Analysis on fractal spaces and heat kernels

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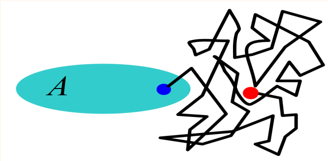
Classical heat kernel

The *heat kernel* in $\mathbb{R}^n$ is the fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$:

$$p_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

This function is also called the Gauss-Weierstrass function. Some applications:

- Solving the Cauchy problem: $u(t, \cdot) = p_t * f$.
- Mollification of functions: $p_t * f \to f$ as $t \to 0$ locally uniformly provided $f \in C_b(\mathbb{R})$.
- Proof of Sobolev embedding theorems.
- $p_t(x)$ is the transition density of Brownian motion in $\mathbb{R}^n$.



- Approximation of the Dirichlet integral: for any $f \in W^{1,2}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla f|^2 dx = \lim_{t \to 0} \frac{1}{2t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_t(x-y) |f(x) - f(y)|^2 dx dy.$$

Analysis on metric spaces: integration

Since the time of Newton and Leibniz, differentiation and integration have been major concepts of mathematics. Nowadays, integration amounts to construction of a measure.

Let (M,d) be a metric space and μ be a Borel measure on M. Assume in what follows that M is *α-regular*, that is, for any metric ball $B(x,r) := \{y \in M : d(x,y) < r\}$ of radius $r < r_0$,

$$\mu(B(x,r)) \simeq r^\alpha, \tag{1}$$

where $\alpha > 0$.

It follows from (1) that

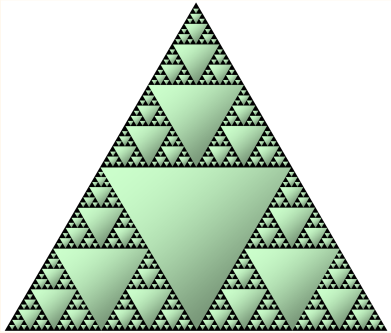
$$\dim_H M = \alpha \quad \text{and} \quad \mathcal{H}_\alpha \simeq \mu.$$

The number α is called also the *fractal dimension* of M. In some sense, α is a numerical characteristic of the integral calculus on M.

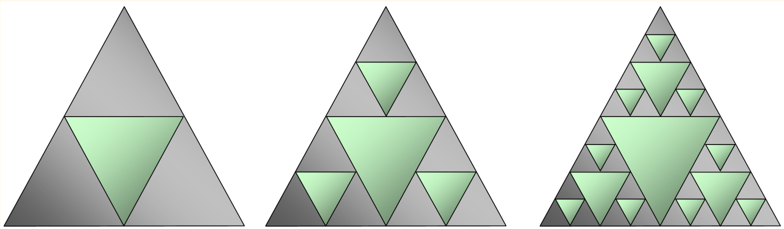
Spaces with fractional α are called *fractals*. They appeared in mathematics as curious examples that initially served as counterexamples to illustrate various theorems.

The most famous fractal – the *Cantor set*, was introduced by Georg Cantor in 1883.

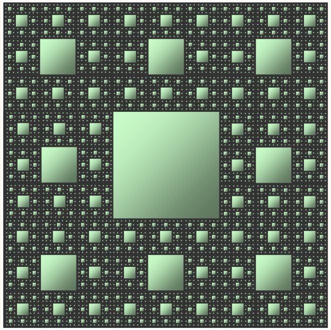
Important examples of fractal sets are the *Sierpinski gasket* (SG) and *Sierpinski carpet* (SC) that were introduced by Wacław Sierpiński in 1915.



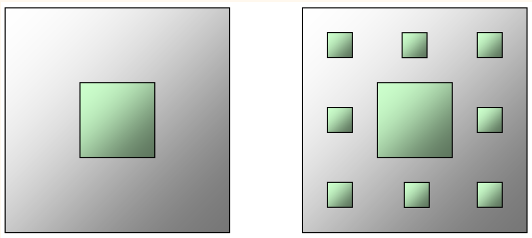
Sierpinski gasket, $\alpha = \frac{\log 3}{\log 2} \approx 1.59$



Three steps of construction of SG

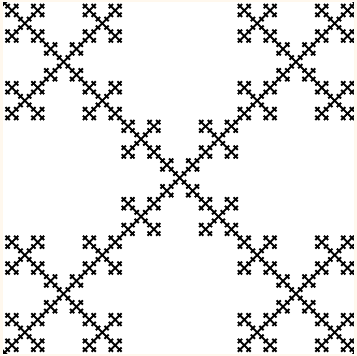


Sierpinski carpet, $\alpha = \frac{\log 8}{\log 3} \approx 1.90$

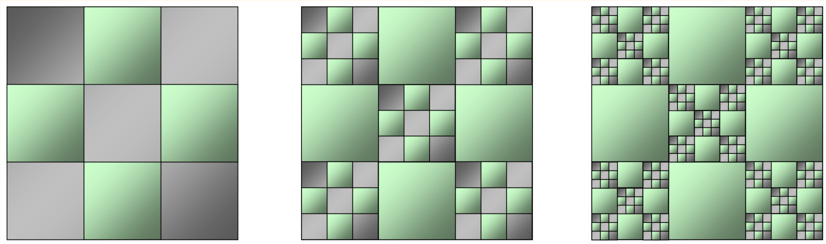


Two steps of construction of SC

Another example of a fractal: the Vicsek snowflake (VS)



Vicsek snowflake, $\alpha = \frac{\log 5}{\log 3} \approx 1.47$



Three steps of construction of VS

Analysis on metric spaces: differentiation

On certain metric spaces, including fractal spaces, it is possible to construct a *Laplace-type* operator, by means of the theory of Dirichlet forms by Beurling–Deny and Fukushima.

A *Dirichlet form* in $L^2(M, \mu)$ is a pair $(\mathcal{E}, \mathcal{F})$ where $\mathcal{F}$ is dense subspace of $L^2(M, \mu)$ and $\mathcal{E}$ is a bilinear form on $\mathcal{F}$ with the following properties:

1. It is positive definite, that is, $\mathcal{E}(f, f) \geq 0$ for all $f \in \mathcal{F}$.

2. It is closed, that is, $\mathcal{F}$ is complete with respect to the norm
$$\int_M f^2 d\mu + \mathcal{E}(f, f).$$

3. It is Markovian, that is, if $f \in \mathcal{F}$ then $\widetilde{f} := \min(f_+, 1) \in \mathcal{F}$ and $\mathcal{E}(\widetilde{f}, \widetilde{f}) \leq \mathcal{E}(f, f)$.

Any Dirichlet form has the generator: a positive definite self-adjoint operator $\mathcal{L}$ in $L^2(M, \mu)$ with domain $\operatorname{dom}(\mathcal{L}) \subset \mathcal{F}$ such that

$$(\mathcal{L}f, g) = \mathcal{E}(f, g) \quad \text{for all } f \in \operatorname{dom}(\mathcal{L}) \text{ and } g \in \mathcal{F}.$$

For example, the bilinear form

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, dx \tag{2}$$

in $\mathcal{F} = W_2^1(\mathbb{R}^n)$ is a Dirichlet form, whose quadratic part is the Dirichlet integral. Its generator is $\mathcal{L} = -\Delta$ with $\operatorname{dom}(\mathcal{L}) = W_2^2(\mathbb{R}^n)$.

Another example of a Dirichlet form in $\mathbb{R}^n$:

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+s}} dxdy, \tag{3}$$

where $s \in (0,2)$ and $\mathcal{F} = B_{2,2}^{s/2}(\mathbb{R}^n)$. It has the generator $\mathcal{L} = (-\Delta)^{s/2}$.

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(f,g) = 0$ whenever $f = \text{const}$ in a neighborhood of $\operatorname{supp} g$. It is called *regular* if $C_0(M) \cap \mathcal{F}$ is dense both in $\mathcal{F}$ and $C_0(M)$.

For example, both Dirichlet forms (2) and (3) are regular, the form (2) is strongly local, while the form (3) is nonlocal.

The generator of any regular Dirichlet form determines the *heat semigroup* $\{e^{-t\mathcal{L}}\}_{t\geq 0}$, as well as a Markov processes $\{X_t\}_{t\geq 0}$ on M with the transition semigroup $e^{-t\mathcal{L}}$, that is,

$$\mathbb{E}_x f(X_t) = e^{-t\mathcal{L}} f(x) \quad \text{for all } f \in C_0(M).$$

If $(\mathcal{E},\mathcal{F})$ is local then $\{X_t\}$ is a diffusion while otherwise the process $\{X_t\}$ contains jumps.

For example, the Dirichlet form (2) determines Brownian motion in $\mathbb{R}^n$, whose transition density is exactly the Gauss-Weierstrass function

$$p_t(x)=\frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right).$$

The Dirichlet form (3) determines a jump process: a symmetric stable Levy process of the index s. In the case $s=1$ its transition density is the Cauchy distribution

$$p_t(x)=\frac{c_n t}{\left(t^2+|x|^2\right)^{\frac{n+1}{2}}}=\frac{c_n}{t^n}\left(1+\frac{|x|^2}{t^2}\right)^{-\frac{n+1}{2}},$$

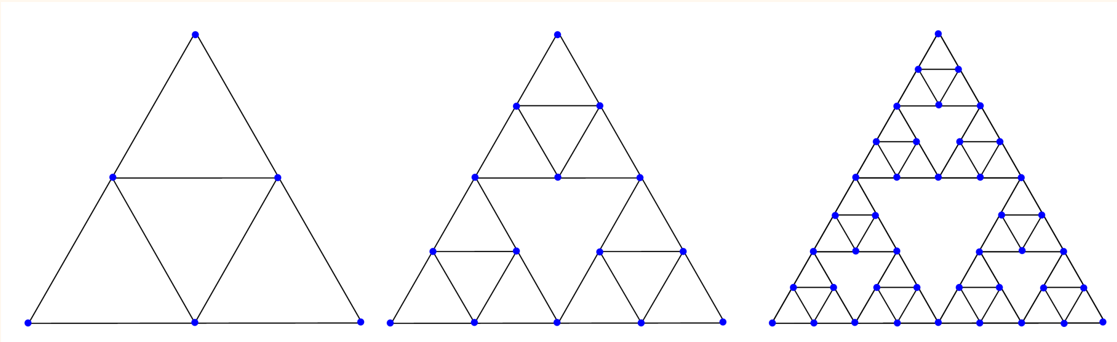
where $c_n=\Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$. For an arbitrary $s\in(0,2)$ we have

$$p_t(x)\simeq\frac{1}{t^{n/s}}\left(1+\frac{|x|}{t^{1/s}}\right)^{-(n+s)}.$$

If a metric measure space M possesses a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ then we consider its generator $\mathcal{L}$ as an analogue of the Laplace operator. In this case the differential calculus is defined on M.

Nontrivial strongly local regular Dirichlet forms have been successfully constructed on large families of fractals, in particular, on SG by Barlow–Perkins '88, Goldstein '87 and Kusuoka '87, on SC by Barlow–Bass '89 and Kusuoka–Zhou '92, on p.c.f. fractals (including VS) by Kigami '93.

In fact, each of these fractals can be regarded as a limit of a sequence of approximating graphs Γ_n.



Approximating graphs $\Gamma_1, \Gamma_2, \Gamma_3$ for SG

Define on each Γ_n a Dirichlet form $\mathcal{E}_n$ by

$$\mathcal{E}_n(f,f) = \sum_{x \sim y} (f(x) - f(y))^2$$

and then consider a scaled limit

$$\mathcal{E}(f,f) = \lim_{n \to \infty} \rho^n \mathcal{E}_n(f,f)$$

with an appropriately chosen scaling parameter ρ. The main difficulty is to ensure the existence of ρ such that this limit exists and is nontrivial for a dense family of f. For example, we have

- on SG: $\rho = \frac{5}{3}$
- on VS: $\rho = 3$
- on SC the exact value of ρ is unknown, $\rho \approx 1.25$.

On SG and VS the limit exists due to monotonicity (Kigami), while on SC it is much harder (Bass–Barlow).

Walk dimension

In all the above examples the heat semigroup $e^{-t\mathcal{L}}$ of the Dirichlet form $(\mathcal{E},\mathcal{F})$ is an integral operator:

$$e^{-t\mathcal{L}}f(x)=\int_M p_t(x,y)\,f(y)\,d\mu(y)$$

whose integral kernel $p_t(x,y)$ is called the *heat kernel* of $(\mathcal{E},\mathcal{F})$ or of $\mathcal{L}$. Moreover, in all the examples the heat kernel satisfies the following estimates

$$p_t(x,y)\asymp\frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{d^{\beta}(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{4}$$

for all $x,y\in M$ and $t\in(0,t_0)$ (Barlow–Perkins '88, Barlow–Bass '92).

Here α is the fractal dimension while β is a new parameter that is called the *walk dimension*. It can be regarded as an numerical characteristic of the differential calculus on M.

It is known that always $\beta\geq2$ and that any pair (α,β) of reals with $\alpha>0$ and $\beta\geq2$ can be realized on some fractal as parameters in the heat kernel bounds (4) (Barlow '04).

Hence, we obtain a large family of metric measure spaces each of them being characterized by a pair (α,β) where α is responsible for integration while β is responsible for differentiation.

The Euclidean space $\mathbb{R}^n$ belongs to this family with $\alpha = n$ and $\beta = 2$. In the case $\beta = 2$ the estimate (4) is called Gaussian, while in the case $\beta > 2$ – sub-Gaussian.

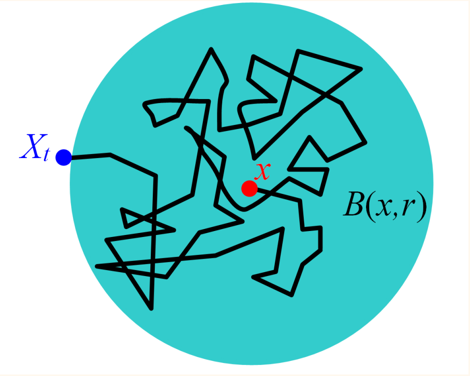
On fractals the values of β is determined by the scaling parameter ρ. It is known that:

- on SG: $\beta = \frac{\log 5}{\log 2} \approx 2.32$
- on VT: $\beta = \frac{\log 15}{\log 3} \approx 2.47$
- on SC the exact value of β is unknown, $\beta \approx 2.09$.

The walk dimension β has the following probabilistic meaning. Denote by τ_Ω the first exit time of X_t from an open set $\Omega \subset M$, that is, $\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}$.

Then in the above setting, for any ball $B(x,r)$ with $r < r_0$, we have

$$\mathbb{E}_x \tau_{B(x,r)} \simeq r^\beta.$$



Besov spaces and characterization of β

Given an α-regular metric measure space (M, d, μ), it is possible to define the family $B^{\sigma}_{p,q}$ of Besov spaces. We need only the following special cases: for any $\sigma > 0$ the space $B^{\sigma}_{2,2}$ consists of functions such that

$$\|f\|^2_{B^{\sigma}_{2,2}} := \|f\|^2_2 + \iint_{M\times M} \frac{|f(x)-f(y)|^2}{d(x,y)^{\alpha+2\sigma}} d\mu(x) d\mu(y) < \infty$$

and $B^{\sigma}_{2,\infty}$ consists of functions such that

$$\|f\|^2_{B^{\sigma}_{2,\infty}} = \|f\|^2_2 + \sup_{0<r<r_0} \frac{1}{r^{\alpha+2\sigma}} \iint_{\{d(x,y)<r\}} |f(x)-f(y)|^2 \, d\mu(x) d\mu(y) < \infty.$$

It is easy to see that $B^{\sigma}_{2,2}$ shrinks as σ increases and that $B^{\sigma}_{2,2}$ contains Lip_0 if $\sigma < 1$. In $\mathbb{R}^n$ the space $B^{\sigma}_{2,2}$ becomes $\{0\}$ if $\sigma > 1$, so that for $\sigma > 1$ the definition of the Besov spaces in $\mathbb{R}^n$ changes. However, in our setting we are interested in the borderline value of σ when the space $B^{\sigma}_{2,2}$ degenerates:

$$\sigma_{crit} = \sup\left\{\sigma > 0 : B^{\sigma}_{2,2} \text{ is dense in } L^2\right\}.$$

Theorem 1 (AG–J.Hu '03) *If* $(\mathcal{E},\mathcal{F})$ *is a Dirichlet form on* M *such that its heat kernel exists and satisfies the sub-Gaussian estimate*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{5}$$

with some α *and* β *then the following is true:*

1. *the space* M *is* α*-regular,* $\alpha = \dim_H M$ *and* $\mu \simeq \mathcal{H}_a$*;*
2. $\sigma_{crit} = \beta/2$ *(consequently,* $\beta \geq 2$*);*
3. $\mathcal{F} = B^{\beta/2}_{2,\infty}$ *and* $\mathcal{E}(f,f) \simeq \|f\|^2_{\dot{B}^{\beta/2}_{2,\infty}}$.

Partial results in this direction: Jonsson '96, Pietruska-Paluba '00.

Corollary 2 *Both* α *and* β *in* (5) *are the invariants of the metric structure* (M,d) *alone.*

Big open question. Let M be an α-regular metric measure space (even self-similar). Set $\beta = 2\sigma_{crit}$. How to construct a strongly local Dirichlet form with the heat kernel satisfying the estimate (5)? Does such a form exist?

Self-similar heat kernels

Let (M, d) be metric space and μ be an α-regular measure on M.

Theorem 3 (AG–Kumagai '08) *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on M. Assume that*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x, y)}{t^{1/\beta}}\right),$$

where $\alpha, \beta > 0$ and Φ is a positive function on $[0, \infty)$. Then the following dichotomy holds:

- *either the Dirichlet form $\mathcal{E}$ is strongly local and $\Phi(s) \asymp C \exp\left(-cs^{\frac{\beta}{\beta-1}}\right)$.*
- *or the Dirichlet form $\mathcal{E}$ is non-local and $\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}$.*

That is, in the first case $p_t(x, y)$ satisfies the sub-Gaussian estimate

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{6}$$

while in the second case we obtain a *stable-like estimate*

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \simeq \min\left(\frac{1}{t^{\alpha.\beta}}, \frac{t}{d(x, y)^{\alpha+\beta}}\right). \tag{7}$$

Estimating heat kernels: strongly local case

Let M be a metric space with precompact balls, μ be an α-regular measure on M and $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on M.

Definition. We say that (M, d) satisfies the *chain condition* (CC) if $\exists C$ such that for all $x, y \in M$ and for $n \in \mathbb{N}$ there exists a sequence $\{x_k\}_{k=0}^n$ of points in M such that $x_0 = x$, $x_n = y$, and

$$d(x_{k-1}, x_k) \leq C \frac{d(x, y)}{n}, \quad \text{for all } k = 1, ..., n.$$

Definition. We say that $(\mathcal{E}, \mathcal{F})$ satisfies the Poincare inequality with exponent β if, for any ball $B = B(x, r)$ on M and for any function $f \in \mathcal{F}$,

$$\mathcal{E}_B(f, f) \geq \frac{c}{r^\beta} \int_{\varepsilon B} \left(f - \overline{f}\right)^2 d\mu, \qquad (PI)$$

where $\overline{f} = ⨍_{\varepsilon B} f d\mu$, and c, ε are small positive constant independent of B and f. For example, in $\mathbb{R}^n$ (PI) holds with $\beta = 2$ and $\varepsilon = 1$.

Let $A \Subset B$ be two open subset of M. Define the capacity of the capacitor (A, B) as follows:

$$\operatorname{cap}(A, B) := \inf \left\{\mathcal{E}(\varphi, \varphi) : \varphi \in \mathcal{F},\ \varphi|_{\overline{A}} = 1,\ \operatorname{supp} \varphi \Subset B \right\}.$$

Definition. We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *capacity condition* if, for any two concentric balls $B_0 := B(x, R)$ and $B := B(x, R + r)$,

$$\operatorname{cap}(B_0, B) \le C \frac{\mu(B)}{r^\beta}. \tag{cap}$$

Conjecture. $(CC) + (PI) + (\text{cap}) \Leftrightarrow (6)$

The implication $\Leftarrow$ is known to be true, so the main difficulty is in $\Rightarrow$.

Let $A \Subset B$ be two open subset of M. For any measurable function u on B, define the *generalized capacity* $\operatorname{cap}_u(A, B)$ by

$$\operatorname{cap}_u(A, B) = \inf \left\{ \mathcal{E}\left(u^2\varphi, \varphi\right) : \varphi \in \mathcal{F},\ \varphi|_{\overline{A}} = 1,\ \operatorname{supp} \varphi \Subset B \right\}.$$

Definition. We say that the *generalized capacity condition* (Gcap) holds if, for any $u \in \mathcal{F}$ and for any two concentric balls $B_0 := B(x, R)$ and $B := B(x, R + r)$,

$$\operatorname{cap}_u(B_0, B) \le \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{Gcap}$$

Theorem 4 (AG–J.Hu–K.S.Lau '15) $(CC) + (PI) + (\text{Gcap}) \Leftrightarrow (6)$.

Estimating heat kernels: jump case

Let now $(\mathcal{E}, \mathcal{F})$ be a jump type Dirichlet form given by

$$\mathcal{E}(f,f) = \iint_{M\times M} (f(x) - f(y))^2 J(x,y)d\mu(x)d\mu(y),$$

where J is a symmetric jump kernel. We use the following condition instead of the Poincaré inequality:

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

Theorem 5 (AG-E.Hu–J. Hu '16 and Z.Q.Chen-Kumagai-J.Wang '16)

$$(J) + (\mathrm{Gcap}) \Leftrightarrow (7).$$

In the case $\beta < 2$ it is easy to show that $(J) \Rightarrow$ (Gcap) so that in this case we obtain the equivalence

$$(J) \Leftrightarrow (7).$$

The latter was also shown by Chen and Kumagai '03, although under some additional assumptions about the metric structure of (M, d).

Conjecture. $(J) + (\mathrm{cap}) \Leftrightarrow (7)$.