Blatt 2. Abgabe bis 03.05.2024

12. Let K be a compact subset of a smooth manifold M and $\{U_j\}_{j=1}^k$ be a finite family of open sets covering K. Prove that there exist non-negative functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_{j=1}^k \varphi_j \equiv 1$ in an open neighbourhood of K and $\sum_{j=1}^k \varphi_j \leq 1$ in M.

    Remark. The family $\{\varphi_j\}$ is called a partition of unity at K subordinate to $\{U_j\}$. If all U_j are charts then the existence of the partition of unity was proved in lectures.

    Hint. Choose first a finite family $\{W_i\}$ of charts covering K and such that each W_i is contained in one of the sets U_j. By a theorem from lectures, there exists a partition of unity $\{\psi_i\}$ of K subordinate to $\{W_i\}$. Use functions ψ_i to construct functions φ_j.

13. Let M be a Riemannian manifold.

    (a) Prove the product rule for the operators d and ∇ on M:
    $$d(uv) = udv + vdu \tag{2}$$
    and
    $$\nabla(uv) = u\nabla v + v\nabla u, \tag{3}$$
    where u and v are smooth function on M.

    (b) Prove the chain rule for the operators d and ∇ on M:
    $$df(u) = f'(u)\,du$$
    and
    $$\nabla f(u) = f'(u)\,\nabla u,$$
    where u and f are smooth functions on M and $\mathbb{R}$, respectively.

14. Let $(M, \mathbf{g})$ be a Riemannian manifold. Let U and V be charts on M with the local coordinates $x^1, ..., x^n$ and $y^1, ..., y^n$, respectively. Denote by g^x and g^y the matrices of $\mathbf{g}$ in U and V, respectively. Let $J = \left(J_i^k\right)_{k,i=1}^n$ be the Jacobian matrix of the change $y = y(x)$ defined in $U \cap V$ by
    $$J_i^k = \frac{\partial y^k}{\partial x^i}, \tag{4}$$
    where k is the row index and i is the column index. Prove the following identity in $U \cap V$:
    $$g^x = J^T g^y J, \tag{5}$$
    where J^T denotes the transposed matrix.

15. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metrics on a smooth manifold M and let g^x and $\widetilde{g}^x$ be the matrices of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively, in some local coordinate system $x^1, ..., x^n$. Prove that the ratio
    $$\frac{\det \widetilde{g}^x}{\det g^x}$$
    does not depend on the choice of the coordinate system (although separately $\det g^x$ and $\det \widetilde{g}^x$ do depend on the coordinate system).