subspace of $\mathbb{R}^n$ as follows: $\mathbb{R}^m = \{x \in \mathbb{R}^n : x^{m+1} = ... = x^n = 0\}$.



Hence, $\varphi|_{S\cap U}$ can be considered as a mapping from $S \cap U$ to $\mathbb{R}^m$, and this mapping is an homeomorphism of $S \cap U$ onto the open set $\varphi(U) \cap \mathbb{R}^m$. Hence, $(S \cap U, \varphi|_{S\cap U})$ is a m-dimensional chart on S, with the local coordinates $x^1, x^2, ..., x^m$. With the atlas consisting of all such charts, the submanifold S becomes a smooth m-dimensional manifold.

Lemma 1.8 *Let M be a smooth manifold of dimension n and $F : M \to \mathbb{R}$ be a smooth function on M. Consider the null set of F, that is*

$$S = \{x \in M : F(x) = 0\}.$$

If

$$dF \neq 0 \text{ on } S \tag{1.29}$$

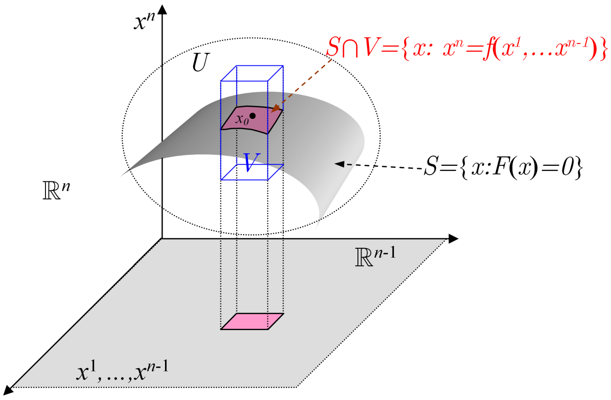
then S is a submanifold of dimension $n-1$.

Proof. Fix a point $x_0 \in S$ and a chart U containing x_0. The condition $dF(x_0) \neq 0$ means that one of the partial derivatives $\frac{\partial F}{\partial x^i}$ does not vanish at x_0. Without loss of generality we can assume that $\frac{\partial F}{\partial x^n}(x_0) \neq 0$.

By the implicit function theorem, there exists an open subset $V \subset U$, containing x_0, such that, for $x \in V$, equation $F(x) = 0$ can be resolved with respect to the coordinate x^n; that is, for all $x \in V$,

$$F(x) = 0 \ \Leftrightarrow \ x^n = f\left(x^1, ..., x^{n-1}\right),$$

where f is a smooth function on an open domain in $\mathbb{R}^{n-1}$.



After the change of coordinates in V

$$y^1 = x^1, \ y^2 = x^2, \ ..., \ y^{n-1} = x^{n-1},$$
$$y^n = x^n - f(x^1, ..., x^{n-1})$$

the equation of S in V becomes $y^n = 0$ and, hence, S is a $(n-1)$-dimensional submanifold. ■

Tangent space on a submanifold. Let S be a submanifold of M of dimension m and ξ be an $\mathbb{R}$-differentiation on S at a point $x_0 \in S$. For any smooth function f on M, its restriction $f|_S$ is a smooth function on S. Hence, by setting

$$\xi(f) := \xi(f|_S), \tag{1.30}$$

we extend ξ to an $\mathbb{R}$-differentiation on M at the same point x_0. In other words, (1.30) defines a linear mapping

$$T_{x_0}S \to T_{x_0}M. \tag{1.31}$$

Lemma 1.9 *The mapping (1.31) is injective and, hence, provides a natural identification of $T_{x_0}S$ as a subspace of $T_{x_0}M$.*

Proof. If $\xi \in T_{x_0}S$ is non-zero then there exists a smooth function $h \in C^\infty(S)$ such that $\xi(h) \neq 0$. In the coordinate system $x^1, ..., x^n$ that is used in the definition of a submanifold, the function h depends on $x^1, ..., x^m$. Setting

$$f\left(x^1, ..., x^m, ..., x^n\right) = h\left(x^1, ..., x^m\right),$$

we obtain a smooth function f in a neighborhood of x_0 in M, such that $f|_S = h$. Therefore, for the extension of ξ to $T_{x_0}M$ we have

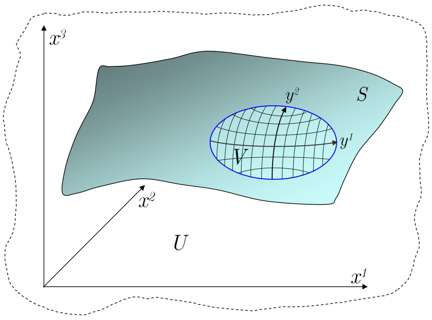
$$\xi(f) = \xi(f|_S) = \xi(h) \neq 0,$$

that is, ξ is non-zero as element of $T_{x_0}M$. Hence, the mapping (1.31) is injective. ∎

Let us describe the mapping (1.31) in local coordinates. Let $x^1, ..., x^n$ be local coordinates in a chart U in M and $y^1, ..., y^m$ be local coordinates in a chart V on S. In the intersection $U \cap V$ we have the relations

$$x^i = x^i\left(y^1, ..., y^m\right), \quad i = 1, ..., n, \tag{1.32}$$

that express the x-coordinates of any point of $U \cap V$ via its y-coordinates.



Local coordinates $x^1, ..., x^n$ and $y^1, ..., y^m$

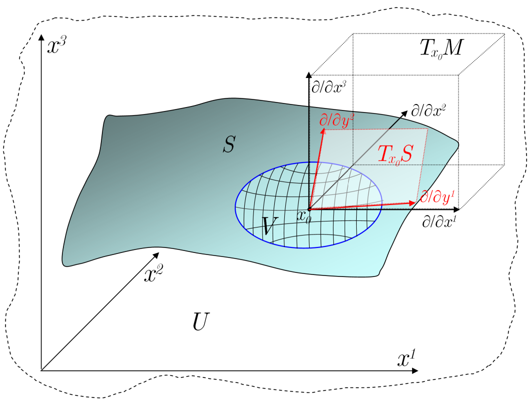
Let x_0 be a point in $U \cap V$. Any smooth function $f = f(x^1, ..., x^n)$ in a neighborhood of x_0 in M can be regarded also as a smooth function of $y^1, ..., y^m$ using (1.32). By the chain rule, we obtain

$$\frac{\partial f}{\partial y^k} = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial y^k} = \frac{\partial x^i}{\partial y^k}\frac{\partial f}{\partial x^i},$$

which can be rewritten in the operator form as follows:

$$\boxed{\frac{\partial}{\partial y^k} = \frac{\partial x^i}{\partial y^k}\frac{\partial}{\partial x^i}}. \tag{1.33}$$

Note that $\left\{\frac{\partial}{\partial x^i}\right\}$ is a basis in $T_{x_0}M$ and $\left\{\frac{\partial}{\partial y^k}\right\}$ is a basis in $T_{x_0}S$, so that (1.33) identifies explicitly $T_{x_0}S$ as a subspace of $T_{x_0}M$.



Tangent space $T_{x_0}S$ as a subspace of $T_{x_0}M$

Cotangent space on a submanifold. Any tangent covector $\omega \in T^*_{x_0}M$ as a linear functional on $T_{x_0}M$ can be restricted to the subspace $T_{x_0}S$ thus yielding an element of $T^*_{x_0}S$ that will also be denoted by ω. Hence, we obtain a surjective linear mapping $T^*_{x_0}M \to T^*_{x_0}S$. Assuming that $x^1, ..., x^n$ and $y^1, ..., y^m$ are the local coordinate systems as above, let us compute $dx^i|_{T_{x_0}S}$ in the basis dy^j. Since by (1.33)

$$\langle dx^i, \frac{\partial}{\partial y^j}\rangle = \langle dx^i, \frac{\partial x^l}{\partial y^j}\frac{\partial}{\partial x^l}\rangle = \frac{\partial x^l}{\partial y^j}\delta^i_l = \frac{\partial x^i}{\partial y^j},$$

it follows that the restriction of dx^i to $T_{x_0}S$ is given by

$$\boxed{dx^i = \frac{\partial x^i}{\partial y^j}dy^j}. \tag{1.34}$$

Alternatively, (1.34) follows from (1.16) considering x^i as a function in the chart $y^1, ..., y^m$.

29.04.24 Lecture 6

Riemannian metric on a submanifold. Let $\mathbf{g}$ be a Riemannian metric on M. For any $x \in S$, we can restrict $\mathbf{g}(x)$ to a bilinear functional on T_xS thus obtaining a Riemannian metric $\mathbf{g}_S$ on S. The metric $\mathbf{g}_S$ is called the *induced metric* of S.

Lemma 1.10 *In the local coordinates $x^1, ..., x^n$ on M and $y^1, ..., y^m$ on S we have the identity*

$$\boxed{(g_S)_{ij} = g_{kl}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}}, \tag{1.35}$$

where g_{kl} are the components of $\mathbf{g}$ in the chart $x^1, ..., x^n$ and $(g_S)_{ij}$ are the components of $\mathbf{g}_S$ in the chart $y^1, ..., y^m$. In the matrix form, we have

$$\boxed{g^y_S = J^T g^x J} \tag{1.36}$$

where $g^x = (g_{kl})$, $g^y_S = ((g_S)_{ij})$ and J is the Jacobi matrix of the change $x = x(y)$, that is,

$$J = (J_{ki}) = \left(\frac{\partial x^k}{\partial y^i}\right). \tag{1.37}$$

Note that, in the matrix J in (1.37), $k = 1, ..., n$ is the row index and $i = 1, ..., m$ is the column index, so that J is an $n \times m$ matrix. Hence, the right hand side of (1.36) is the product of the three matrices of the following dimensions: $m \times n$, $n \times n$, $n \times m$, which results in a matrix $m \times m$.

Proof. Restricting $\mathbf{g} = g_{kl}dx^k dx^l$ to $T_{x_0}S$, we obtain by (1.34)

$$\mathbf{g}_S = g_{kl}dx^k dx^l = \left(\frac{\partial x^k}{\partial y^i}dy^i\right)\left(\frac{\partial x^l}{\partial y^j}dy^j\right) = g_{kl}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}dy^i dy^j.$$

Comparing with

$$\mathbf{g}_S = (g_S)_{ij}\, dy^i dy^j$$

we obtain (1.35). Next, we have by (1.35) and (1.37)

$$(g_S)_{ij} = J_{ki} g_{kl} J_{lj} = J_{ik}^T g_{kl} J_{lj} = \left(J^T g^x J\right)_{ij}$$

whence (1.36) follows. ■

In a particular case $m = n$, S is an open subset of M and the induced metric $\mathbf{g}_S$ coincides with the original metric $\mathbf{g}$, so that (1.36) provides the relation between the matrices g^x and g^y of $\mathbf{g}$ in two coordinate systems $x^1, ..., x^n$ and $y^1, ..., y^n$, respectively (cf. Exercise 14).

Example. Consider in $\mathbb{R}^{n+1}$ the following equation

$$\left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 = 1,$$

which defines the unit sphere $\mathbb{S}^n$. Since $\mathbb{S}^n$ is the null set of the function

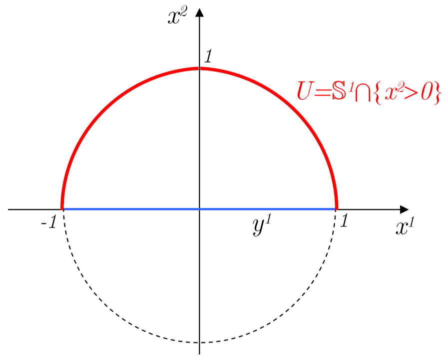
$$F(x) = \left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 - 1,$$

whose differential $dF = (2x^1, ..., 2x^{n+1})$ does not vanish on $\mathbb{S}^n$, we conclude that $\mathbb{S}^n$ is a submanifold of $\mathbb{R}^{n+1}$ of dimension n. Furthermore, considering $\mathbb{R}^{n+1}$ as a Riemannian manifold with the canonical Euclidean metric $\mathbf{g}_{\mathbb{R}^{n+1}}$, we see that $\mathbb{S}^n$ can be regarded as Riemannian manifold with the induced metric that is called the *canonical spherical metric* and is denoted by $\mathbf{g}_{\mathbb{S}^n}$.

Let us compute $\mathbf{g}_{\mathbb{S}^1}$ using the following chart on $\mathbb{S}^1$ (see also Exercise 17). The upper semi-circle

$$U := \mathbb{S}^1 \cap \left\{x^2 > 0\right\}$$

is the graph of a function $f(x^1) = \sqrt{1 - (x^1)^2}$ on the interval $(-1, 1)$ and, hence, is a chart on $\mathbb{S}^1$ with the local coordinate $y^1 = x^1$.



The upper semi-circle

Clearly, the relations between the coordinates x^1, x^2 in $\mathbb{R}^2$ and y^1 in $\mathbb{S}^1$ are

$$x^1 = y^1 \quad \text{and} \quad x^2 = \sqrt{1 - (y^1)^2}.$$

It follows that

$$dx^1 = dy^1 \quad \text{and} \quad dx^2 = \frac{-y^1}{\sqrt{1 - (y^1)^2}} dy^1.$$

Since

$$\mathbf{g}_{\mathbb{R}^2} = \left(dx^1\right)^2 + \left(dx^2\right)^2$$

we obtain that

$$\mathbf{g}_{\mathbb{S}^1} = \left(dy^1\right)^2 + \frac{\left(y^1\right)^2}{1-\left(y^1\right)^2}\left(dy^1\right)^2 = \frac{\left(dy^1\right)^2}{1-\left(y^1\right)^2}.$$

Alternatively, the same follows from (1.35) as $\mathbf{g}_{\mathbb{S}^1}$ has only one component:

$$\left(g_{\mathbb{S}^1}\right)_{11} = \left(g_{\mathbb{R}^2}\right)_{kl}\frac{\partial x^k}{\partial y^1}\frac{\partial x^l}{\partial y^1} = \left(\frac{\partial x^1}{\partial y^1}\right)^2 + \left(\frac{\partial x^2}{\partial y^1}\right)^2 = 1 + \frac{\left(y^1\right)^2}{1-\left(y^1\right)^2} = \frac{1}{1-\left(y^1\right)^2}.$$

1.7 Riemannian measure

Let us recall the definition of the notion of measure. Let X be an arbitrary set. A σ-algebra $\mathcal{A}$ on X is a family of subsets of X such that $\mathcal{A}$ contains $\emptyset$, X and $\mathcal{A}$ is closed under taking complement and countable unions (hence, also intersections). A measure μ on a σ-algebra $\mathcal{A}$ is a mapping $\mu : \mathcal{A} \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and μ is σ-additive, that is,

$$\mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty}\mu(A_i)$$

for all $A_i \in \mathcal{A}$. Given a measure μ, one can define the notion of the integral $\int_X f d\mu$ for a class of measurable functions.

The most famous example of a measure is the Lebesgue measure λ defined on the σ-algebra $\mathcal{L}(\mathbb{R}^n)$ of Lebesgue measurable subsets of $\mathbb{R}^n$. Recall that the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ is defined as the minimal σ-algebra containing all open subsets of $\mathbb{R}^n$, and the elements of $\mathcal{B}(\mathbb{R}^n)$ are called Borel sets. It is known that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{L}(\mathbb{R}^n)$ and that any Lebesgue measurable set is a union of a Borel set and a null set (=a set of measure zero).

Let M be a smooth manifold of dimension n. Denote by $\mathcal{B}(M)$ the smallest σ-algebra containing all open sets in M. The elements of $\mathcal{B}(M)$ are called *Borel sets.* We say that a set $E \subset M$ is *measurable* if, for any chart U, the intersection $E \cap U$ is Lebesgue measurable in U. Obviously, the family of all measurable sets in M forms a σ-algebra, that will be denoted by $\mathcal{L}(M)$. Since any open subset of M is measurable, it follows that also all Borel sets are measurable, that is, $\mathcal{B}(M) \subset \mathcal{L}(M)$.

The purpose of this section is to show that any Riemannian manifold $(M, \mathbf{g})$ features a canonical measure ν that is defined on $\mathcal{L}(M)$ and that is called the *Riemannian measure (or volume).* This measure is defined by means of the following theorem.

For any chart U on M with the local coordinates $x^1, ..., x^n$, consider the matrix $g^x = (g_{ij})$ where g_{ij} are the components of the metric $\mathbf{g}$ in coordinates $x^1, ..., x^n$.

Theorem 1.11 *For any Riemannian manifold* $(M, \mathbf{g})$, *there exists a unique measure* ν *on* $\mathcal{L}(M)$ *such that, in any chart* U *on* M *with coordinates* $x^1, ..., x^n$,

$$d\nu = \sqrt{\det g^x}\, dx, \tag{1.38}$$

where dx denotes the Lebesgue measure in U.

Furthermore, the measure ν has the following properties: $\nu(K) < \infty$ for any compact set $K \subset M$ and $\nu(\Omega) > 0$ for any non-empty open set $\Omega \subset M$.

Note that $\det g^x > 0$ by the positive definiteness of g^x. The condition (1.38) means that, for any measurable set $A \subset U$,

$$\nu(A) = \int_A \sqrt{\det g^x}\, dx, \tag{1.39}$$

where A in the right hand side is regarded as a subset of $\mathbb{R}^n$. This identity implies that, for any non-negative measurable function f on U,

$$\boxed{\int_U f d\nu = \int_U f\sqrt{\det g^x} dx}.$$

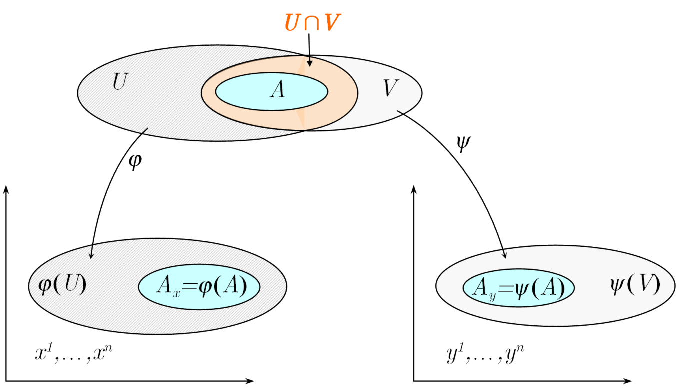
Proof. We need to construct measure ν with the domain $\mathcal{L}(M)$ that satisfies (1.39) in any chart U. Let us use (1.39) as definition of ν on the σ-algebra $\mathcal{L}(U)$ of Lebesgue measurable sets in U. We need to show that the measure ν defined by (1.39) in each chart, can be extended to $\mathcal{L}(M)$ and, moreover, this extension is unique.

Step 1. Let us first prove that the measures that are defined by (1.39) in different charts, are compatible. That is, if U and V are two charts on M and A is a measurable set in $W := U \cap V$ then the integral in (1.39) has the same values in the both charts.

Let $x^1, ..., x^n$ and $y^1, ..., y^n$ be the local coordinate systems in U and V, respectively. Denote by g^x and g^y the matrices of $\mathbf{g}$ in the coordinates $x^1, ..., x^n$ and $y^1, ..., y^n$, respectively. We need to show that, for any measurable set $A \subset W$,

$$\int_{A_x} \sqrt{\det g^x} dx = \int_{A_y} \sqrt{\det g^y} dy,$$

where dx and dy stand for the Lebesgue measures in U and V, respectively, and the notations A_x and A_y mean that A is considered as a subset of U with coordinates $x^1, ..., x^n$, and that of V with coordinates $y^1, ..., y^n$, respectively.



A set A in the intersection of two charts (U, φ) and (V, ψ).

Let J be the Jacobi matrix of the change $x = x(y)$, that is, $J = \left(\frac{\partial x^k}{\partial y^i}\right)$ (cf. (1.37)). By (1.36) we have

$$g^y = J^T g^x J,$$

which implies

$$\det g^y = \det J^T \det g^x \det J = \det g^x (\det J)^2. \tag{1.40}$$

Next, let us use the following formula for change of variables in the Lebesgue integral in $\mathbb{R}^n$: if f is a non-negative measurable function in W then

$$\int_{W_x} f(x)\,dx = \int_{W_y} f(x(y))\,|\det J|\,dy. \tag{1.41}$$

Applying this for $f = 1_A\sqrt{\det g^x}$ and using (1.40), we obtain

$$\int_{A_x} \sqrt{\det g^x} dx = \int_{A_y} \sqrt{\det g^x}\,|\det J|\,dy = \int_{A_y} \sqrt{\det g^x (\det J)^2} dy = \int_{A_y} \sqrt{\det g^y} dy,$$

which proves the claim.

Step 2. Let us prove that the measure ν on $\mathcal{L}(M)$ that satisfies (1.38) in all charts, is unique.

By Lemma 1.1, there is a countable family $\{U_i\}_{i=1}^{\infty}$ of relatively compact charts covering M and such that each $\overline{U}_i$ is contained in a chart. Consider the sets

$$\begin{aligned} V_1 &= U_1, \\ V_2 &= U_2 \setminus U_1 = U_2 \cap U_1^c \\ V_3 &= U_3 \setminus U_2 \setminus U_1 = U_3 \cap U_2^c \cap U_1^c \\ &\dots \\ V_i &= U_i \cap U_{i-1}^c \cap \dots \cap U_1^c \\ &\dots \end{aligned}$$

Clearly we have

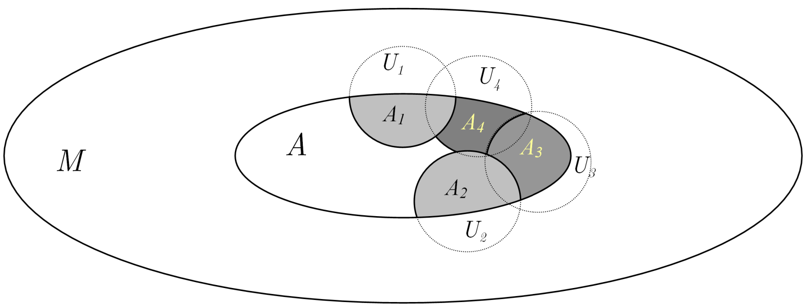
$$M = \bigsqcup_i V_i$$

because for any point $x \in M$ there is a (unique) minimal i such that $x \in U_i$ and, hence, $x \in U_i \cap U_{i-1}^c \dots \cap U_1^c$.

For any measurable set A on M, define the sets

$$A_i = A \cap V_i \tag{1.42}$$

Then we have $A_i \in \mathcal{L}(U_i)$ and $A = \bigsqcup_i A_i$.



Splitting A into disjoint sets A_i.

Therefore, for any measure ν, we should have

$$\nu(A) = \sum_i \nu(A_i). \tag{1.43}$$

However, the value $\nu(A_i)$ is uniquely determined by (1.38) because A_i is contained in the chart U_i. Hence, $\nu(A)$ is also uniquely defined, which was to be proved.

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Step 3. Let us prove the existence of ν. For that fix some covering $\{U_i\}$ as above, and, for any measurable set A, define $\nu(A)$ by (1.39), using the fact that $\nu(A_i)$ is already defined. Let us show that ν is a measure, that is, ν is σ-additive. Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of disjoint measurable sets in M such that

$$A = \bigsqcup_k B_k.$$

We need to prove that

$$\nu(A) = \sum_k \nu(B_k). \tag{1.44}$$

Defining the sets B_{ki} similarly to (1.42), that is,

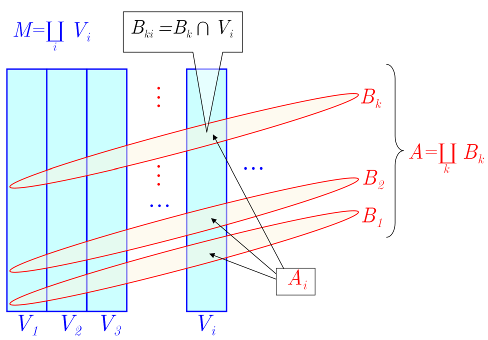
$$B_{ki} = B_k \cap V_i$$

we obtain that

$$B_k = \bigsqcup_i B_{ki}$$

as well as

$$A_i = A \cap V_i = \bigsqcup_k (B_k \cap V_i) = \bigsqcup_k B_{ki}.$$



Sets A_i and B_{ki}

Since ν is σ-additive in each chart U_i, we obtain

$$\nu(A_i) = \sum_k \nu(B_{ki})$$

Adding up in i and interchanging the summation in i and k, we obtain

$$\nu(A) \overset{\text{def}}{=} \sum_i \nu(A_i) = \sum_i \sum_k \nu(B_{ki}) = \sum_k \sum_i \nu(B_{ki}) \overset{\text{def}}{=} \sum_k \nu(B_k),$$

which proves (1.44).

Step 4. Any compact set $K \subset M$ can covered by a finite number of charts U_i. Applying (1.39) in a chart containing $\overline{U}_i$ and noticing $\sqrt{\det g}$ is bounded on $\overline{U}_i$, we obtain $\nu(U_i) < \infty$, which implies $\nu(K) < \infty$.

Any non-empty open set $\Omega \subset M$ contains some non-empty chart U, whence it follows from (1.39) that

$$\nu(\Omega) \geq \nu(U) = \int_U \sqrt{\det g}\, d\lambda > 0.$$

■

* **Remark** The extension of measure ν from the charts to the whole manifold can also be done using the Carathéodory extension of measures. Consider the following family of subsets of M:

$$S = \left\{A \subset M : A \text{ is a relatively compact measurable set and } \overline{A} \text{ is contained in a chart}\right\}.$$

Observe that S is a semi-ring and, by the above Claim, ν is defined as a measure on S. Hence, the Carathéodory extension of ν exists and is a complete measure on M. It is not difficult to check that the domain of this measure is exactly $\mathcal{L}(M)$. Since the union of sets U_i from Lemma 1.1 is M and $\nu(U_i) < \infty$, the measure ν on S is σ-finite and, hence, its extension to $\mathcal{L}(M)$ is unique.

Since the Riemannian measure ν is finite on compact sets, any continuous function with compact support is integrable against ν. Let us record the following simple property of measure ν, which will be used in the next section.

Lemma 1.12 *If $f \in C(M)$ and*

$$\int_M f\varphi\, d\nu = 0 \tag{1.45}$$

for all $\varphi \in C_0^\infty(M)$ then $f \equiv 0$.

Proof. See Exercise 21. ■

1.8 Divergence theorem

Recall that the divergence of a smooth vector field $v(x)$ in $\mathbb{R}^n$ (or in a domain in $\mathbb{R}^n$) is a function defined by

$$\operatorname{div} v(x) = \sum_{i=1}^n \frac{\partial v^i}{\partial x^i}.$$

Divergence satisfies the following identity any smooth vector field v in $\mathbb{R}^n$ and a smooth scalar function u with compact support in $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} (\operatorname{div} v)\, u\, dx = -\int_{\mathbb{R}^n} v \cdot \nabla u\, dx,$$

which can be deduced from the divergence theorem of Gauss. Alternatively, this identity is a consequence of Fubini's theorem and the integration by part formula: for all $w \in C^\infty(\mathbb{R}^n)$ and $u \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{\partial w}{\partial x^i} u dx = -\int_{\mathbb{R}^n} w \frac{\partial u}{\partial x^i} dx \tag{1.46}$$

applied with $w = v^i$.

For any smooth vector field $v(x)$ on a Riemannian manifold $(M, \mathbf{g})$, its *divergence* $\operatorname{div} v(x)$ is a smooth function on M, defined by means of the following statement.

Theorem 1.13 (The divergence theorem) *For any smooth vector field $v(x)$ on a Riemannian manifold $(M, \mathbf{g})$, there exists a unique smooth function on M, denoted by $\operatorname{div} v$, such that the following identity holds*

$$\int_M (\operatorname{div} v)\, u\, d\nu = -\int_M \langle v, \nabla u \rangle d\nu, \tag{1.47}$$

for all $u \in C_0^\infty(M)$.

Both gradient ∇ and divergence div depend on the metric $\mathbf{g}$. In the cases when this dependence should be emphasized, we will use the extended notations $\nabla_{\mathbf{g}}$ and $\operatorname{div}_{\mathbf{g}}$.

The expression $\langle v, \nabla u \rangle = \langle v, \nabla u \rangle_{\mathbf{g}}$ is the inner product of the tangent vectors v and ∇u. By (1.25), we have

$$\langle v, \nabla u \rangle_{\mathbf{g}} = \langle \nabla u, v \rangle_{\mathbf{g}} = \langle du, v \rangle = \frac{\partial u}{\partial x^i} v^i,$$

where $\langle du, v \rangle$ is the pairing of the tangent covector du and vector v.

Proof. The uniqueness of $\operatorname{div} v$ is simple: if there are two candidates for $\operatorname{div} v$, say $(\operatorname{div} v)'$ and $(\operatorname{div} v)''$ then, for all $u \in C_0^\infty(M)$,

$$\int_M (\operatorname{div} v)' \, u\, d\nu = \int_M (\operatorname{div} v)'' \, u\, d\nu,$$

which implies $(\operatorname{div} v)' = (\operatorname{div} v)''$ by Lemma 1.12.

To prove the existence of $\operatorname{div} v$, let us first show that $\operatorname{div} v$ exists in any chart. Namely, if U is a chart on M with the coordinates $x^1, ..., x^n$ then, using (1.25), (1.38), and the integration-by-parts formula in U as a subset of $\mathbb{R}^n$, we obtain, for any $u \in C_0^\infty(U)$,

$$\begin{aligned}
\int_U \langle v, \nabla u \rangle d\nu &= \int_U \langle du, v \rangle d\nu \\
&= \int_U \frac{\partial u}{\partial x^i} v^i \sqrt{\det g}\, d\lambda \\
&= -\int_U \frac{\partial}{\partial x^i} \left(v^i \sqrt{\det g} \right) u\, d\lambda \\
&= -\int_U \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(v^i \sqrt{\det g} \right) u\, d\nu,
\end{aligned} \tag{1.48}$$

where $g = (g_{ij})$ is the matrix of the metric $\mathbf{g}$ in U. Comparing with (1.47) we see that the divergence in U can be defined by

$$\boxed{\operatorname{div} v = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} v^i \right)}. \tag{1.49}$$

If U and V are two charts then (1.49) defines the divergences in U and in V, which agree in $U \cap V$ by the uniqueness statement. Hence, (1.49) defines $\operatorname{div} v$ as a function on the entire manifold M. Moreover, the divergence defined in this way satisfies the identity (1.47) for all test functions u compactly supported in one of the charts.

We are left to extend the identity (1.47) to all functions $u \in C_0^\infty(M)$. Let $\{\Omega_\alpha\}$ be any family of charts covering M. By Corollary 1.4, any function $u \in C_0^\infty(M)$ can be represented as a finite sum $u_1 + ... + u_m$, where each u_j is smooth and compactly supported in one of Ω_α. Hence, (1.47) holds for each of the functions u_j. By adding up all such identities, we obtain (1.47) for the function u. ■

It follows from (1.49) that

$$\operatorname{div} v = \frac{\partial v^i}{\partial x^i} + v^i \frac{\partial}{\partial x^i} \ln \sqrt{\det g}.$$

In particular, if $\det g \equiv 1$ then we obtain the same formula as in $\mathbb{R}^n$: $\operatorname{div} v = \frac{\partial v^i}{\partial x^i}$.

Corollary 1.14 *The identity (1.47) holds also if $u(x)$ is any smooth function on M and $v(x)$ is a compactly supported smooth vector field on M.*

Proof. Let $K = \operatorname{supp} v$. By Theorem 1.3, there exists a cutoff function of K, that is, a function $\varphi \in C_0^\infty(M)$ such that $\varphi \equiv 1$ in a neighbourhood of K. Then $u\varphi \in C_0^\infty(M)$, and we obtain by Theorem 1.13

$$\int_M \operatorname{div} v \, u \, d\nu = \int_M \operatorname{div} v \, (u\varphi) \, d\nu = -\int_M \langle v, \nabla (u\varphi) \rangle d\nu = -\int_M \langle v, \nabla u \rangle d\nu.$$

■

* **Alternative definition of divergence.** Let us define the divergence $\operatorname{div} v$ in any chart by

$$\operatorname{div} v = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} v^i \right), \tag{1.50}$$

and show by a direct computation that, in the intersection of any two charts, (1.50) defines the same function. This approach allows to avoid integration in the definition of divergence but it is more technically involved (besides, we need integration and Theorem 1.13 anyway).

We will use the following formula: if $a = (a_j^i)$ is a non-singular $n \times n$ matrix smoothly depending on a real parameter t and (b_j^i) is its inverse (where i is the row index and j is the column index) then

$$\frac{\partial}{\partial t} \ln \det a = b_k^l \frac{\partial a_l^k}{\partial t}. \tag{1.51}$$

In the common domain of two coordinate systems $x^1, ..., x^n$ and $y^1, ..., y^n$, set

$$J_i^k = \frac{\partial y^k}{\partial x^i} \text{ and } I_k^i = \frac{\partial x^i}{\partial y^k},$$

so that the matrices I and J are mutually inverse. Let g be the matrix of the tensor $\mathbf{g}$ and v^i be the components of the vector v in coordinates $x^1, ..., x^n$, and let $\widetilde{g}$ be the matrix of $\mathbf{g}$ and $\widetilde{v}^k$ be the components of the vector v in coordinates $y^1, ..., y^n$. Then we have

$$v = v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k} = v^i J_i^k \frac{\partial}{\partial y^k}$$

so that

$$\widetilde{v}^k = v^i J_i^k.$$

Since by (1.40)

$$\sqrt{\det \widetilde{g}} = \sqrt{\det g}\, |\det J|^{-1},$$

the divergence of v in the coordinates $y^1, ..., y^n$ is given by

$$\begin{aligned}
\operatorname{div} v &= \frac{1}{\sqrt{\det \widetilde{g}}} \frac{\partial}{\partial y^k} \left(\sqrt{\det \widetilde{g}} \widetilde{v}^k \right) = \frac{\det J}{\sqrt{\det g}} I_k^j \frac{\partial}{\partial x^j} \left(\sqrt{\det g} v^i (\det J)^{-1} J_i^k \right) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} \left(\sqrt{\det g} v^i \right) I_k^j J_i^k + v^i I_k^j J_i^k \det J \frac{\partial}{\partial x^j} (\det J)^{-1} + v^i I_k^j \frac{\partial J_i^k}{\partial x^j} \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} v^i \right) - v^i \frac{\partial}{\partial x^i} \ln \det J + v^i I_k^j \frac{\partial J_i^k}{\partial x^j},
\end{aligned}$$

where we have used the fact that the matrices J and I are mutually inverse and, hence, $I_k^j J_i^k = \delta_i^j$. To finish the proof, it suffices to show that, for any index i,

$$-\frac{\partial}{\partial x^i} \ln \det J + I_k^j \frac{\partial J_i^k}{\partial x^j} = 0. \tag{1.52}$$

By (1.51), we have

$$\frac{\partial}{\partial x^i} \ln \det J = I_k^j \frac{\partial J_j^k}{\partial x^i}.$$

Noticing that

$$\frac{\partial J_j^k}{\partial x^i} = \frac{\partial^2 y^k}{\partial x^j \partial x^i} = \frac{\partial^2 y^k}{\partial x^i \partial x^j} = \frac{\partial J_i^k}{\partial x^j},$$

we obtain (1.52).

1.9 Laplace-Beltrami operator

Recall that the Laplace operator in $\mathbb{R}^n$ is given by

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2}. \tag{1.53}$$

It is also easy to see that

$$\Delta f = \sum_{i=1}^n \frac{\partial}{\partial x^i} \left(\frac{\partial f}{\partial x^i} \right) = \operatorname{div} (\nabla f).$$

Having defined gradient and divergence, we can now define the *Laplace-Beltrami operator* (frequently referred to simply as the Laplace operator) on any Riemannian manifold $(M, \mathbf{g})$ as follows:

$$\boxed{\Delta = \operatorname{div} \circ \nabla}.$$

Strictly speaking, one should use the notations $\Delta_{\mathbf{g}}$, $\operatorname{div}_{\mathbf{g}}$ and $\nabla_{\mathbf{g}}$ but the index $\mathbf{g}$ is usually skipped when there is no danger of confusion.

Hence, for any smooth function f on M, we have

$$\Delta f = \operatorname{div}(\nabla f), \tag{1.54}$$

so that Δf is also a smooth function on M. In local coordinates, we have

$$(\nabla f)^i = g^{ij}\frac{\partial f}{\partial x^j},$$

where $(g^{ij}) = (g_{ij})^{-1}$, which yields

$$\boxed{\Delta f = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial f}{\partial x^j}\right)}, \tag{1.55}$$

For example, if $(g_{ij}) \equiv \mathrm{id}$ then also $(g^{ij}) \equiv \mathrm{id}$, and (1.55) takes the form (1.53). Hence, the classical Laplace operator in $\mathbb{R}^n$ is a particular case of the Laplace-Beltrami operator. Since the matrix (g^{ij}) is symmetric and positive definite, the operator Δ in (1.55) is an elliptic second order operator in the divergence form.

Proposition 1.15 (The Green formula) *If u and v are smooth functions on a Riemannian manifold M and one of them has a compact support then*

$$\int_M u\Delta v\,d\nu = -\int_M \langle\nabla u,\nabla v\rangle d\nu = \int_M v\Delta u\,d\nu. \tag{1.56}$$

Proof. Consider the vector field ∇v. Clearly, $\operatorname{supp}\nabla v \subset \operatorname{supp} v$ so that either $\operatorname{supp} u$ or $\operatorname{supp}\nabla v$ is compact. By Theorem 1.13, Corollary 1.14, and (1.54), we obtain

$$\int_M u\Delta v\,d\nu = \int_M u\operatorname{div}(\nabla v)\,d\nu = -\int_M \langle\nabla u,\nabla v\rangle d\nu.$$

The second identity in (1.56) is proved similarly. ■