Blatt 4. Abgabe bis 17.05.2024

21. Let $(M, \mathbf{g})$ be a Riemannian manifold and ν be the Riemannian measure on M. Assume that a function $f \in C(M)$ satisfies the following identity

$$\int_M f\varphi d\nu = 0 \tag{9}$$

for all $\varphi \in C_0^\infty(M)$. Prove that $f \equiv 0$ on M.

22. *(Product rule for divergence)* Let $(M, \mathbf{g})$ be a Riemannian manifold, and let $\nabla, \operatorname{div}$ be the gradient and divergence associated with $\mathbf{g}$, respectively. Prove that, for any smooth function u and any smooth vector field ω on M,

$$\operatorname{div}(u\omega) = \langle \nabla u, \omega \rangle_{\mathbf{g}} + u \operatorname{div} \omega. \tag{10}$$

Hint. Use the divergence theorem and the product rule of gradient of Exercise 13a

23. The Laplace-Beltrami operator Δ on a Riemannian manifold $(M, \mathbf{g})$ is defined for any function $u \in C^\infty(M)$ by

$$\Delta u = \operatorname{div}(\nabla u).$$

(a) *(Product rule for the Laplacian)* Prove that, for any two smooth functions u and v on M,

$$\Delta(uv) = u\Delta v + 2\langle \nabla u, \nabla v \rangle_{\mathbf{g}} + (\Delta u) v. \tag{11}$$

(b) *(Chain rule for the Laplacian)* Prove that

$$\Delta f(u) = f''(u) |\nabla u|_{\mathbf{g}}^2 + f'(u) \Delta u,$$

where u and f are smooth functions on M and $\mathbb{R}$, respectively.

24. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metric tensors on a smooth n-dimensional manifold M. Assume that, for some constant C,

$$\widetilde{\mathbf{g}} \leq C\mathbf{g}, \tag{12}$$

that is, for all $x \in M$ and $\xi \in T_xM$,

$$\widetilde{\mathbf{g}}(x)(\xi, \xi) \leq C\mathbf{g}(x)(\xi, \xi). \tag{13}$$

(a) Prove that if ν and $\widetilde{\nu}$ are the Riemannian measures of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively, then

$$\frac{d\widetilde{\nu}}{d\nu} \leq C^{n/2}.$$

(b) Prove that, for any smooth function f on M,

$$|\nabla f|_{\mathbf{g}}^2 \leq C |\nabla f|_{\widetilde{\mathbf{g}}}^2.$$

Hint. Fix $x_0 \in M$ and consider $T_{x_0}M$ as a Euclidean space with the inner product $\mathbf{g}$. Since $\widetilde{\mathbf{g}}$ is a symmetric bilinear form in this space, there exists a $\mathbf{g}$-orthonormal basis $\{e_1, ..., e_n\}$ in $T_{x_0}M$ in which $\widetilde{\mathbf{g}}$ has a diagonal form, that is, $(\widetilde{g_{ij}}) = \operatorname{diag}\{\alpha_1, ..., \alpha_n\}$ with some reals α_i. By a linear change of coordinates in a neighborhood of x_0, you can assume that $\frac{\partial}{\partial x^i} = e_i$. For (a) note also that, by Exercise 15, the ratio $\frac{\det \widetilde{g}(x_0)}{\det g(x_0)}$ does not depend on the choice of local coordinates.

25. * Fix n reals $a_1, ..., a_n$ and consider the matrix

$$B = \begin{pmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & \dots & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \dots & a_2a_n \\ a_3a_1 & a_3a_2 & 1+a_3^2 & \dots & a_3a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \dots & 1+a_n^2 \end{pmatrix}$$

that is, $B = (b_{ij})$ where $b_{ii} = 1 + a_i^2$ and $b_{ij} = a_ia_j$ for $i \neq j$. The purpose of this question is to prove the identity

$$\det B = 1 + a_1^2 + ... + a_n^2. \tag{14}$$

(a) Consider an auxiliary $(n+1) \times (n+1)$ matrix

$$A = \begin{pmatrix} 1 & -a_1 & -a_2 & \dots & \dots & -a_n \\ a_1 & 1 & & & & \\ a_2 & & 1 & & \mathbf{0} & \\ \vdots & & & \ddots & & \\ \vdots & & \mathbf{0} & & \ddots & \\ a_n & & & & & 1 \end{pmatrix},$$

where all the entries of the matrix outside the first column, the first row and the main diagonal are zeros. Prove that

$$\det A = 1 + a_1^2 + ... + a_n^2.$$

(b) Prove the identity (14)

Hint. Prove first that the matrix AA^T has the block diagonal form

$$AA^T = \begin{pmatrix} c & \mathbf{0} \\ \mathbf{0} & \boxed{B} \end{pmatrix},$$

where B is the above matrix and $c = 1 + a_1^2 + ... + a_n^2$.

Remark. The identity (14) will be used later on in order to compute measure on certain manifolds.