Heat kernel and Lipschitz-Besov spaces

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Abstract

On a metric measure space (M, ρ, μ) we consider a family of the Lipschitz-Besov spaces $\Lambda^s_{p,q}$ that is defined only using the metric ρ and measure μ, and a family of Besov spaces $B^s_{p,q}$ that is defined using an auxiliary self-adjoint operator L and the associated heat semigroup. Under certain assumptions about the heat kernel of L, we prove the identity of the two families of the function spaces.

Contents

1 Introduction and main results

1.1 Function spaces on a metric measure space

Let (M, ρ) be a locally compact complete separable metric space and μ be a non-negative Borel measure with full support on M (that is, $0 < \mu(E) < \infty$ for any non-void relatively compact open set $E \subset M$). We

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will refer to the triple (M,ρ,μ) as a *metric measure space*. For $p \in (0,\infty)$, let $L^p = L^p(M,\mu)$ be the space of functions whose absolute value raised to the p-th power has finite integral with respect to μ. The space $L^\infty = L^\infty(M,\mu)$ consists of all essentially bounded μ-measurable functions on M.

Let us recall the definition of the Lipschitz-Besov space on M. We use $\fint_E$ to denote $\frac{1}{\mu(E)}\int_E$, for any measurable set $E \subset M$.

Definition 1.1. For $s \in (0,\infty)$, $p \in [1,\infty)$ and $q \in (0,\infty)$, the *Lipschitz-Besov space* $\Lambda^s_{p,q} = \Lambda^s_{p,q}(M,\mu)$ is defined to be the collection of all $f \in L^p$ such that

$$\|f\|_{\dot{\Lambda}^s_{p,q}} := \left\{\int_0^\infty \left[\int_M \fint_{B(x,r)} \frac{|f(x)-f(y)|^p}{r^{sp}}\, d\mu(y)\, d\mu(x)\right]^{q/p} \frac{dr}{r}\right\}^{1/q} < \infty.$$

If $q = \infty$, then $\Lambda^s_{p,\infty}$ is defined to be the collection of all $f \in L^p$ such that

$$\|f\|_{\dot{\Lambda}^s_{p,\infty}} := \sup_{r\in(0,\infty)} \left[\int_M \fint_{B(x,r)} \frac{|f(x)-f(y)|^p}{r^{sp}}\, d\mu(y)\, d\mu(x)\right]^{1/p} < \infty.$$

Endow the space $\Lambda^s_{p,q}$ with the norm

$$\|f\|_{\Lambda^s_{p,q}} := \|f\|_{L^p} + \|f\|_{\dot{\Lambda}^s_{p,q}}.$$

Clearly, $\Lambda^s_{p,q}$ is a Banach space. The above Lipschitz-Besov spaces $\Lambda^s_{p,q}$ were first introduced by Jonsson [16] when M is a d-set of $\mathbb{R}^n$. If M is a general α-regular metric measure space (i. e., $\mu(B(x,r)) \simeq r^\alpha$ for all $x \in M$ and $r > 0$), these spaces were introduced in [12] for the case $p = 2$, $q = \infty$ and by Yang and Lin [26] for general p, q. Extensions of the result of [26] to the *RD*-spaces (i. e., spaces of homogeneous type satisfies the reverse doubling condition; see [14]) was due to Müller and Yang [19].

Notice that for large enough s the space $\Lambda^s_{p,q}$ may degenerate to trivial spaces consisting only of constant functions. For example, if (M,ρ,μ) is the classical Euclidean space then $\Lambda^s_{2,q} = \{0\}$ if $s > 1$. Also, if (M,ρ,μ) is a fractal space admitting the heat kernel that satisfies (1.10) below, then $\Lambda^s_{2,q} = \{0\}$ if $s > \beta/2$ (see [16]). The same property remains valid if the condition (1.10) is weakened to (1.5) below (see [11, 12, 22]). Thus, the value of β in (P3) and (P4) (see Subsection 1.2 below) illustrates an intrinsic property of the Lipschitz-Besov space.

Let us emphasize that the above definition of the spaces $\Lambda^s_{p,q}$ requires only a metric measure structure on M. However, as we would like to avoid considering a degenerate space $\Lambda^s_{p,q}$ for large s, we introduce more general Lipschitz-Besov spaces $\Lambda^{m,s}_{p,q}$ using an additionally given operator L.

Let L be a positive definite self-adjoint operator in L^2. Then its power $L^{m/2}$ for $m \in (0,\infty)$ is well-defined as a self-adjoint operator in L^2. Denote by $\mathrm{Dom}(L^{m/2})$ its domain in L^2. For $m = 0$, we understand $\mathrm{Dom}(L^{m/2})$ as the collection of all measurable functions on (M,ρ,μ).

Definition 1.2. Let $m \in [0,\infty)$, $s \in (0,\infty)$, $p \in [1,\infty)$ and $q \in (0,\infty]$. Define the *Lipschitz-Besov space* $\Lambda^{m,s}_{p,q} = \Lambda^{m,s}_{p,q}(M,\mu)$ to be the collection of all $f \in \mathrm{Dom}(L^{m/2}) \cap L^p$ such that

$$\|f\|_{\Lambda^{m,s}_{p,q}} := \|f\|_{L^p} + \|L^{m/2} f\|_{\dot{\Lambda}^s_{p,q}} < \infty.$$

Denote by $\widetilde{\Lambda}^{m,s}_{p,q}$ the completion of $\Lambda^{m,s}_{p,q}$ with respect to the norm $\|\cdot\|_{\Lambda^{m,s}_{p,q}}$.

For $m = 0$, the spaces $\Lambda^{0,s}_{p,q}$ and $\widetilde{\Lambda}^{0,s}_{p,q}$ coincide with $\Lambda^{s}_{p,q}$.

Let us introduce another notion of the Besov spaces via the heat semigroup. Let L be a positive definite self-adjoint operator in L^2 as above. Then, for the spectral resolution $\{E_\lambda\}_{\lambda\ge 0}$ of L and for all $f \in \mathrm{Dom}(L)$, we have

$$Lf = \int_0^\infty \lambda \, dE_\lambda f.$$

Given any $\gamma \in (0,\infty)$, one can define the family of operators $\{(tL)^\gamma e^{-tL}\}_{t>0}$ via the functional calculus:

$$(tL)^\gamma e^{-tL} f = \int_0^\infty (t\lambda)^\gamma e^{-t\lambda} \, dE_\lambda f, \quad f \in L^2.$$

Fix some value $\beta > 0$ (that later will be the same as in $(U\Phi)_\beta$ but so far β is arbitrary).

Definition 1.3. For given $r \in (0,\infty)$, $p \in [1,\infty)$, $q \in (0,\infty]$ choose some $k \in (r/\beta,\infty)$ and define the *Besov space* $B^r_{p,q} = B^r_{p,q}(M,\mu)$ as the collection of all $f \in L^p$ such that

$$\|f\|_{B^r_{p,q}} := \|f\|_{L^p} + \left(\int_0^\infty \left[t^{-r/\beta}\|(tL)^k e^{-tL} f\|_{L^p}\right]^q \frac{dt}{t}\right)^{1/q} \tag{1.1}$$

is finite, where a usual modification is made when $q = \infty$.

Remark 1.4. For a general operator L, note that $(tL)^k e^{-tL} f$ might not be well-defined for functions $f \in L^p$. However, under certain assumptions about the operator L (see Subsection 1.2 below), we shall prove in Proposition 2.4 below that for all $k \in [0,\infty)$,

$$\|(tL)^k e^{-tL} f\|_{L^p} \le C\|f\|_{L^p}$$

uniformly in $t \in (0,\infty)$. Consequently, for any $a \in (0,\infty)$,

$$\left(\int_a^\infty \left[t^{-r/\beta}\|(tL)^k e^{-tL} f\|_{L^p}\right]^q \frac{dt}{t}\right)^{1/q} \le C(a)\,\|f\|_{L^p},$$

and the integral $\int_0^\infty$ in (1.1) can be replaced with $\int_0^c$ thus leading to an equivalent norm. As it will be proved in Proposition 2.9, under certain assumptions about the operator L the norms $\|\cdot\|_{B^r_{p,q}}$ are equivalent for different values of k provided $k > r/\beta$. It is known that $B^r_{p,q}$ are complete (quasi)Banach spaces; see, for example [5, Theorem 4.1].

The main purpose of this paper is investigation of the relation between the spaces $\Lambda^{m,s}_{p,q}$ and $B^r_{p,q}$. Since in the case $m = 0$ the space $\Lambda^s_{p,q}$ is defined independently of the operator L, one cannot expect any relation between them unless L satisfies certain hypotheses.

In the next two Subsections we state the necessary hypotheses in terms of the *heat kernel* of the heat semigroup e^{-tL} and give some examples. Then we come back to the space $\Lambda^{m,s}_{p,q}$, $B^r_{p,q}$ and state our main result about their identity.

Notation. Throughout the paper we use the following notation. Let $\mathbb{N} = \{1,2,\dots\}$, $\mathbb{Z}_+ = \{0,1,2,\dots\}$ and $\mathbb{Z} = \{0,\pm1,\pm2,\dots\}$. We shall write C for various positive constants that are independent of the main variables involved. Occasionally we use $C(\alpha,\beta,\gamma,\dots)$ to denote a positive constant depending on the parameters $\alpha,\beta,\gamma,\dots$. Given any two nonnegative functions F and G, the notation $F \lesssim G$ (equivalently, $G \gtrsim F$) means that the inequality $F \le CG$ holds for some constant C in a specified domain of the functions F, G. If $F \lesssim G \lesssim F$, then we write $F \simeq G$.

1.2 The notion of the heat kernel

Let L be a self-adjoint positive definite operator on L^2 with the domain $\mathrm{Dom}(L)$ that is a dense subspace of L^2. The *heat semigroup* $\{P_t\}_{t\geq 0}$ generated by L is defined by

$$P_t = e^{-tL},$$

so that P_t is a bounded self-adjoint operator in L^2. Assume that, for any $t > 0$, the operator P_t has an *integral kernel* p_t that is a continuous function on $M \times M$ such that, for all $f \in L^2$ and $x \in M$,

$$P_t f(x) = e^{-tL} f(x) = \int_M p_t(x,y) f(y)\, d\mu(y). \tag{1.2}$$

The function $(t,x,y) \mapsto p_t(x,y)$ is called *the heat kernel* of L.

It follows from (1.2) that the heat kernel is *symmetric*, that is, for all $t > 0$ and $x, y \in M$,

$$p_t(x,y) = p_t(y,x),$$

and satisfies the *semigroup property*: for all $s, t > 0$ and $x, y \in M$,

$$p_{s+t}(x,y) = \int_M p_s(x,z) p_t(z,y)\, d\mu(z).$$

We assume that the heat kernel satisfies in addition the following conditions:

(P1) Stochastic completeness: $\int_M p_t(x,y)\, d\mu(y) \equiv 1$ for all $t > 0$ and $x \in M$.

(P2) There exists a positive constant C such that, for all $t > 0$ and $x \in M$,

$$\int_M |p_t(x,y)|\, d\mu(y) \leq C. \tag{1.3}$$

(P3) Upper bound: for all $t \in (0,1]$ and $x, y \in M$,

$$|p_t(x,y)| \leq \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{\rho(x,y)}{t^{1/\beta}}\right), \tag{$U\Phi$}_\beta$$

where $\alpha, \beta > 0$ and Φ is a non-negative monotone decreasing function on $[0,\infty)$ such that, for any $\gamma < \beta$,

$$\int^\infty \tau^{\alpha+\gamma} \Phi(\tau) \frac{d\tau}{\tau} < \infty. \tag{1.4}$$

(P4) Hölder continuity: for all $t \in (0,1]$ and $x, y, y' \in M$ such that $\rho(y,y') \leq t^{1/\beta}$,

$$\left|p_t(x,y) - p_t(x,y')\right| \leq \left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^\Theta \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{\rho(x,y)}{t^{1/\beta}}\right), \tag{$H\Phi$}_\Theta$$

where α, β and Φ are the same as in (P3), and Θ is some positive constant (that can be assumed to be sufficiently small).

Throughout the article, we will fix the aforementioned parameters α, β and Θ.

In addition, we assume that the parameter α is related to the metric measure structure as follows: for all $x \in M$ and $0 < r < \operatorname{diam} M$,

$$\mu(B(x,r)) \le C r^{\alpha}, \tag{V_α}_\le$$

where $B(x,r)$ is an open metric ball in (M,ρ).

Notice that the heat kernel p_t might be signed; see example (vii) in Subsection 1.3 below. Properties (P1) and (P3) imply that p_t is an *approximation of identity*, that is, for any $u \in L^p$ with $p \in [1,\infty)$,

$$\int_M p_t(x,y) u(y)\, d\mu(y) \stackrel{L^p}{\longrightarrow} u(x) \ \text{ as } t \to 0_+;$$

see Lemma 2.8 below for its proof. Observe that (P2) is a rather weak requirement. For example, if $p_t \ge 0$ then (P2) follows trivially from (P1). Also, if the upper bound estimate (P3) is true for all $t > 0$, then it implies (P2); see (2.3) and (2.1) below.

It is worth mentioning, that (P2) implies that the operator L is positive definite. Indeed, by (1.3) we obtain that for all $t > 0$

$$\|e^{-tL}\|_{2\to 2} \le C,$$

whereas if the operator L is not positive definite then $\|e^{-tL}\|_{2\to 2} \to \infty$ as $t \to \infty$.

1.3 Some examples of heat kernel

Let us give some examples of heat kernels in different setups.

(i) Consider the Euclidean space $\mathbb{R}^n$ with standard metric and measure, which satisfies $(V_\alpha)_\le$ with $\alpha = n$. Consider also the Laplace operator $L = -\Delta = -\sum_{j=1}^n \partial_{x_j}^2$. Then the heat semigroup $\{e^{-tL}\}_{t\ge 0}$ has the classical Gauss-Weierstrass kernel:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Obviously, (P1)-(P4) hold with $\alpha = n, \beta = 2, \Theta = 1$ and

$$\Phi(\tau) = \frac{1}{(4\pi)^{n/2}} \exp(-\tau^2/4).$$

(ii) Consider the Euclidean space $\mathbb{R}^n$ and the operator $L = (-\Delta)^{1/2}$. Then the semigroup $\{e^{-tL}\}_{t\ge 0}$ has the Poisson kernel:

$$p_t(x,y) = C_n \frac{1}{t^n}\left(1 + \frac{|x-y|^2}{t}\right)^{-\frac{n+1}{2}},$$

where $C_n = \pi^{-\frac{n+1}{2}} \Gamma(\frac{n+1}{2})$. Again (P1)-(P4) hold with $\alpha = n, \beta = 1, \Theta = 1$ and

$$\Phi(\tau) = C_n (1+\tau^2)^{-\frac{n+1}{2}}.$$

(iii) Let L be a self-adjoint positive definite operator in L^2. Assume that the kernel p_t of L satisfies (P1) and (P4) for all $t \in (0,\infty)$, as well a stronger version of (P3): for all $t \in (0,\infty)$ and $x, y \in M$,

$$\frac{1}{t^{\alpha/\beta}} \Phi_1\left(\frac{\rho(x,y)}{t^{1/\beta}}\right) \le p_t(x,y) \le \frac{1}{t^{\alpha/\beta}} \Phi_2\left(\frac{\rho(x,y)}{t^{1/\beta}}\right), \tag{1.5}$$

where Φ_1 and Φ_2 are non-negative monotone decreasing functions on $[0,\infty)$ such that $\Phi_1(c) > 0$ for some $c > 0$ and Φ_2 satisfies (1.4). In particular, the heat kernel is non-negative and (P2) is trivially satisfied.

Fix some $\delta \in (0,1)$ and consider the subordinated semigroup $\{e^{-tL^\delta}\}_{t\geq 0}$. It is known that this semigroup has the heat kernel

$$q_t(x,y) = \int_0^\infty \eta_t(s) p_s(x,y)\, ds, \tag{1.6}$$

where $\eta_t(s)$ is the subordinator. It is known that $\eta_t(s)$ is positive and satisfies the following identities

$$\int_0^\infty \eta_t(s)\, ds = 1 \tag{1.7}$$

and

$$\eta_t(s) = \frac{1}{t^{1/\delta}} \eta\left(\frac{s}{t^{1/\delta}}\right),$$

where η is a positive function on $(0,\infty)$ such that for any $\gamma > 0$,

$$\eta(s) = o(s^\gamma) \quad \text{as } s \to 0 \tag{1.8}$$

and

$$\eta(s) \simeq \frac{1}{s^{1+\delta}}, \qquad s \geq 1; \tag{1.9}$$

(see [11, Section 5.4], [27]). As p_t satisfies (P1), so does q_t by means of (1.6) and (1.7). Since $q_t \geq 0$, it satisfies also (P2). Condition (P3) for q_t follows from [11, Lemma 5.4] where it was proved that, for $\beta' = \delta\beta$,

$$q_t(x,y) \simeq \frac{1}{t^{\alpha/\beta'}} \left(1 + \frac{\rho(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')},$$

for all $t \in (0,\infty)$ and $x, y \in M$. Let us verify that q_t satisfies the Hölder continuity (P4). Since p_t satisfies (1.5) and (P4), we see that for all $t \in (0,\infty)$ and $x, y, y' \in M$ such that $\rho(y,y') \leq t^{1/\beta'}$,

$$\begin{aligned}
&|q_t(x,y) - q_t(x,y')| \\
&\leq \int_0^\infty \eta_t(s) |p_s(x,y) - p_s(x,y')|\, ds \\
&\leq C \int_0^\infty \eta_t(s) \frac{1}{s^{\alpha/\beta}} \left(\frac{\rho(y,y')}{s^{1/\beta}}\right)^\Theta \left[\Phi_2\left(\frac{\rho(x,y)}{s^{1/\beta}}\right) + \Phi_2\left(\frac{\rho(x,y')}{s^{1/\beta}}\right)\right] ds \\
&= C \frac{1}{t^{\alpha/\beta'}} \left(\frac{\rho(y,y')}{t^{1/\beta'}}\right)^\Theta \int_0^\infty \eta\left(\frac{s}{t^{1/\delta}}\right) \left(\frac{t^{1/\delta}}{s}\right)^{(\alpha+\Theta)/\beta} \left[\Phi_2\left(\frac{\rho(x,y)}{s^{1/\beta}}\right) + \Phi_2\left(\frac{\rho(x,y')}{s^{1/\beta}}\right)\right] \frac{ds}{t^{1/\delta}}.
\end{aligned}$$

It follows from (1.4) that $\Phi_2(\tau) \leq C(1+\tau)^{-(\alpha+\beta')}$, so

$$\Phi_2\left(\frac{\rho(x,y)}{s^{1/\beta}}\right) \leq C\left(1 + \frac{\rho(x,y)}{s^{1/\beta}}\right)^{-(\alpha+\beta')} \leq C\left(1 + \frac{t^{1/\delta}}{s}\right)^{(\alpha/\beta+\delta)} \left(1 + \frac{\rho(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')}$$

and thus

$$\begin{aligned}
&\int_0^\infty \eta\left(\frac{s}{t^{1/\delta}}\right)\left(\frac{t^{1/\delta}}{s}\right)^{(\alpha+\Theta)/\beta} \Phi_2\left(\frac{\rho(x,y)}{s^{1/\beta}}\right)\frac{ds}{t^{1/\delta}} \\
&\le C\left(1+\frac{\rho(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')}\int_0^\infty \eta\left(\frac{s}{t^{1/\delta}}\right)\left(\frac{t^{1/\delta}}{s}\right)^{(\alpha+\Theta)/\beta}\left(1+\frac{t^{1/\delta}}{s^{1/\beta}}\right)^{(\alpha/\beta+\delta)}\frac{ds}{t^{1/\delta}} \\
&\simeq \left(1+\frac{\rho(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')}\int_0^\infty \eta(\tau)\tau^{-(\alpha+\Theta)/\beta}(1+\tau)^{-\alpha/\beta-\delta}\,d\tau \\
&\simeq \left(1+\frac{\rho(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')},
\end{aligned}$$

where the last integral converges by means of (1.8) and (1.9). Similarly one obtains

$$\int_0^\infty \eta\left(\frac{s}{t^{1/\delta}}\right)\left(\frac{t^{1/\delta}}{s}\right)^{(\alpha+\Theta)/\beta}\Phi_2\left(\frac{\rho(x,y')}{s^{1/\beta}}\right)\frac{ds}{t^{1/\delta}} \le C\left(1+\frac{\rho(x,y')}{t^{1/\beta'}}\right)^{-(\alpha+\beta')} \simeq \left(1+\frac{\rho(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')},$$

whence it follows that

$$\left|q_t(x,y)-q_t(x,y')\right| \le C\frac{1}{t^{\alpha/\beta'}}\left(\frac{\rho(y,y')}{t^{1/\beta'}}\right)^{\Theta}\left(1+\frac{\rho(x,y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')},$$

which proves (P4) for q_t.

(iv) Consider the operator $L=(-\Delta)^{m/2}$ in $\mathbb{R}^n$, where $0<m<2$. As a consequence of (i) and (iii), the heat kernel of e^{-tL} satisfies (P1)-(P4). Moreover, it satisfies the two-sided estimate

$$p_t(x,y) \simeq \frac{1}{t^{n/m}}\left(1+\frac{|x-y|}{t^{1/m}}\right)^{-(n+m)}$$

for all $t\in(0,\infty)$ and $x,y\in\mathbb{R}^n$, and (P4) holds for $\beta=m$ and $\Theta=1$.

(v) Let M be the unbounded Sierpinski gasket SG in $\mathbb{R}^n$, ρ be the induces metric on SG and μ be the Hausdorff measure on SG of dimension

$$\alpha = \dim_H \mathrm{SG} = \log_2(n+1).$$

It is known that SG satisfies $(V_\alpha)_\le$ (see [3]). It is also known that SG admits a local Dirichlet form $\mathcal{E}$ whose heat kernel satisfies the *sub-Gaussian* estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{\rho(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right), \tag{1.10}$$

where the parameter $\beta=\log_2(n+3)$ is called the *walk dimension* (see Barlow [3] for more details). The sign $\asymp$ means here that both $\le$ and $\ge$ hold but with different values of the positive constants C,c. The operator L in this case is the generator of the Dirichlet form $\mathcal{E}$.

The sub-Gaussian estimate (1.10) is valid on many other fractal spaces, with various values of α and β; more precisely, any couple α,β in the range $2\le\beta\le\alpha+1$ is possible. In all these cases the heat kernel is known to be stochastically complete, so that (P1) and (P2) are satisfied. Clearly, (P3) follows from (1.10).

By [4, Theorem 3.1 and Corollary 4.2], the estimate (1.10) implies the Hölder continuity (P4) with some small Θ (see also [13, Theorem 7.4]).

(vi) Let M be a geodesically complete Riemannian manifold, ρ be the geodesic distance, and μ be the Riemannian measure. The Laplace-Beltrami operator $L = -\Delta$ on M can be made into a self-adjoint operator in L^2 by appropriately defining its domain. As was proved by Li and Yau [17], if the Ricci curvature of M is non-negative, then the heat kernel of the heat semigroup e^{-tL} satisfies the estimate

$$p_t(x,y) \asymp \frac{C}{\mu(B(x,\sqrt{t}))}\exp\left(-c\frac{\rho(x,y)^2}{t}\right). \tag{1.11}$$

The heat kernel is in this case stochastically complete (see, for example, [10]), and (1.11) implies (P4) by [4, Theorem 3.1 and Corollary 4.2]. Hence, all (P1)-(P4) are satisfied.

(vii) Let $m \in \mathbb{N}$ and α, β be n-dimensional multi-indices. Consider in $\mathbb{R}^n$ the elliptic operator of order $2m$ of the form

$$L = \sum_{|\alpha|\le m,\, |\beta|\le m} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha,\beta}D^{\beta})$$

with the leading part

$$L_0 = (-1)^m \sum_{|\alpha|=|\beta|=m} D^{\alpha}(a_{\alpha,\beta}D^{\beta}),$$

where $A := \{a_{\alpha,\beta}(x)\}$ is a symmetric matrix of complex-valued bounded measurable functions on $\mathbb{R}^n$. Thus operators L_0 and L admit self-adjoint extensions in L^2. Denote by p_t^0 the heat kernel of the semigroup e^{-tL_0} and by p_t the heat kernel of the semigroup e^{-tL}. It is known that if $m > 1$ then the heat kernels p_t^0 and p_t cannot be non-negative functions; see Davies [9, Section 5.5]. Upper bound and Hölder continuity estimates of the kernel p_t^0 and p_t are studied in Auscher and Qafsaqui [2]; see also [1, 9].

For $p \in [1,\infty)$, denote by $L^p(\mathbb{R}^n)$ the Lebesgue space with respect to the Lebesgue measure. Assume that L_0 satisfies the so-called *strong Garding inequality*: for all functions u in the Sobolev space $W^{m,2}(\mathbb{R}^n)$,

$$\mathrm{Re}(L_0u,\ u) \ge \delta_0\|\nabla^m u\|^2_{L^2(\mathbb{R}^n)}$$

for some constant $\delta_0 > 0$. Let $\mathrm{BMO}(\mathbb{R}^n)$ be the space of locally integrable functions f satisfying the condition

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{\text{balls } B \text{ in } \mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|\,dx < \infty,$$

where f_B denotes the arithmetic mean of f in the ball B. Obviously, the constant functions are in $\mathrm{BMO}(\mathbb{R}^n)$. If every $a_{\alpha,\beta} \in \mathrm{BMO}(\mathbb{R}^n)$, then the following estimates of p_t^0 and p_t were given in [2, Proposition 47]. For any multi-indices $|\gamma| \le m-1$, there exist positive constants C and c such that for all $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$,

$$|D_x^{\gamma}p_t^0(x,y)| \le C\frac{1}{t^{(n+|\gamma|)/(2m)}}\exp\left(-c\left(\frac{|x-y|}{t^{1/(2m)}}\right)^{\frac{2m}{2m-1}}\right).$$

In particular, p_t^0 satisfies (P3) for all $t > 0$ with $\alpha = n$ and $\beta = 2m$. It follows that p_t^0 satisfies also (P2). If $m = 1$, then for all $t \in (0,\infty)$ and $x, y, y' \in \mathbb{R}^n$ such that $|y - y'| \le t^{1/(2m)}$,

$$|p_t^0(x,y) - p_t^0(x,y')| \le C\frac{1}{t^{n/(2m)}}\exp\left(-c\left(\frac{|x-y|}{t^{1/(2m)}}\right)^{\frac{2m}{2m-1}}\right),$$

that is, p_t^0 satisfies (P4) for all $t > 0$. By [2, Theorem 16], the kernel p_t satisfies (P3) and (P4) for short time $t \in (0, T_0]$ with some $T_0 \in (0, \infty)$. Unfortunately, it remains unclear when p_t^0 or p_t are in general stochastically complete.

1.4 Main results

As above, let (M, ρ, μ) be a metric measure space, L be a positive definite self-adjoint operator in L^2, and $p_t(x, y)$ be the heat kernel of L.

Our main result is the next theorem.

Theorem 1.5. *Let (M, ρ, μ) be a metric measure space that satisfies $(\mathrm{V}_\alpha)_\le$. Assume that L is a positive definite operator whose heat semigroup $\{P_t\}_{t\ge 0}$ has the heat kernel $\{p_t\}_{t>0}$ satisfying (P1)-(P4). Fix the parameters $p \in (1, \infty)$, $q \in (0, \infty]$, $m \in [0, \infty)$ as well as*

$$s \in (0, \Theta \wedge (\beta/2))$$

and set

$$r = m\beta/2 + s.$$

Then the following assertions hold:

(a) *For all $f \in \mathrm{Dom}(L^{m/2}) \cap L^p$,*

$$\|f\|_{B^r_{p,q}} \simeq \|f\|_{\Lambda^{m,s}_{p,q}}.$$

(b) *If in addition $q < \infty$ the following two spaces are identical:*

$$B^r_{p,q} = \widetilde{\Lambda}^{m,s}_{p,q} \tag{1.12}$$

with equivalent norms.

(c) *If (P2) holds for all $t \in (0, \infty)$, then the assertions of (a) and (b) are valid also for $p = 1$.*

Under the hypotheses of Theorem 1.5, we see that, for any two pairs of (m, s) and (m', s') such that $m, m' \in [0, \infty)$,

$$s, s' \in (0, \Theta \wedge (\beta/2)) \quad \text{and} \quad m\beta/2 + s = m'\beta/2 + s',$$

we have the identity of the spaces

$$\widetilde{\Lambda}^{m,s}_{p,q} = \widetilde{\Lambda}^{m',s'}_{p,q}$$

with equivalent norms. This provides a way of defining the higher order Sobolev spaces W^r_p with $r = m\beta/2 + s$ on α-regular metric measure spaces as follows:

$$\|f\|_{W^r_p} := \|f\|_{\Lambda^{m,s}_{p,p}} \simeq \|f\|_{L^p} + \left[\int_M \int_M \frac{|L^{m/2} f(x) - L^{m/2} f(y)|^p}{\rho(x, y)^{\alpha+sp}}\, d\mu(x)\, d\mu(y)\right]^{1/p},$$

which may be an interesting topic of research in the future.

For the case $p = q = 2$ and under some additional assumptions the identity (1.12) holds for the maximal range $s \in (0, \beta/2)$, as is stated in the next theorem.

Theorem 1.6. *Let (M, ρ, μ) be a metric measure space that satisfies $(\mathrm{V}_\alpha)_\leq$. Assume that L is a positive definite operator whose heat semigroup $\{P_t\}_{t\geq 0}$ has the non-negative heat kernel $\{p_t\}_{t>0}$ satisfying (P1) and (P3) for all $t \in (0, \infty)$, and (P4) for $t \in (0, 1]$. Fix the parameters $m \in [0, \infty)$,*

$$s \in (0, \beta/2),$$

and set

$$r = m\beta/2 + s.$$

Then the following two spaces are identical:

$$B^r_{2,2} = \widetilde{\Lambda}^{m,s}_{2,2} \tag{1.13}$$

with equivalent norms.

Remark 1.7. Assuming that the heat kernel p_t is non-negative and satisfies (P1) and (1.5), Hu and Zähle [15, Theorem 5.2 and Corollary 3.4] proved that, for any $s \in (0, \beta/2)$,

$$B^s_{2,2} = \Lambda^s_{2,2}$$

with equivalent norms. Under a stronger assumption (1.10) instead of (1.5) the same equivalence was also obtained by Pietruska-Pałuba [23, Section 4.2].

However, the relation between the spaces $\Lambda^s_{p,q}$ and $B^s_{p,q}$ for general p, q and s remained unknown even when $p = 2$, which was posed in [23] as an open question. Our Theorem 1.5 answers this question for small values of s, as well as characterises $B^r_{p,q}$ for arbitrarily large r in terms of $\widetilde{\Lambda}^{m,s}_{p,q}$.

Remark 1.8. Consider the case when M is the Euclidean space $\mathbb{R}^n$, ρ is the Euclidean distance and μ is the Lebesgue measure. Let $L = -\Delta$ be the Laplace operator and p_t as above the Gauss-Weierstrass kernel, so that $\alpha = n, \beta = 2$ and $\Theta = 1$. Theorem 1.5 says in this case that, for all $m \in (0, \infty)$, $s \in (0, 1)$, $p \in [1, \infty)$ and $q \in (0, \infty]$,

$$\begin{aligned} &\|f\|_{L^p(\mathbb{R}^n)} + \left\{ \int_0^\infty \left[\int_{\mathbb{R}^n} \fint_{B(x,r)} \frac{|(-\Delta)^{m/2} f(x) - (-\Delta)^{m/2} f(y)|^p}{r^{sp}} \, dy \, dx \right]^{q/p} \frac{dr}{r} \right\}^{1/q} \\ &\simeq \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_0^\infty \left[t^{-(m+s)/2} \|(-t\Delta)^k e^{t\Delta} f\|_{L^p(\mathbb{R}^n)} \right]^q \frac{dt}{t} \right)^{1/q} \end{aligned} \tag{1.14}$$

for functions $f \in \mathrm{Dom}((-\Delta)^{m/2}) \cap L^p(\mathbb{R}^n)$, where $k > (m + s)/2$.

For the operator $L = \sqrt{-\Delta}$ and for the Poisson kernel p_t we have $\alpha = n, \beta = 1$ and $\Theta = 1$. By Theorem 1.5 we see that, for all $m \in (0, \infty)$, $s \in (0, 1/2)$, $p \in [1, \infty)$ and $q \in (0, \infty]$,

$$\begin{aligned} &\|f\|_{L^p(\mathbb{R}^n)} + \left\{ \int_0^\infty \left[\int_{\mathbb{R}^n} \fint_{B(x,r)} \frac{|(-\Delta)^{m/2} f(x) - (-\Delta)^{m/2} f(y)|^p}{r^{sp}} \, dy \, dx \right]^{q/p} \frac{dr}{r} \right\}^{1/q} \\ &\simeq \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_0^\infty \left[t^{-(m+s)} \|(t\sqrt{-\Delta})^k e^{-t\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)} \right]^q \frac{dt}{t} \right)^{1/q} \end{aligned} \tag{1.15}$$

for functions $f \in \mathrm{Dom}((-\Delta)^{m/2}) \cap L^p(\mathbb{R}^n)$, where $k > m + s$.

For $m = 0$ and $s \in (0,1)$, the norm equivalences (1.14) and (1.15) are well known; see Triebel [25, Theorem 1.7.3 and Theorem 1.8.3]. The right hand sides of (1.14) and (1.15) are the Gauss-Weierstrass heat semigroup and Poisson semigroup characterizations of the Besov space $B^s_{p,q}(\mathbb{R}^n)$, respectively. They are both equivalent to the norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_0^1 t^{-sq}\|\phi_t * f\|^q_{L^p(\mathbb{R}^n)}\,\frac{dt}{t}\right)^{1/q},$$

where ϕ can be taken a smooth function such that

$$\operatorname{supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^n : 1/4 \le |\xi| \le 4\}$$

and

$$\hat{\phi} > c > 0 \ \text{ on } \ \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\},$$

and $\phi_t(\cdot) = t^n\phi(t\cdot)$. We refer the reader to Triebel [24, 25] and the references therein for detailed discussions of the various characterizations of the classical Besov spaces.

It should be remarked that, assuming that $(\mathrm{V}_\alpha)_\le$, (P1), (P3) and (P4) hold for $\beta = 2$ and $t \in (0,\infty)$, Bui, Duong and Yan [5] systematically studied the Besov spaces $B^r_{p,q}$ (for small r only) defined in Definition 1.3 and proved that when $L = -\Delta$ the space $B^r_{p,q}$ is equivalent to the classical Besov space.

Remark 1.9. Let (M,ρ,μ) be the space of homogeneous type with the measure μ satisfying the reverse doubling condition, that is, for all $x \in M$ and $0 < r \le \frac{\operatorname{diam} M}{3}$,

$$\mu(B(x,2r)) \simeq \mu(B(x,r)).$$

Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, and $\epsilon_3 > 0$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on L^2 is called an *approximation of the identity of order* $(\epsilon_1,\epsilon_2,\epsilon_3)$, if there exists a positive constant C such that, for all $k \in \mathbb{Z}$ and x, x', y and $y' \in M$, the integral kernel, $S_k(x,y)$, of S_k is a measurable function, from $M \times M$ into $\mathbb{C}$, satisfying that

(i) $|S_k(x,y)| \le C\frac{1}{\mu(B(x,2^{-k}+d(x,y)))}\frac{2^{-k\epsilon_2}}{[2^{-k}+d(x,y)]^{\epsilon_2}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C\frac{d(x,x')^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}}\frac{1}{\mu(B(x,2^{-k}+d(x,y)))}\frac{2^{-k\epsilon_2}}{[2^{-k}+d(x,y)]^{\epsilon_2}}$ for all $d(x,x') \le [2^{-k}+d(x,y)]/2$;

(iii) S_k satisfies (ii) with x and y interchanged;

(iv) for all $d(x,x') \le [2^{-k}+d(x,y)]/3$ and $d(y,y') \le [2^{-k}+d(x,y)]/3$,
$|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \le C\frac{d(x,x')^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}}\frac{d(y,y')^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}}\frac{1}{\mu(B(x,2^{-k}+d(x,y)))}\frac{2^{-k\epsilon_3}}{[2^{-k}+d(x,y)]^{\epsilon_3}}$;

(v) $\int_M S_k(x,w)\,d\mu(w) = 1 = \int_M S_k(w,y)\,d\mu(w)$.

Obviously, an approximation of the identity has similar properties as a heat kernel, except the second order difference property in (iv). For the existence of such $\{S_k\}_{k\in\mathbb{Z}}$, we refer the reader to Han, Müller and Yang [14] (see also David, Journé and Semmes [6] when (M,ρ,μ) is an α-regular metric measure space).

Via the above approximation of the identity, the authors of [14] build a framework for the theory of Besov and Triebel-Lizorkin spaces on (M, ρ, μ). In particular, for $s \in (0, \epsilon_1)$, $p \in [1, \infty]$ and $q \in (0, \infty]$, the Besov space $\mathcal{B}^s_{p,q}$ can be defined by means of the following norm:

$$\|f\|_{\mathcal{B}^s_{p,q}} = \|S_0 f\|_{L^p} + \left(\sum_{k=1}^{\infty} 2^{ksq} \|D_k f\|^q_{L^p} \right)^{1/q}$$

where $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$, and a usual modification is made when $q = \infty$. It was proved by Müller and Yang [19] that the space $\Lambda^s_{p,q}$ coincides with the Besov space $\mathcal{B}^s_{p,q}$; see also Yang and Lin [26] when (M, ρ, μ) is an α-regular metric measure space.

Remark 1.10. Let (M, ρ, μ) be as in Remark 1.9. Assume that the non-collapsing condition holds: there exists a positive constant c such that

$$\inf_{x \in M} \mu(B(x, 1)) \geq c.$$

Assume that L is a positive defined self-adjoint operator in L^2 such that the kernel p_t of the heat semigroup $\{e^{-tL}\}_{t \geq 0}$ satisfies (P1) and the following stronger conditions:

(P3)′ for all $t \in (0, 1]$ and $x, y \in M$,

$$|p_t(x, y)| \leq C \frac{\exp\left(-c \frac{[\rho(x,y)]^2}{t}\right)}{\sqrt{\mu(B(x, \sqrt{t}))\, \mu(B(y, \sqrt{t}))}};$$

(P4)′ for all $t \in (0, 1]$ and $x, y, y' \in M$ satisfying that $\rho(y, y') \leq \sqrt{t}$,

$$|p_t(x, y) - p_t(x, y')| \leq C \left(\frac{\rho(y, y')}{\sqrt{t}} \right)^{\Theta} \frac{\exp\left(-c \frac{[\rho(x,y)]^2}{t}\right)}{\sqrt{\mu(B(x, \sqrt{t}))\, \mu(B(y, \sqrt{t}))}}.$$

In this case $\beta = 2$. Let $\Phi_0, \Phi \in C^\infty_c(\mathbb{R}^+)$ such that

(i) $\operatorname{supp} \Phi_0 \subset [0, 2]$, $|\Phi_0(\lambda)| \geq c > 0$ on $[0, 2^{3/4}]$ and $\Phi_0^{(2\nu+1)}(0) = 0$ for all $\nu \in \mathbb{N}$;

(ii) $\operatorname{supp} \Phi \subset [1/2, 2]$ and $|\Phi(\lambda)| \geq c > 0$ on $[2^{-3/4}, 2^{3/4}]$;

(iii) $\Phi_j(\cdot) = \Phi(2^{-j} \cdot)$ for all $j \geq 1$ and $\sum_{j=0}^{\infty} \Phi_j(\lambda) = 1$ for all $\lambda \in \mathbb{R}^+$.

For any $s \in (0, \Theta \wedge 1)$, $p \in (1, \infty)$ and $q \in (0, \infty)$, it was proved in [20, Theorem 6.7, Remark 6.8] and [18] that

$$\|f\|_{B^s_{p,q}} \simeq \left\{ \sum_{j=0}^{\infty} 2^{jsq} \|\Phi_j(\sqrt{L}) f\|^q_{L^p} \right\}^{1/q} \simeq \|f\|_{\mathcal{B}^s_{p,q}},$$

where $\Phi_j(\sqrt{L})$ are operators defined via the spectral resolution of L. Applying Remark 1.9 yields that all the norms in the last formulae coincide to $\|\cdot\|_{\Lambda^s_{p,q}}$. In particular $B^s_{p,q} = \Lambda^s_{p,q}$ with equivalent norms.

The article is organized as follows. In Section 2 we make necessary preparation for the proof. In Subsection 2.1 we obtain the estimates of fractional derivatives of the heat semigroup. In Subsection 2.2 we deduce an inhomogeneous version of the continuous Calderón reproducing formulae. In Subsection 2.3, we prove that the definition of $\|\cdot\|_{B^r_{p,q}}$ in (1.1) is independent of the choices of k. Finally, in Section 3, we prove Theorems 1.5 and 1.6.

2 Auxiliary estimates

2.1 Fractional derivatives of the heat semigroup

We start with the following basic estimates of the metric measure space (M,ρ,μ) that satisfies $(V_\alpha)_\le$.

Lemma 2.1. *Let $\gamma\in(0,\infty)$. There exists a constant $C>0$ such that for all $x,y\in M$ and $t>0$,*

$$\int_M \frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(y)\le C \tag{2.1}$$

and

$$\int_M \left[1+\frac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}\left[1+\frac{\rho(z,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(z)\le Ct^{\alpha/\beta}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{2.2}$$

Proof. By $(\mathrm{V}_\alpha)_\le$, we see that for all $t>0$ and $y\in M$,

$$\begin{aligned}\int_M \left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(y) &= \left(\int_{B(x,t^{1/\beta})}+\sum_{j=1}^{\infty}\int_{2^{j-1}t^{1/\beta}\le\rho(x,y)<2^jt^{1/\beta}}\right)\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(y)\\ &\le Ct^{\alpha/\beta},\end{aligned}$$

which implies (2.1). For any $x,y,z\in M$, one has either $\rho(x,z)\ge\frac12\rho(x,y)$ or $\rho(z,y)\ge\frac12\rho(x,y)$, so (2.1) implies that

$$\begin{aligned}&\int_M \left[1+\frac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}\left[1+\frac{\rho(z,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(z)\\ &\le\left(\int_{\rho(x,z)\ge\frac12\rho(x,y)}+\int_{\rho(z,y)\ge\frac12\rho(x,y)}\right)\left[1+\frac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}\left[1+\frac{\rho(z,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(z)\\ &\le Ct^{\alpha/\beta}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}.\end{aligned}$$

Thus, (2.2) holds. □

Let p_t be the heat kernel satisfying (P1)-(P4). The upper bound and Hölder continuity estimates of p_t have some self-improvement properties in the time t, as follows.

Lemma 2.2. *Let $\gamma\in(0,\beta)$ and $T_0\in[1,\infty)$. Then, there exists a positive constant $C=C(\gamma,T_0,\alpha,\beta,\Theta)$ such that the following hold:*

(i) *For all* $t \in (0, T_0]$ *and* $x, y \in M$,

$$|p_t(x,y)| \le C \frac{1}{t^{\alpha/\beta}} \left[1 + \frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{2.3}$$

(ii) *For all* $t \in (0, T_0]$ *and* $x, y, y' \in M$ *such that* $\rho(y, y') \le t^{1/\beta}$,

$$|p_t(x,y) - p_t(x,y')| \le C \left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta} \frac{1}{t^{\alpha/\beta}} \left[1 + \frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{2.4}$$

Proof. Let the function Φ be as in (P3) and (P4). Given any $\gamma \in (0, \beta)$, it is easy to verify that

$$\Phi(\tau) \le C(1+\tau)^{-(\alpha+\gamma)}, \qquad \tau > 0,$$

for some positive constant $C = C(\gamma)$. Thus, from (P3) and (P4), it follows that (i) and (ii) hold when $T_0 = 1$.

Now we let $1 < T_0 \le 2$ and prove that (i) and (ii) hold. For any $t \in (0, T_0]$, by the semigroup property, we write

$$p_t(x,y) = \int_M p_{t/2}(x,z) p_{t/2}(z,y)\, d\mu(z).$$

Since now $0 < t/2 \le T_0/2 \le 1$, the kernel $p_{t/2}$ satisfies (2.3) and (2.4). Thus, by (2.2), we have

$$\begin{aligned} |p_t(x,y)| &\le C \frac{1}{t^{2\alpha/\beta}} \int_M \left[1 + \frac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} \left[1 + \frac{\rho(z,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(z) \\ &\le C \frac{1}{t^{\alpha/\beta}} \left[1 + \frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} \end{aligned} \tag{2.5}$$

and, for $\rho(y, y') \le (t/2)^{1/\beta}$,

$$\begin{aligned} |p_t(x,y) - p_t(x,y')| &= \left| \int_M p_{t/2}(x,z) \left[p_{t/2}(z,y) - p_{t/2}(z,y')\right] d\mu(z) \right| \\ &\le C \frac{1}{t^{2\alpha/\beta}} \left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta} \int_M \left[1 + \frac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} \left[1 + \frac{\rho(z,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(z) \\ &\le C \frac{1}{t^{2\alpha/\beta}} \left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta} \left[1 + \frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \end{aligned}$$

For $(t/2)^{1/\beta} < \rho(y,y') \le t^{1/\beta}$, we have $1 + \frac{\rho(x,y')}{t^{1/\beta}} \simeq 1 + \frac{\rho(x,y)}{t^{1/\beta}}$, which combined with (2.5) implies that

$$\begin{aligned} |p_t(x,y) - p_t(x,y')| &\le C \left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta} (|p_t(x,y)| + |p_t(x,y')|) \\ &\le C \left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta} \frac{1}{t^{\alpha/\beta}} \left[1 + \frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \end{aligned} \tag{2.6}$$

Altogether, we deduce that (2.3) and (2.4) hold for $t \in (0, T_0]$ with $1 < T_0 \le 2$.

Let $N \in \mathbb{N}$ such that $2^{N-1} < T_0 \le 2^N$. Repeating the above arguments N times, we get (2.3) and (2.4) for $t \in (0, T_0]$. This finishes the proof of the lemma. □

Given any number $\nu \in (0,\infty)$, denote by $q_{\nu,t}(x,y)$ the kernel of the operator $(tL)^\nu e^{-tL}$. If $\nu = k$ is a positive integer, since

$$(tL)^k e^{-tL} = t^k \partial_t^k e^{-tL} = t^k \partial_t^k P_t,$$

so we occasionally write $t^k\partial_t^k P_t(x,y)$ as $q_{k,t}(x,y)$. Indeed, the kernel $q_{k,t}$ has short time upper bound and Hölder continuity similar to that of p_t, as follows.

Proposition 2.3. *Let $k \in \mathbb{Z}_+$, $\gamma \in (0,\beta)$ and $T_0 \in [1,\infty)$. Then, there exists a positive constant $C = C(k,\gamma,T_0,\alpha,\beta,\Theta)$ such that the following hold:*

(i) *For all $t \in (0,T_0]$ and $x,y \in M$,*

$$|q_{k,t}(x,y)| \le C\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{2.7}$$

(ii) *For all $t \in (0,T_0]$ and $x,y,y' \in M$ such that $\rho(y,y') \le t^{1/\beta}$,*

$$|q_{k,t}(x,y)-q_{k,t}(x,y')| \le C\left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta}\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}.$$

Proof. For $k = 0$, we see that (i) and (ii) are given exactly in Lemma 2.2. For $k \ge 1$, (i) follows directly from [8, Theorem 4] and Lemma 2.2(i).

Now we show (ii) for $k \ge 1$. For any $t > 0$, the semigroup property of p_t implies that

$$q_{k,t}(x,y) = 2^k\int_M q_{k,t/2}(x,z)p_{t/2}(z,y)\,d\mu(z).$$

For all $t \in (0,T_0]$ and $x,y,y' \in M$ such that $\rho(y,y') \le (t/2)^{1/\beta}$, we apply (i), Lemma 2.2(ii) and (2.2) to derive that

$$\begin{aligned}|q_{k,t}(x,y)-q_{k,t}(x,y')| &\le 2^k\int_M |q_{k,t/2}(x,z)||p_{t/2}(z,y)-p_{t/2}(z,y')|\,d\mu(z) \qquad (2.8)\\ &\le C\frac{1}{t^{2\alpha/\beta}}\left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta}\int_M\left[1+\frac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}\left[1+\frac{\rho(z,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}d\mu(z)\\ &\le C\frac{1}{t^{\alpha/\beta}}\left(\frac{\rho(y,y')}{t^{1/\beta}}\right)^{\Theta}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)},\end{aligned}$$

which proves (ii) for $\rho(y,y') \le (t/2)^{1/\beta}$. For $(t/2)^{1/\beta} < \rho(y,y') \le t^{1/\beta}$, by (i) and the same method used in (2.6), we deduce that (ii) also holds. □

For fractional derivatives of the heat semigroup, we have the following norm estimates, both for short and large time. Under the current assumptions (P1)-(P4) we can only obtain that (2.11) below holds for $p \in (1,\infty)$. To obtain (2.11) for $p = 1$ or ∞, we need the large time upper bound estimates of p_t; see Lemma 3.3 below for how to deal with it.

Proposition 2.4. (i) *If $p \in [1,\infty]$, then for all $t \in (0,\infty)$ and $f \in L^p$,*

$$\|e^{-tL}f\|_{L^p} \le C\|f\|_{L^p}. \tag{2.9}$$

(ii) *If* $T_0 \geq 1$, $k \in \mathbb{Z}_+$ *and* $p \in [1, \infty]$, *then for all* $t \in (0, T_0]$ *and* $f \in L^p$,

$$\|(tL)^k e^{-tL} f\|_{L^p} \leq C\|f\|_{L^p}. \tag{2.10}$$

(iii) *If* $k \in \mathbb{Z}_+$, $\nu \in [0, 1)$ *and* $p \in (1, \infty)$, *then for all* $t \in (0, \infty)$ *and* $f \in L^p$,

$$\|(tL)^{k+\nu} e^{-tL} f\|_{L^p} \leq C\|f\|_{L^p}. \tag{2.11}$$

Here the constant C *appearing in* (i)-(iii) *is positive and depends on* k, ν, p *and* T_0, *but independent of* t *and* f.

Proof. First we prove (i). For $p \in [1, \infty)$, by (P2) and Hölder's inequality, we see that for all $t > 0$,

$$\begin{aligned}
|e^{-tL} f(x)| &\leq \int_M |p_t(x, y)||f(y)|\, d\mu(y) \\
&\leq \left[\int_M |p_t(x, y)|\, d\mu(y)\right]^{1/p'} \left[\int_M |p_t(x, y)||f(y)|^p\, d\mu(y)\right]^{1/p} \\
&\leq C \left[\int_M |p_t(x, y)||f(y)|^p\, d\mu(y)\right]^{1/p},
\end{aligned}$$

which combined with Fubini's theorem further implies that

$$\|e^{-tL} f\|_{L^p}^p \leq C \int_M \int_M |p_t(x, y)||f(y)|^p\, d\mu(y)\, d\mu(x) \leq C\|f\|_{L^p}^p.$$

A modification of the above arguments yields (2.9) for $p = \infty$. This proves (i).

Now we prove (ii). If $k \in \mathbb{Z}_+$, $\nu = 0$ and $t \in (0, T_0]$ with $T_0 \geq 1$, then (2.1) and (2.7) imply that for any given $\gamma \in (0, \beta)$,

$$\int_M |q_{k,t}(x, y)|\, d\mu(y) \leq C \frac{1}{t^{\alpha/\beta}} \int_M \left[1 + \frac{\rho(x, y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\mu(y) \leq C.$$

Arguing as in the proof of (i), with the kernel p_t there replaced by $q_{k,t}$, we obtain (ii).

Next we prove (iii) for the case $k \in \mathbb{Z}_+$, $\nu = 0$ and $t \in (0, \infty)$. It was known from [8, Lemma 2] that the kernel $\{p_t\}_{t>0}$ can be analytically continued to complex times $z = t + is$ such that $t > 0$. We claim that there exists a constant C such that for all $f \in L^p$ with $p \in (1, \infty)$,

$$\|e^{-zL} f\|_{L^p} \leq C\|f\|_{L^p} \tag{2.12}$$

uniformly in the complex time z satisfying $|\arg z| < \theta_p$ with

$$\theta_p = \frac{\pi}{2}(1 - |2/p - 1|). \tag{2.13}$$

Assuming this claim for the moment, we show (2.11). Let Γ be the circle in the complex plane with center t and radius $\frac{1}{2} t \sin\theta_p$, so Cauchy's theorem states that

$$q_{k,t}(x, y) = t^k \partial_t^k P_t(x, y) = \frac{t^k\, k!}{2\pi i} \int_\Gamma \frac{p_z(x, y)}{(t - z)^{k+1}}\, dz$$

and thus

$$t^k L^k e^{-tL} f(x) = \frac{t^k\, k!}{2\pi i} \int_\Gamma \frac{e^{-zL} f(x)}{(t-z)^{k+1}}\, dz. \tag{2.14}$$

Observe that $|\arg z| < \theta_p$ for any $z \in \Gamma$. Taking L^p-norm in both sides of (2.14), we use (2.12) to derive that

$$\|(tL)^k e^{-tL} f\|_{L^p} \le \frac{t^k\, k!}{2\pi} \int_\Gamma \frac{\|e^{-zL} f\|_{L^p}}{|t-z|^{k+1}}\, |dz| \le C t^k \int_\Gamma \frac{\|f\|_{L^p}}{|t-z|^{k+1}}\, |dz| = C\|f\|_{L^p},$$

which proves (2.11) for $k \in \mathbb{Z}_+$, $\nu = 0$ and $t \in (0, \infty)$.

To verify the claim (2.12), we proceed as in the proof of [7, Theorem 1.4.2]. Let $r > 0$, $\theta \in (-\pi/2, \pi/2)$, $f \in L^1 \cap L^2$ and $g \in L^2 \cap L^\infty$. Consider the operator A_z defined on the strip $\{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$ by

$$\langle A_z f, g\rangle = \langle e^{-h(z)L} f, g\rangle \qquad with \;\; h(z) = re^{i\theta z}.$$

By the functional calculus of self-adjoint operators, we see that

$$|\langle A_z f, g\rangle| \le \|f\|_{L^2} \|g\|_{L^2}, \quad if \;\; \mathrm{Re}\, z = 1.$$

Property (P4) implies that

$$|\langle A_z f, g\rangle| \le C\|f\|_{L^1} \|g\|_{L^\infty}, \quad if \;\; \mathrm{Re}\, z = 0,$$

where the constant $C > 0$ is independent of z. Applying the interpolation theorem for analytic families of operators (see, for example [7, Section 1.1.6]) yields that for $0 < t < 1$,

$$\|A_t f\|_{L^{p(t)}} \le C\|f\|_{L^{p(t)}},$$

where $1/p(t) = 1 - t/2$. Equivalently, for $p \in (1, 2)$, we obtain (2.12) when $|\arg z| < \theta_p$ with θ_p defined as in (2.13). For $p \in (2, \infty)$, the corresponding result is obtained by duality. The case $p = 2$ follows from the functional calculus of L. Altogether, we obtain (2.12).

Finally we prove (iii) for $k \in \mathbb{Z}_+$, $\nu \in (0, 1)$ and $t \in (0, \infty)$. For all $\lambda \in (0, \infty)$, one has

$$\lambda^\nu = C_\nu \lambda^2 \int_0^\infty \xi^{1-\nu} e^{-\xi\lambda}\, d\xi,$$

where C_ν is some positive number depending only on ν. Consequently, for all $t \in (0, \infty)$,

$$(t\lambda)^{k+\nu} e^{-t\lambda} = C_\nu \int_0^\infty \frac{t^{k+\nu} \xi^{1-\nu}}{(t+\xi)^{k+2}} [(t+\xi)\lambda]^{k+2} e^{-(t+\xi)\lambda}\, d\xi$$

and the functional calculus gives us that

$$(tL)^{k+\nu} e^{-tL} = C_\nu \int_0^\infty \frac{t^{k+\nu} \xi^{1-\nu}}{(t+\xi)^{k+2}} [(t+\xi)L]^{k+2} e^{-(t+\xi)L}\, d\xi. \tag{2.15}$$

For all $f \in L^p$ with $p \in (1, \infty)$, since (2.11) holds for $k \in \mathbb{Z}_+$, $\nu = 0$ and $t \in (0, \infty)$ was already proved, it follows that

$$\|[(t+\xi)L]^{k+2} e^{-(t+\xi)L} f\|_{L^p} \le C\|f\|_{L^p}$$

uniformly in t and ξ, which further implies that

$$\begin{aligned}\|(tL)^{k+\nu}e^{-tL}f\|_{L^p} &\le C_\nu \int_0^\infty \frac{t^{k+\nu}\xi^{1-\nu}}{(t+\xi)^{k+2}}\|[(t+\xi)L]^{k+2}e^{-(t+\xi)L}f\|_{L^p}\,d\xi\\ &\le C\|f\|_{L^p}\int_0^\infty \frac{t^{k+\nu}\xi^{1-\nu}}{(t+\xi)^{k+2}}\,d\xi\\ &= C\|f\|_{L^p}.\end{aligned}$$

This proves (2.11) for the case $k \in \mathbb{Z}_+$, $\nu \in (0,1)$ and $t \in (0,\infty)$. □

As a simple consequence of Proposition 2.4, we have the following two corollaries, which will be useful in the following sections.

Corollary 2.5. *Let $T_0 \ge 1$, $p \in [1,\infty]$, $\nu \in [0,\infty)$ and $N, N_1 \in \mathbb{Z}_+$ such that $N - N_1 \ge \nu$. Then, there exists a positive constant C such that for all $\lambda, t \in (0, T_0]$ and $f \in L^p$,*

$$\|(\lambda L)^N e^{-2\lambda L}(tL)^\nu e^{-tL}f\|_{L^p} \le C\min\left\{\left(\frac{t}{\lambda}\right)^\nu, \left(\frac{\lambda}{t}\right)^{N-N_1-\nu}\right\}\|(\lambda L)^{N_1+\nu}e^{-\lambda L}f\|_{L^p}. \tag{2.16}$$

Proof. Write

$$(\lambda L)^N e^{-2\lambda L}(tL)^\nu e^{-tL}f = \frac{\lambda^{N-N_1-\nu}t^\nu}{(\lambda+t)^{N-N_1}}((\lambda+t)L)^{N-N_1}\,e^{-(\lambda+t)L}(\lambda L)^{N_1+\nu}e^{-\lambda L}f.$$

Since $N - N_1 \in \mathbb{Z}_+$, applying (2.10) implies that

$$\begin{aligned}\|(\lambda L)^N e^{-2\lambda L}(tL)^\nu e^{-tL}f\|_{L^p} &\le C\frac{\lambda^{N-N_1-\nu}t^\nu}{(\lambda+t)^{N-N_1}}\|(\lambda L)^{N_1+\nu}e^{-\lambda L}f\|_{L^p}\\ &\le C\min\left\{\left(\frac{t}{\lambda}\right)^\nu, \left(\frac{\lambda}{t}\right)^{N-N_1-\nu}\right\}\|(\lambda L)^{N_1+\nu}e^{-\lambda L}f\|_{L^p}.\end{aligned}$$

Thus, (2.16) holds. □

Corollary 2.6. *Let $p \in [1,\infty]$, $\nu \in [0,\infty)$ and $q \in (0,1]$. Then, there exists a positive constant C such that for all $j \in \mathbb{Z}_+$ and $f \in L^p$,*

$$\left[\int_{2^{-j}}^{2^{-j+1}}\|(tL)^\nu e^{-tL}f\|_{L^p}\,\frac{dt}{t}\right]^q \le C\int_{2^{-j-1}}^{2^{-j}}\|(tL)^\nu e^{-tL}f\|_{L^p}^q\,\frac{dt}{t}. \tag{2.17}$$

Proof. For each $t \in [2^{-j}, 2^{-j+1}]$, we apply (2.9) to derive that

$$\|(tL)^\nu e^{-tL}f\|_{L^p} \le 2^\nu\|e^{-(t-2^{-j})L}(2^{-j}L)^\nu e^{-2^{-j}L}f\|_{L^p} \le C\|(2^{-j}L)^\nu e^{-2^{-j}L}f\|_{L^p}.$$

Thus,

$$\left[\int_{2^{-j}}^{2^{-j+1}}\|(tL)^\nu e^{-tL}f\|_{L^p}\,\frac{dt}{t}\right]^q \le C\|(2^{-j}L)^\nu e^{-2^{-j}L}f\|_{L^p}^q. \tag{2.18}$$

Further, for any $\tau \in [2^{-j-1}, 2^{-j}]$, again using (2.9) yields that

$$\|(2^{-j}L)^{\gamma}e^{-2^{-j}L}f\|_{L^p}^q \le 2^{\gamma q}\|e^{-(2^{-j}-\tau)L}(\tau L)^{\gamma}e^{-\tau L}f\|_{L^p}^q \le C\|(\tau L)^{\gamma}e^{-\tau L}f\|_{L^p}^q,$$

which further gives us that

$$\|(2^{-j}L)^{\gamma}e^{-2^{-j}L}f\|_{L^p}^q \le C\int_{2^{-j-1}}^{2^{-j}}\|(\tau L)^{\gamma}e^{-\tau L}f\|_{L^p}^q\,\frac{d\tau}{\tau}. \tag{2.19}$$

Combining (2.18) and (2.19) yields (2.17). □

2.2 An inhomogeneous version of the continuous Calderón reproducing formulae

It was proved in [5, Theorem 2.3] that for all $p \in (1, \infty)$, $k \in \mathbb{N}$ and $f \in L^p$,

$$f = \frac{1}{(k-1)!}\int_0^{\infty} t^k L^k e^{-tL}f\,\frac{dt}{t} \qquad in\ \ L^p.$$

This is usually refereed to as the homogeneous version of the continuous Calderón-reproducing formulae. In this paper, since the upper bound and Hölder continuity are assumed only for short times, we therefore need the following inhomogeneous version.

Proposition 2.7. *Let $p \in [1, \infty)$ and $k \in \mathbb{N}$. Then, for any $f \in L^p$,*

$$f = \sum_{m=0}^{k-1}\frac{1}{m!}L^m e^{-L}f + \frac{1}{(k-1)!}\int_0^1 t^k L^k e^{-tL}f\,\frac{dt}{t}, \tag{2.20}$$

where the integral converges strongly in L^p.

To prove Proposition 2.7, we need the following lemma.

Lemma 2.8. *Let $m \in \mathbb{N}$, $p \in [1, \infty)$ and $f \in L^p$. Then,*

$$\lim_{t\to 0_+}\|e^{-tL}f - f\|_{L^p} = 0 \tag{2.21}$$

and

$$\lim_{t\to 0_+}\|(tL)^m e^{-tL}f\|_{L^p} = 0. \tag{2.22}$$

Proof. Since we are considering the behavior of e^{-tL} as $t \to 0_+$, we may restrict $t \in (0, 1]$. Denote by $C_c(M)$ the space of continuous functions with compact supports on M. Due to (2.10) and the density of $C_c(M)$ in L^p for $p \in [1, \infty)$, we only need to prove (2.21) and (2.22) for $f \in C_c(M)$.

First we let $f \in C_c(M)$ and prove (2.21). Assume that supp $f \subset B(x_0, R)$ for some $x_0 \in M$ and $R > 0$. For $\delta > 0$ to be determined later, by (P1), we write

$$e^{-tL}f(x) - f(x) = \int_M p_t(x, y)[f(y) - f(x)]\,d\mu(y), \tag{2.23}$$

so that

$$\begin{aligned}\|e^{-tL}f-f\|_{L^p}^p &= \int_{M\setminus B(x_0,2R)}\left|\int_M p_t(x,y)[f(y)-f(x)]\,d\mu(y)\right|^p d\mu(x)\\ &\quad+\int_{B(x_0,2R)}\left|\int_{\rho(x,y)<\delta} p_t(x,y)[f(y)-f(x)]\,d\mu(y)\right|^p d\mu(x)\\ &\quad+\int_{B(x_0,2R)}\left|\int_{\rho(x,y)\ge\delta} p_t(x,y)[f(y)-f(x)]\,d\mu(y)\right|^p d\mu(x)\\ &=: J_1+J_2+J_3.\end{aligned}$$

By (P2), we see that $\int_M |p_t(x,y)|\,d\mu(y)\le C$. For J_1, since supp $f\subset B(x_0,R)$, we use Hölder's inequality and Fubini's theorem to derive that

$$\begin{aligned}J_1 &= \int_{M\setminus B(x_0,2R)}\left|\int_M p_t(x,y)f(y)\,d\mu(y)\right|^p d\mu(x)\\ &\le C\int_{M\setminus B(x_0,2R)}\left[\int_{B(x_0,R)}|p_t(x,y)||f(y)|^p\,d\mu(y)\right]d\mu(x)\\ &\le C\int_M\left[\int_{\rho(x,y)>R}|p_t(x,y)|\,d\mu(x)\right]|f(y)|^p\,d\mu(y).\end{aligned}$$

Choose $\gamma\in(0,\beta)$. Then Lemma 2.2(i) and (2.1) imply that

$$\begin{aligned}\int_{\rho(x,y)>R}|p_t(x,y)|\,d\mu(x) &\le C\int_{\rho(x,y)>R}\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-\alpha-\gamma}d\mu(x)\\ &\le C\left(1+\frac{R}{t^{1/\beta}}\right)^{-\gamma/2}\int_{\rho(x,y)>R}\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-\alpha-\gamma/2}d\mu(x)\\ &= C\left(1+\frac{R}{t^{1/\beta}}\right)^{-\gamma/2},\end{aligned}\tag{2.24}$$

and thus

$$J_1\le C\|f\|_{L^p}^p\left(1+\frac{R}{t^{1/\beta}}\right)^{-\gamma/2},$$

which tends to zero as $t\to 0_+$. For any $\epsilon>0$, since $f\in C_c(M)$, there exists some $\delta>0$ such that $|f(x)-f(y)|<\epsilon$ when $\rho(x,y)<\delta$. From this, it follows that $J_2\le\epsilon^p\mu(B(x_0,2R))\le C\epsilon^pR^\alpha$, which tends to 0 if $\epsilon\to 0_+$. With the above chosen δ, we use (2.24), obtaining

$$\begin{aligned}J_3 &\le \int_{B(x_0,2R)}\int_{\rho(x,y)\ge\delta}|p_t(x,y)||f(y)-f(x)|^p\,d\mu(y)\,d\mu(x)\\ &\le C\int_M\int_{\rho(x,y)\ge\delta}|p_t(x,y)||f(y)|^p\,d\mu(x)\,d\mu(y)+C\int_M\int_{\rho(x,y)\ge\delta}|p_t(x,y)||f(x)|^p\,d\mu(y)\,d\mu(x)\\ &\le C\|f\|_{L^p}^p\left(1+\frac{\delta}{t^{1/\beta}}\right)^{-\gamma/2},\end{aligned}$$

which also tends to zero as $t\to 0_+$. Combining the estimates of J_1, J_2 and J_3 yields (2.21).

Now we prove (2.22). For $t \in (0, 1]$ and $x \in M$, it follows from (P1) that $\int_M q_{m,t}(x, y)\, d\mu(y) = 0$ when $m \in \mathbb{N}$. Thus, instead of (2.23), we have

$$(tL)^m e^{-tL} f(x) = \int_M q_{m,t}(x, y)[f(y) - f(x)]\, d\mu(y).$$

Next, following the proof of (2.21), we just need to replace the kernel p_t by $q_{m,t}$ and use Proposition 2.3(i) instead of Lemma 2.2(i), which leads to (2.22). The details are omitted. □

Proof of Proposition 2.7. Let $p \in [1, \infty)$ and $f \in L^p$. We claim that, for all $m \in \mathbb{Z}_+$,

$$\lim_{t \to 1_-} (tL)^m e^{-tL} f = L^m e^{-L} f \qquad \text{in } L^p. \tag{2.25}$$

To see the claim, we write

$$(tL)^m e^{-tL} f - L^m e^{-L} f = (tL)^m e^{-tL}(1 - e^{-(1-t)L}) f + (t^m - 1)L^m e^{-L} f.$$

By (2.10), we have $\|L^m e^{-L} f\|_{L^p} \le C\|f\|_{L^p}$ and thus

$$\lim_{t \to 1_-} \|(t^m - 1)L^m e^{-L} f\|_{L^p} = 0.$$

Again (2.10) implies that

$$\|(tL)^m e^{-tL}(1 - e^{-(1-t)L}) f\|_{L^p} \le C\|(1 - e^{-(1-t)L}) f\|_{L^p} = C\|f - e^{-(1-t)L} f\|_{L^p},$$

which tends to 0 by means of (2.21). This proves (2.25) and the claim.

By (2.21), (2.22) and (2.25), we use integration by parts to obtain

$$f = e^{-L} f - \int_0^1 \partial_t e^{-tL} f\, dt = e^{-L} f + \int_0^1 tLe^{-tL} f \,\frac{dt}{t}$$

in L^p, that is, (2.20) holds for $k = 1$. Now we assume that (2.20) holds in L^p for some $k \in \mathbb{N}$ and prove its validity for $k + 1$. Indeed, integration by parts again gives us that

$$\begin{aligned}
\frac{1}{k!} \int_0^1 t^{k+1} L^{k+1} e^{-tL} f \,\frac{dt}{t} &= \frac{1}{k!} \int_0^1 \left[k t^{k-1} L^k e^{-tL} f - \partial_t \left(t^k L^k e^{-tL} f \right) \right] dt \\
&= \frac{1}{k!} \left[k \int_0^1 t^k L^k e^{-tL} f \,\frac{dt}{t} - L^k e^{-L} f \right] \\
&= f - \sum_{m=0}^{k-1} \frac{1}{m!} L^m e^{-L} f - \frac{1}{k!} L^k e^{-L} f \\
&= f - \sum_{m=0}^{k} \frac{1}{m!} L^m e^{-L} f.
\end{aligned}$$

By (2.21), (2.22) and (2.25), all the equalities in the above formula hold in L^p. This proves that (2.20) holds in L^p for $k + 1$. This finishes the proof of the proposition. □

2.3 Norm equivalence

Proposition 2.9. *Let $s \in (0,\infty)$, $p \in (1,\infty)$ and $q \in (0,\infty]$. Then, the norms $\|\cdot\|_{B^s_{p,q}}$ defined in (1.1) are equivalent for any two values of k satisfying $k > s/\beta$.*

Proof. Let $s/\beta < k_1 < k_2 < \infty$ and define

$$\|f\|_{B^s_{p,q}(k_i)} = \|f\|_{L^p} + \left(\int_0^\infty \left[t^{-s/\beta}\|(tL)^{k_i}e^{-tL}f\|_{L^p}\right]^q \frac{dt}{t}\right)^{1/q}, \qquad i = 1,\ 2. \tag{2.26}$$

We need to show that

$$\|f\|_{B^s_{p,q}(k_1)} \simeq \|f\|_{B^s_{p,q}(k_2)}.$$

Because of (2.11), the integral sign $\int_0^\infty$ in (2.26) can be equivalently replaced by $\int_0^c$, where c can be any constant in $(0,\infty]$. By $k_2 > k_1$ and (2.11), we see that

$$\|(tL)^{k_2}e^{-tL}f\|_{L^p} = \|(tL)^{k_2-k_1}e^{-\frac{1}{2}tL}(tL)^{k_1}e^{-\frac{1}{2}tL}f\|_{L^p} \le C\|(tL)^{k_1}e^{-\frac{1}{2}tL}f\|_{L^p},$$

so that

$$\|f\|_{B^s_{p,q}(k_2)} \le C\|f\|_{L^p} + C\left(\int_0^c \left[t^{-s/\beta}\|(tL)^{k_1}e^{-\frac{1}{2}tL}f\|_{L^p}\right]^q \frac{dt}{t}\right)^{1/q} \simeq \|f\|_{B^s_{p,q}(k_1)}.$$

It remains to prove

$$\|f\|_{B^s_{p,q}(k_1)} \le C\|f\|_{B^s_{p,q}(k_2)}. \tag{2.27}$$

Fix some integer $N > k_2$. By Proposition 2.7 and a change of variables, we write

$$(tL)^{k_1}e^{-tL}f = \sum_{m=0}^{N-1}\frac{1}{m!}\,L^m e^{-L}(tL)^{k_1}e^{-tL}f + \frac{2^N}{(N-1)!}\int_0^{\frac{1}{2}}(\lambda L)^N e^{-2\lambda L}(tL)^{k_1}e^{-tL}f\,\frac{d\lambda}{\lambda}. \tag{2.28}$$

For any $0 \le m \le N-1$, applying (2.10) yields that

$$\|L^m e^{-L}(tL)^{k_1}e^{-tL}f\|_{L^p} = t^{k_1}\|L^{m+k_1}e^{-L}e^{-tL}f\|_{L^p} \le Ct^{k_1}\|f\|_{L^p}.$$

With this, taking the L^p-norm on both sides of (2.28) and applying (2.16), we obtain

$$\begin{aligned}
\|(tL)^{k_1}e^{-tL}f\|_{L^p} &\le C\left[t^{k_1}\|f\|_{L^p} + \int_0^{\frac{1}{2}}\|(\lambda L)^N e^{-2\lambda L}(tL)^{k_1}e^{-tL}f\|_{L^p}\frac{d\lambda}{\lambda}\right]\\
&\le C\left[t^{k_1}\|f\|_{L^p} + \int_0^{\frac{1}{2}}\min\left\{\left(\frac{t}{\lambda}\right)^{k_1}, \left(\frac{\lambda}{t}\right)^{N-k_2}\right\}\|(\lambda L)^{k_2}e^{-\lambda L}f\|_{L^p}\frac{d\lambda}{\lambda}\right]\\
&=: J_1(t) + J_2(t).
\end{aligned}$$

Since $k_1 > s/\beta$, it follows that

$$\left(\int_0^1 \left[t^{-s/\beta}J_1(t)\right]^q \frac{dt}{t}\right)^{1/q} \le C\|f\|_{L^p}\left(\int_0^1 t^{(k_1-s/\beta)q}\frac{dt}{t}\right)^{1/q} \le C\|f\|_{L^p}.$$

Thus, to obtain (2.27), it suffices to prove that

$$\left(\int_0^1 \left[t^{-s/\beta} J_2(t)\right]^q \frac{dt}{t}\right)^{1/q} \le C\|f\|_{B^s_{p,q}(k_2)}. \tag{2.29}$$

First we prove (2.29) for $q > 1$. Write

$$\left[t^{-s/\beta} J_2(t)\right]^q = C\left[\int_0^{\frac{1}{2}} \left(\frac{t}{\lambda}\right)^{-s/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{k_1}, \left(\frac{\lambda}{t}\right)^{N-k_2}\right\} \lambda^{-s/\beta}\|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p} \frac{d\lambda}{\lambda}\right]^q. \tag{2.30}$$

Since $k_1 > s/\beta$, one can verify that

$$\int_0^{\frac{1}{2}} \left(\frac{t}{\lambda}\right)^{-s/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{k_1}, \left(\frac{\lambda}{t}\right)^{N-k_2}\right\} \frac{d\lambda}{\lambda} < \infty.$$

Further, applying Hölder's inequality to (2.30) yields that

$$\left[t^{-s/\beta} J_2(t)\right]^q \le C\int_0^{\frac{1}{2}} \left(\frac{t}{\lambda}\right)^{-s/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{k_1}, \left(\frac{\lambda}{t}\right)^{N-k_2}\right\} \left[\lambda^{-s/\beta}\|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda},$$

which combined with Fubini's theorem implies that

$$\begin{aligned}\int_0^1 \left[t^{-s/\beta} J_2(t)\right]^q \frac{dt}{t} &\le C\int_0^1\int_0^1 \left(\frac{t}{\lambda}\right)^{-s/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{k_1}, \left(\frac{\lambda}{t}\right)^{N-k_2}\right\} \left[\lambda^{-s/\beta}\|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda}\frac{dt}{t}\\ &\le C\int_0^1 \left[\lambda^{-s/\beta}\|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda}.\end{aligned}$$

This proves (2.29) for $q > 1$. If $q = \infty$, one may easily verify (2.29) by using the above argument.

Now we show (2.29) for $q \in (0, 1]$. We split the integral interval in the right of (2.8) and apply the fact

$$\left(\sum |a_j|\right)^\kappa \le \sum |a_j|^\kappa \qquad \textit{when } \kappa \in (0,1], \tag{2.31}$$

and Corollary 2.6. It follows that

$$\begin{aligned}\left[t^{-s/\beta} J_2(t)\right]^q &\simeq \sum_{j=1}^\infty \left[\left(\frac{t}{2^{-j}}\right)^{-s/\beta} \min\left\{\left(\frac{t}{2^{-j}}\right)^{k_1}, \left(\frac{2^{-j}}{t}\right)^{N-k_2}\right\} 2^{js/\beta}\right]^q \int_{2^{-j-1}}^{2^{-j}} \|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p}^q \frac{d\lambda}{\lambda}\\ &\le C\int_0^{\frac{1}{4}} \left(\frac{t}{\lambda}\right)^{-sq/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{k_1 q}, \left(\frac{\lambda}{t}\right)^{(N-k_2)q}\right\} \left[\lambda^{-s/\beta}\|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda}.\end{aligned}$$

By Fubini's theorem and the fact

$$\int_0^{\frac{1}{4}} \left(\frac{t}{\lambda}\right)^{-sq/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{k_1 q}, \left(\frac{\lambda}{t}\right)^{(N-k_2)q}\right\} \frac{d\lambda}{\lambda} < \infty,$$

we further deduce that

$$\begin{aligned}\int_0^1 \left[t^{-s/\beta} J_2(t)\right]^q \frac{dt}{t} &\le C\int_0^1\int_0^{\frac{1}{4}} \left(\frac{t}{\lambda}\right)^{-sq/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{k_1 q}, \left(\frac{\lambda}{t}\right)^{(N-k_2)q}\right\} \left[\lambda^{-s/\beta}\|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda}\frac{dt}{t}\\ &\le C\int_0^{\frac{1}{4}} \left[\lambda^{-s/\beta}\|(\lambda L)^{k_2} e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda}.\end{aligned}$$

This proves (2.29) for $q \in (0, 1]$. We complete the proof the proposition. ◻

3 Proofs of Theorems 1.5 and 1.6

3.1 Proof of Theorem 1.5(a)

Proof of Theorem 1.5(a). According to Proposition 2.9, the norm $\|\cdot\|_{B^s_{p,q}}$ defined in (1.1) is independent of the choices of k satisfying $k > s/\beta$. Thus we choose $k_0 \in \mathbb{N}$ such that $k_0 > s/\beta + 2$. Fix $k = k_0 + m/2$, so that $k > m/2 + s/\beta$. In the proof below, we shall consider the norm $\|\cdot\|_{B^s_{p,q}}$ defined via such a k as in (1.1).

Let $f \in \mathrm{Dom}(L^{m/2}) \cap L^p$. First we will prove that

$$\|f\|_{B^{m\beta/2+s}_{p,q}} \le C\|f\|_{\Lambda^{m,s}_{p,q}}. \tag{3.1}$$

It follows from (P1) that the kernel of the operator $(tL)^{k_0}e^{-tL}$, which is denoted by $q_{k_0,t}(x,y)$, satisfies that

$$\int_M q_{k_0,t}(x,y)\, d\mu(y) = 0.$$

By $k = k_0 + m/2$, we have $(tL)^k e^{-tL} f = t^{m/2}(tL)^{k_0}e^{-tL}(L^{m/2} f)$ and thus

$$(tL)^k e^{-tL} f(x) = t^{m/2}\int_M q_{k_0,t}(x,y)\,[L^{m/2} f(y) - L^{m/2} f(x)]\, d\mu(y).$$

Since $0 < s < \min\{\Theta, \beta/2\}$, we choose $\gamma \in (s,\beta)$. With this γ, applying Proposition 2.3(i) to $q_{k_0,t}(x,y)$ yields that for $t \in (0,1]$ and $x \in M$,

$$\begin{aligned}|(tL)^k e^{-tL} f(x)| &\le Ct^{m/2}\int_M \frac{1}{t^{\alpha/\beta}}\left[1 + \frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} |L^{m/2} f(y) - L^{m/2} f(x)|\, d\mu(y)\\ &\le Ct^{m/2}\sum_{i=0}^{\infty} 2^{-i\gamma} \fint_{\rho(x,y)\sim 2^i t^{1/\beta}} |L^{m/2} f(y) - L^{m/2} f(x)|\, d\mu(y).\end{aligned}$$

where the notation $\rho(x,y) \sim 2^i t^{1/\beta}$ means $2^{i-1}t^{1/\beta} \le \rho(x,y) < 2^i t^{1/\beta}$ when $i \ge 1$ and $\rho(x,y) < t^{1/\beta}$ when $i = 0$. Further, by the fact $\sum_{i=0}^{\infty} 2^{-i\gamma} < \infty$, applying Hölder's inequality twice yields that

$$|(tL)^k e^{-tL} f(x)|^p \le Ct^{mp/2}\sum_{i=0}^{\infty} 2^{-i\gamma}\fint_{\rho(x,y)<2^i t^{1/\beta}} |L^{m/2} f(y) - L^{m/2} f(x)|^p\, d\mu(y),$$

and

$$\begin{aligned}&t^{-m/2-s/\beta}\|(tL)^k e^{-tL} f\|_{L^p}\\ &\le C\left[\sum_{i=0}^{\infty} 2^{i(s-\gamma)}\int_M \fint_{\rho(x,y)<2^i t^{1/\beta}} \frac{|L^{m/2} f(y) - L^{m/2} f(x)|^p}{(2^i t^{1/\beta})^{sp}}\, d\mu(y)\, d\mu(x)\right]^{1/p}.\end{aligned} \tag{3.2}$$

Consider first the case $q > p$. Raising both sides of (3.2) to the power q, and then applying $s < \gamma$ and Hölder's inequality with exponents $\frac{1}{q/p} + \frac{1}{(q/p)'} = 1$, we conclude that

$$\left[t^{-m/2-s/\beta}\|(tL)^k e^{-tL} f\|_{L^p}\right]^q \le C\sum_{i=0}^{\infty} 2^{i(s-\gamma)}\left[\int_M \fint_{\rho(x,y)<2^i t^{1/\beta}} \frac{|L^{m/2} f(y) - L^{m/2} f(x)|^p}{(2^i t^{1/\beta})^{sp}}\, d\mu(y)\, d\mu(x)\right]^{q/p},$$

so that

$$
\begin{aligned}
&\int_0^1 \left[t^{-m/2-s/\beta}\|(tL)^k e^{-tL} f\|_{L^p}\right]^q \frac{dt}{t} \\
&\le C\sum_{i=0}^{\infty} 2^{i(s-\gamma)} \int_0^1 \left[\int_M \fint_{\rho(x,y)<2^i t^{1/\beta}} \frac{|L^{m/2} f(y) - L^{m/2} f(x)|^p}{(2^i t^{1/\beta})^{sp}}\, d\mu(y)\, d\mu(x)\right]^{q/p} \frac{dt}{t} \\
&\le C\sum_{i=0}^{\infty} 2^{i(s-\gamma)} \int_0^{2^i} \left[\int_M \fint_{\rho(x,y)<r} \frac{|L^{m/2} f(y) - L^{m/2} f(x)|^p}{r^{sp}}\, d\mu(y)\, d\mu(x)\right]^{q/p} \frac{dr}{r} \\
&\le C\|f\|^q_{\dot{\Lambda}^{m,s}_{p,q}},
\end{aligned}
$$

which proves (3.1) for the case $q > p$. When $q = \infty$, a modification of the above arguments also implies (3.1). If $q \in (0, p]$, then by (3.2) and (2.31), we see that

$$
\begin{aligned}
&\int_0^1 \left[t^{-m/2-s/\beta}\|(tL)^k e^{-tL} f\|_{L^p}\right]^q \frac{dt}{t} \\
&\le C\int_0^1 \sum_{i=0}^{\infty} 2^{i(s-\gamma)q/p} \left[\int_M \fint_{\rho(x,y)<2^i t^{1/\beta}} \frac{|L^{m/2} f(y) - L^{m/2} f(x)|^p}{(2^i t^{1/\beta})^{sp}}\, d\mu(y)\, d\mu(x)\right]^{q/p} \frac{dt}{t} \\
&= C\sum_{i=0}^{\infty} 2^{i(s-\gamma)q/p} \int_0^{2^i} \left[\int_M \fint_{\rho(x,y)<r} \frac{|L^{m/2} f(y) - L^{m/2} f(x)|^p}{r^{sp}}\, d\mu(y)\, d\mu(x)\right]^{q/p} \frac{dr}{r} \\
&\le C\|f\|^q_{\dot{\Lambda}^{m,s}_{p,q}},
\end{aligned}
$$

so (3.1) also holds when $q \in (0, p]$. Altogether, we obtain (3.1).

To finish the proof of Theorem 1.5(a), we still need to prove that

$$
\|f\|_{\dot{\Lambda}^{m,s}_{p,q}} \le C\|f\|_{B^{m\beta/2+s}_{p,q}}. \tag{3.3}
$$

To this end, let $N \in \mathbb{N}$ such that $N > k_0 + k$, where $k = k_0 + m/2$ with k_0 being a large positive integer. For $f \in \mathrm{Dom}(L^{m/2}) \cap L^p$, we write

$$
L^{m/2} f = \sum_{i=0}^{N-1} \frac{1}{i!} L^{i+m/2} e^{-L} f + \frac{2^N}{(N-1)!} \int_0^{\frac{1}{2}} t^N L^{N+m/2} e^{-2tL} f\, \frac{dt}{t}.
$$

Noticing that

$$
L^{i+m/2} e^{-L} f = L^i e^{-\frac{1}{2}L} L^{m/2} e^{-\frac{1}{2}L} f
$$

and

$$
t^N L^{N+m/2} e^{-2tL} f = t^{-m/2} (tL)^{N-k_0} e^{-tL} (tL)^k e^{-tL} f,
$$

we have

$$
L^{m/2} f(x) = \sum_{i=0}^{N-1} \frac{2^i}{i!} \int_M q_{i,1/2}(x,z)\, L^{m/2} e^{-\frac{1}{2}L} f(z)\, d\mu(z) + C\int_0^{\frac{1}{2}} \int_M t^{-m/2} q_{N-k_0,t}(x,z)\, (tL)^k e^{-tL} f(z)\, d\mu(z)\, \frac{dt}{t}
$$

and thus

$$
\begin{aligned}
|L^{m/2} f(x) - L^{m/2} f(y)| &= \sum_{i=0}^{N-1} \frac{2^i}{i!} \int_M [q_{i,1/2}(x,z) - q_{i,1/2}(y,z)]\, L^{m/2} e^{-\frac{1}{2}L} f(z)\, d\mu(z) \\
&\quad + C \int_0^{\frac{1}{2}} \int_M t^{-m/2} [q_{N-k_0,t}(x,z) - q_{N-k_0,t}(y,z)]\, (tL)^k e^{-tL} f(z)\, d\mu(z) \frac{dt}{t}.
\end{aligned}
$$

Let $\gamma \in (s,\beta)$. For any $x,y,z \in M$ such that $\rho(x,y) < r$, by Proposition 2.3, one has

$$
\begin{aligned}
|q_{N-k_0,t}(x,z) - q_{N-k_0,t}(y,z)| &\le \begin{cases} C\dfrac{1}{t^{\alpha/\beta}} \left(\dfrac{\rho(x,y)}{t^{1/\beta}}\right)^{\Theta} \left[1 + \dfrac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} & if\ r \le t^{1/\beta} \\ C\dfrac{1}{t^{\alpha/\beta}} \left(\left[1 + \dfrac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} + \left[1 + \dfrac{\rho(y,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}\right) & if\ r > t^{1/\beta} \end{cases} \\
&\le C \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\} \mathcal{J}_t(x,y,z),
\end{aligned}
\tag{3.4}
$$

where

$$
\mathcal{J}_t(x,y,z) := \frac{1}{t^{\alpha/\beta}} \left(\left[1 + \frac{\rho(x,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} + \left[1 + \frac{\rho(y,z)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}\right).
$$

Similar to (3.4), one has

$$
|q_{i,1/2}(x,z) - q_{i,1/2}(y,z)| \le C \min\left\{1,\ r^{\Theta}\right\} \mathcal{J}_{\frac{1}{2}}(x,y,z), \qquad 0 \le i \le N-1.
$$

Therefore,

$$
\begin{aligned}
& r^{-s} |L^{m/2} f(x) - L^{m/2} f(y)| \\
&\quad \le C r^{-s} \min\left\{1,\ r^{\Theta}\right\} \int_M \mathcal{J}_{\frac{1}{2}}(x,y,z) |L^{m/2} e^{-\frac{1}{2}L} f(z)|\, d\mu(z) \\
&\qquad + C \int_0^{\frac{1}{2}} \int_M (r^{-1} t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\} \mathcal{J}_t(x,y,z)\, t^{-(m/2+s/\beta)} |(tL)^k e^{-tL} f(z)|\, d\mu(z) \frac{dt}{t} \\
&\quad =: Z_1 + Z_2.
\end{aligned}
$$

From (2.1), it follows that for all $t \in (0,\infty)$,

$$
\int_M \mathcal{J}_t(x,y,z)\, d\mu(z) \le C < \infty.
$$

Applying Hölder's inequality yields that

$$
Z_1 \le C r^{-s} \min\left\{1,\ r^{\Theta}\right\} \left[\int_M \mathcal{J}_{\frac{1}{2}}(x,y,z) |L^{m/2} e^{-\frac{1}{2}L} f(z)|^p\, d\mu(z)\right]^{1/p}.
$$

By the assumption $s < \Theta$, we see that for all $r \in (0,\infty)$,

$$
\int_0^{\infty} (r^{-1} t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\} \frac{dt}{t} \le C, \tag{3.5}
$$

so

$$\int_0^{\frac{1}{2}} \int_M (r^{-1}t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\} \mathcal{J}_t(x,y,z)\ d\mu(z)\, \frac{dt}{t} \le C < \infty$$

uniformly in the variables z and r. Applying this and Hölder's inequality yields that

$$Z_2 \le C\left[\int_0^{\frac{1}{2}} \int_M (r^{-1}t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\} \mathcal{J}_t(x,y,z)\ \left[t^{-(m/2+s/\beta)}|(tL)^k e^{-tL} f(z)|\right]^p\ d\mu(z)\, \frac{dt}{t}\right]^{1/p}$$

Combining the estimates of Z_1 and Z_2 implies that

$$\begin{aligned}
&r^{-sp}|L^{m/2} f(x) - L^{m/2} f(y)|^p \\
&\le Cr^{-sp} \min\left\{1,\ r^{\Theta p}\right\} \int_M \mathcal{J}_{\frac{1}{2}}(x,y,z)|L^{m/2}e^{-\frac{1}{2}L} f(z)|^p\, d\mu(z) \\
&+C \int_0^{\frac{1}{2}} \int_M (r^{-1}t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\}\ \mathcal{J}_t(x,y,z)\ \left[t^{-(m/2+s/\beta)}|(tL)^k e^{-tL} f(z)|\right]^p\ d\mu(z)\, \frac{dt}{t}.
\end{aligned}$$

By this, Fubini's theorem and $\int_M \fint_{\rho(x,y)<r} \mathcal{J}_t(x,y,z)\, d\mu(y)\, d\mu(x) \le C$ uniformly in $r \in (0,\infty)$, $t \in (0, \frac{1}{2}]$ and $z \in M$, we deduce that

$$\begin{aligned}
&\int_M \fint_{\rho(x,y)<r} \frac{|L^{m/2} f(y) - L^{m/2} f(x)|^p}{r^{sp}}\, d\mu(y)\, d\mu(x) \\
&\le Cr^{-sp} \min\left\{1,\ r^{\Theta p}\right\} \|L^{m/2}e^{-\frac{1}{2}L} f\|_{L^p}^p \\
&\quad +C \int_0^{\frac{1}{2}} (r^{-1}t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\}\ \|t^{-(m/2+s/\beta)}(tL)^k e^{-tL} f\|_{L^p}^p\, \frac{dt}{t} \\
&=: Y_1(r) + Y_2(r).
\end{aligned}$$

Since (2.10) implies that $\|L^{m/2}e^{-\frac{1}{2}L} f\|_{L^p} \le C\|f\|_{L^p}$, we have $\|Y_1\|_{L^\infty} \le \|f\|_{L^p}$ and

$$\left(\int_0^\infty [Y_1(r)]^{q/p}\ \frac{dr}{r}\right)^{1/q} \le C\|f\|_{L^p} \left(\int_0^\infty r^{-sq} \min\left\{1,\ r^{\Theta q}\right\}\ \frac{dr}{r}\right)^{1/q} \le C\|f\|_{L^p}.$$

Thus, to obtain (3.3), we only need to verify that

$$\left(\int_0^\infty [Y_2(r)]^{q/p}\ \frac{dr}{r}\right)^{1/q} \le C\|f\|_{B^{m\beta/2+s}_{p,q}}. \tag{3.6}$$

If $q/p \ge 1$, then by (3.5) and Hölder's inequality with exponents $\frac{1}{q/p} + \frac{1}{(q/p)'} = 1$, we obtain

$$[Y_2(r)]^{q/p} \le C \int_0^{\frac{1}{2}} (r^{-1}t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\}\ \|t^{-(m/2+s/\beta)}(tL)^k e^{-tL} f\|_{L^p}^q\, \frac{dt}{t}.$$

Applying Fubini's theorem and the fact that

$$\int_0^\infty (r^{-1}t^{1/\beta})^s \min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\}\ \frac{dr}{r} \le C$$

uniformly in t, we obtain

$$\int_0^\infty [Y_2(r)]^{q/p}\ \frac{dr}{r} \le C\int_0^{\frac{1}{2}} \|t^{-(m/2+s/\beta)}(tL)^k e^{-tL} f\|_{L^p}^q\ \frac{dt}{t} \le C\|f\|^q_{B^{m\beta/2+s}_{p,q}}.$$

This proves (3.6) for the case $\infty > q/p \ge 1$. The case $q = \infty$ follows by a simple modification of the above arguments.

Now we prove (3.6) for the case $q/p < 1$. Using (2.31) and Corollary 2.6, we have that

$$\begin{aligned}
[Y_2(r)]^{q/p} &\simeq \sum_{j=1}^\infty \left[(r2^{j/\beta})^{-s}\min\left\{1,\ (r2^{j/\beta})^{\Theta}\right\}\right]^{q/p}\ 2^{jq(m/2+s/\beta)}\left[\int_{2^{-j-1}}^{2^{-j}} \|(tL)^k e^{-tL} f\|_{L^p}^p\ \frac{dt}{t}\right]^{q/p}\\
&\le C\sum_{j=1}^\infty \left[(r2^{j/\beta})^{-s}\min\left\{1,\ (r2^{j/\beta})^{\Theta}\right\}\right]^{q/p}\ 2^{jq(m/2+s/\beta)}\int_{2^{-j-2}}^{2^{-j-1}} \|(tL)^k e^{-tL} f\|_{L^p}^q\ \frac{dt}{t}\\
&\le C\int_0^{\frac{1}{4}} \left[(rt^{-1/\beta})^{-s}\min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\}\right]^{q/p}\ t^{-q(m/2+s/\beta)}\|(tL)^k e^{-tL} f\|_{L^p}^q\ \frac{dt}{t}.
\end{aligned}$$

Notice that

$$\int_0^\infty \left[(rt^{-1/\beta})^{-s}\min\left\{1,\ (rt^{-1/\beta})^{\Theta}\right\}\right]^{q/p}\ \frac{dr}{r} \le C$$

uniformly in t. Applying Fubini's theorem implies that

$$\int_0^\infty [Y_2(r)]^{q/p}\ \frac{dr}{r} \le C\int_0^{\frac{1}{4}} \|t^{-(m/2+s/\beta)}(tL)^k e^{-tL} f\|_{L^p}^q\ \frac{dt}{t} \le C\|f\|^q_{B^{m\beta/2+s}_{p,q}}.$$

This proves (3.6) and thus (3.3). We complete the proof of the Theorem 1.5(a). □

3.2 Proof of Theorem 1.5(b)

To prove Theorem 1.5(b), we shall use Theorem 1.5(a) and the following two density lemmas.

Lemma 3.1. *Let all the assumptions be as in Theorem 1.5(b). Let $m \in (0,\infty)$, $s \in \mathbb{R}$, $p \in (1,\infty)$ and $q \in (0,\infty)$. Then,* $\mathrm{Dom}(L^{m/2}) \cap B^s_{p,q}$ *is a dense subset of $B^s_{p,q}$.*

Proof. Let $f \in B^s_{p,q}$. Obviously, $f \in L^p$. Given a large number $N \in \mathbb{N}$, which will be determined later, we apply Proposition 2.7 to deduce that

$$f = \sum_{i=0}^{N-1} \frac{1}{i!} L^i e^{-L} f + \frac{1}{(N-1)!}\int_0^1 (\lambda L)^N e^{-\lambda L} f\ \frac{d\lambda}{\lambda} \quad \textit{in}\ \ L^p. \tag{3.7}$$

Let ϕ be a smooth function defined on $[0,\infty)$ such that $0 \le \phi \le 1$, $\phi(t) = 1$ on $[0,1]$ and $\phi(t) = 0$ when $t \notin [0,2]$. Fix $x_0 \in M$. Given any $\sigma, \eta \in (0,1)$, let $\phi_\eta(x) = \phi(\eta\rho(x,x_0))$ and

$$f_{\sigma,\eta} := \sum_{i=0}^{N-1} \frac{1}{i!} L^i e^{-L}(f\phi_\eta) + \frac{1}{(N-1)!}\int_\sigma^1 (\lambda L)^N e^{-tL}(f\phi_\eta)\ \frac{d\lambda}{\lambda}.$$

To verify that every $f_{\sigma,\eta} \in \mathrm{Dom}(L^{m/2})$, we apply (2.11) to derive that

$$\begin{aligned}\|L^{m/2} f_{\sigma,\eta}\|_{L^2} &\le \sum_{i=0}^{N-1} \frac{1}{i!}\|L^{i+m/2}e^{-L}(f\phi_\eta)\|_{L^2} + \frac{1}{(N-1)!}\int_\sigma^1 \lambda^{-m/2}\|(\lambda L)^{N+m/2}e^{-\lambda L}(f\phi_\eta)\|_{L^2}\,\frac{d\lambda}{\lambda}\\ &\le C_{N,\sigma}\,\|f\phi_\eta\|_{L^2},\end{aligned}$$

which if finite since $f\phi_\eta \in L^2$. Thus $f_{\sigma,\eta} \in \mathrm{Dom}(L^{m/2})$.

Now we prove that $f_{\sigma,\eta}$ tends to f in L^p as $\sigma,\eta \to 0_+$. According to Proposition 2.7, the sequence $\{g_\sigma\}_{\sigma>0}$ given by

$$g_\sigma := \sum_{i=0}^{N-1} \frac{1}{i!}L^i e^{-L} f + \frac{1}{(N-1)!}\int_\sigma^1 (\lambda L)^N e^{-\lambda L} f\,\frac{d\lambda}{\lambda}$$

converges to f in L^p as $\sigma \to 0_+$. Thus, it suffices to show that for any fixed small $\sigma > 0$,

$$\lim_{\eta\to 0_+} \|f_{\sigma,\eta} - g_\sigma\|_{L^p} = 0. \tag{3.8}$$

By (2.10), we obtain

$$\begin{aligned}\|f_{\sigma,\eta} - g_\sigma\|_{L^p} &\le \sum_{i=0}^{N-1} \frac{1}{i!}\|L^i e^{-L}(f(1-\phi_\eta))\|_{L^p} + \frac{1}{(N-1)!}\int_\sigma^1 \|(\lambda L)^N e^{-\lambda L}(f(1-\phi_\eta))\|_{L^p}\,\frac{d\lambda}{\lambda}\\ &\le C_{N,\sigma}\,\|f(1-\phi_\eta)\|_{L^p},\end{aligned}$$

which tends to 0 as $\eta \to 0_+$. This proves (3.8). Thus $f_{\sigma,\eta}$ tends to f in L^p as $\sigma,\eta \to 0_+$.

Due to the above arguments and Proposition 2.9, to prove that $f_{\sigma,\eta}$ tends to f in $B^s_{p,q}$ as $\sigma,\eta \to 0_+$, we only need to show that

$$\lim_{\sigma,\eta\to 0_+} \left(\int_0^1 \left[t^{-s/\beta}\|(tL)^k e^{-tL}(f - f_{\sigma,\eta})\|_{L^p}\right]^q \frac{dt}{t}\right)^{1/q} = 0$$

for some fixed $k \in \mathbb{N}$ such that $k > s/\beta$. To this end, write

$$|(tL)^k e^{-tL}(f - f_{\sigma,\eta})| \le |(tL)^k e^{-tL}(f - g_\sigma)| + |(tL)^k e^{-tL}(g_\sigma - f_{\sigma,\eta})|.$$

Thus, it suffices to show that

$$\lim_{\sigma\to 0_+} \int_0^1 \left[t^{-s/\beta}\|(tL)^k e^{-tL}(f - g_\sigma)\|_{L^p}\right]^q \frac{dt}{t} = 0 \tag{3.9}$$

and that, for any fixed σ,

$$\lim_{\eta\to 0_+} \int_0^1 \left[t^{-s/\beta}\|(tL)^k e^{-tL}(g_\sigma - f_{\sigma,\eta})\|_{L^p}\right]^q \frac{dt}{t} = 0. \tag{3.10}$$

First we prove (3.9). Applying (3.7) with some integer $N > k$ and the definition of g_σ yields that

$$\begin{aligned}t^{-s/\beta}\|(tL)^k e^{-tL}(f - g_\sigma)\|_{L^p} &\le \frac{t^{-s/\beta}}{(N-1)!}\int_0^\sigma \|(\lambda L)^N e^{-\lambda L}(tL)^k e^{-tL} f\|_{L^p}\,\frac{d\lambda}{\lambda}\\ &= Ct^{-s/\beta}\int_0^{\frac{1}{2}\sigma} \|(\lambda L)^N e^{-2\lambda L}(tL)^k e^{-tL} f\|_{L^p}\,\frac{d\lambda}{\lambda}.\end{aligned}$$

Then, by Corollary 2.6, one has

$$\begin{aligned} &t^{-s/\beta}\|(tL)^k e^{-tL}(f-g_\sigma)\|_{L^p} \\ &\quad\le C\int_0^{\frac12\sigma}\left(\frac{t}{\lambda}\right)^{-s/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^k,\ \left(\frac{\lambda}{t}\right)^{N-k}\right\}\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}\,\frac{d\lambda}{\lambda}. \end{aligned} \tag{3.11}$$

If $q \ge 1$, then the assumption $k > s/\beta$ implies that

$$\int_0^{\frac12\sigma}\left(\frac{t}{\lambda}\right)^{-s/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^k,\ \left(\frac{\lambda}{t}\right)^{N-k}\right\}\frac{d\lambda}{\lambda} < \infty$$

uniformly in σ and t, which together with Hölder's inequality further implies that the right side of (3.11) is bounded by

$$\left[\int_0^{\frac12\sigma}\left(\frac{t}{\lambda}\right)^{-s/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^k,\ \left(\frac{\lambda}{t}\right)^{N-k}\right\}\left[\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}\right]^q\frac{d\lambda}{\lambda}\right]^{1/q}.$$

With this and Fubini's theorem, one has

$$\begin{aligned} &\int_0^1\left[t^{-s/\beta}\|(tL)^k e^{-tL}(f-g_\sigma)\|_{L^p}\right]^q\frac{dt}{t} \\ &\quad\le C\int_0^1\int_0^{\frac12\sigma}\left(\frac{t}{\lambda}\right)^{-s/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^k,\ \left(\frac{\lambda}{t}\right)^{N-k}\right\}\left[\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}\right]^q\frac{d\lambda}{\lambda}\frac{dt}{t} \\ &\quad\le C\int_0^{\frac12\sigma}\left[\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}\right]^q\frac{d\lambda}{\lambda}\to 0 \end{aligned}$$

as $\sigma \to 0_+$, where the last inequality follows from

$$\int_0^1\left(\frac{t}{\lambda}\right)^{-s/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^k,\ \left(\frac{\lambda}{t}\right)^{N-k}\right\}\frac{dt}{t}\le C<\infty$$

by means of the assumption $k > s/\beta$. This proves (3.9) for the case $q \ge 1$.

Next we show that (3.9) remains valid when $q \in (0,1)$. Denote by J the unique number such that $2^{-J} < \frac12\sigma \le 2^{-J+1}$. By (2.31) and Corollary 2.6, we obtain

$$\begin{aligned} &\left[\int_0^{\frac12\sigma}\left(\frac{t}{\lambda}\right)^{-s/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^k,\ \left(\frac{\lambda}{t}\right)^{N-k}\right\}\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}\,\frac{d\lambda}{\lambda}\right]^q \\ &\quad\le\left[\sum_{j=J}^{\infty}\int_{2^{-j}}^{2^{-j+1}}\left(\frac{t}{\lambda}\right)^{-s/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^k,\ \left(\frac{\lambda}{t}\right)^{N-k}\right\}\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}\,\frac{d\lambda}{\lambda}\right]^q \\ &\quad\le C\sum_{j=J}^{\infty}\left(\frac{t}{2^{-j}}\right)^{-sq/\beta}\min\left\{\left(\frac{t}{2^{-j}}\right)^{kq},\ \left(\frac{2^{-j}}{t}\right)^{(N-k)q}\right\}2^{jsq/\beta}\int_{2^{-j-1}}^{2^{-j}}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}^q\,\frac{d\lambda}{\lambda} \\ &\quad\le C\int_0^{\frac12\sigma}\left(\frac{t}{\lambda}\right)^{-sq/\beta}\min\left\{\left(\frac{t}{\lambda}\right)^{kq},\ \left(\frac{\lambda}{t}\right)^{(N-k)q}\right\}\left[\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L}f\|_{L^p}\right]^q\frac{d\lambda}{\lambda}. \end{aligned}$$

From this and (3.11), it follows that when $\sigma \to 0_+$,

$$\int_0^1 \left[t^{-s/\beta}\|(tL)^k e^{-tL}(f - g_\sigma)\|_{L^p}\right]^q \frac{dt}{t}$$
$$\le C \int_0^1 \int_0^{\frac{1}{2}\sigma} \left(\frac{t}{\lambda}\right)^{-sq/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{kq}, \left(\frac{\lambda}{t}\right)^{(N-k)q}\right\} \left[\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda}\frac{dt}{t}$$
$$\le C \int_0^{\frac{1}{2}\sigma} \left[\lambda^{-s/\beta}\|(\lambda L)^k e^{-\lambda L} f\|_{L^p}\right]^q \frac{d\lambda}{\lambda} \to 0,$$

where the last inequality is due to Fubini's theorem and the fact that

$$\int_0^1 \left(\frac{t}{\lambda}\right)^{-sq/\beta} \min\left\{\left(\frac{t}{\lambda}\right)^{kq}, \left(\frac{\lambda}{t}\right)^{(N-k)q}\right\} \frac{dt}{t} \le C < \infty;$$

while the latter holds because of $k > s/\beta$. This proves that (3.9) still holds when $q \in (0, 1)$.

Finally we prove (3.10) by fixing some σ. Notice that

$$\|(tL)^k e^{-tL}(f_{\sigma,\eta} - g_\sigma)\|_{L^p} \le \sum_{i=0}^{N-1} \frac{1}{i!}\|L^i e^{-L}(tL)^k e^{-tL}(f(1-\phi_\eta))\|_{L^p}$$
$$+\frac{1}{(N-1)!}\int_\sigma^1 \|(\lambda L)^N e^{-\lambda L}(tL)^k e^{-tL}(f(1-\phi_\eta))\|_{L^p} \frac{d\lambda}{\lambda}.$$

For $t \in (0, 1]$ and $\lambda \in [\sigma, 1]$, applying Corollary 2.6 yields that

$$\|L^i e^{-L}(tL)^k e^{-tL}(f(1-\phi_\eta))\|_{L^p} \le C \min\left\{t^k, t^{-i}\right\} \|f(1-\phi_\eta)\|_{L^p} \le Ct^k\|f(1-\phi_\eta)\|_{L^p}$$

and

$$\|(\lambda L)^N e^{-\lambda L}(tL)^k e^{-tL}(f(1-\phi_\eta))\|_{L^p} \le C \min\left\{\left(\frac{t}{\lambda}\right)^k, \left(\frac{\lambda}{t}\right)^N\right\} \|f(1-\phi_\eta)\|_{L^p} \le C_\sigma t^k\|f(1-\phi_\eta)\|_{L^p}.$$

Combining the last two formulae gives that

$$\|(tL)^k e^{-tL}(f_{\sigma,\eta} - g_\sigma)\|_{L^p} \le C_\sigma t^k\|f(1-\phi_\eta)\|_{L^p}.$$

From this and $k > s/\beta$, we see that the left hand side of (3.10) is bounded by

$$C_\sigma\|f(1-\phi_\eta)\|_{L^p}\left(\int_0^1 t^{q(k-s/\beta)}\frac{dt}{t}\right)^{1/q} = C_\sigma\|f(1-\phi_\eta)\|_{L^p},$$

which tends to 0 as $\eta \to 0_+$. Thus (3.10) holds and we complete the proof of the lemma. □

Lemma 3.2. *Let all the assumptions be as in Theorem 1.5(b). Let* $m \in [0, \infty)$, $m^* \in [m, \infty) \cap (0, \infty)$, $s \in (0, \Theta \wedge (\beta/2))$, $p \in (1, \infty)$ *and* $q \in (0, \infty)$. *Then the space* $\mathrm{Dom}(L^{m^*/2}) \cap B^{m\beta/2+s}_{p,q}$ *is dense in* $\widetilde{\Lambda}^{m,s}_{p,q}$.

Proof. Notice that

$$\mathrm{Dom}(L^{m^*/2}) = \left\{ f \in L^2 : \int_0^\infty \lambda^{m^*} d\|E_\lambda f\|^2 < \infty \right\} = \left\{ f \in L^2 : \int_0^\infty (1+\lambda)^{m^*} d\|E_\lambda f\|^2 < \infty \right\}.$$

Since $m^* \in [m, \infty)$, it follows that

$$\mathrm{Dom}(L^{m^*/2}) \subset \mathrm{Dom}(L^{m/2}). \tag{3.12}$$

By this and Theorem 1.5(a), we see that

$$\mathrm{Dom}(L^{m^*/2}) \cap B^{m\beta/2+s}_{p,q} \subset \mathrm{Dom}(L^{m/2}) \cap B^{m\beta/2+s}_{p,q} \subset \Lambda^{m,s}_{p,q}.$$

To see the density of $\mathrm{Dom}(L^{m^*/2}) \cap B^{m\beta/2+s}_{p,q}$ in $\widetilde{\Lambda}^{m,s}_{p,q}$, we let $f \in \widetilde{\Lambda}^{m,s}_{p,q}$. For any $\epsilon > 0$, since $\widetilde{\Lambda}^{m,s}_{p,q}$ is the completion of $\Lambda^{m,s}_{p,q}$, there exists $g_\epsilon \in \Lambda^{m,s}_{p,q}$ such that

$$\|f - g_\epsilon\|_{\widetilde{\Lambda}^{m,s}_{p,q}} < \epsilon.$$

Clearly, such g_ϵ belongs to $\mathrm{Dom}(L^{m/2})$ and thus $g_\epsilon \in B^{m\beta/2+s}_{p,q}$ by Theorem 1.5(a). According to Lemma 3.1, the space $\mathrm{Dom}(L^{m^*/2}) \cap B^{m\beta/2+s}_{p,q}$ is dense in $B^{m\beta/2+s}_{p,q}$. Thus, there exists some $f_\epsilon \in \mathrm{Dom}(L^{m^*/2}) \cap B^{m\beta/2+s}_{p,q}$ such that $\|g_\epsilon - f_\epsilon\|_{B^{m\beta/2+s}_{p,q}} < \epsilon$. By (3.12), we see that $f_\epsilon \in \mathrm{Dom}(L^{m/2})$, so does $g_\epsilon - f_\epsilon$. Applying Theorem 1.5(a) yields that

$$\|g_\epsilon - f_\epsilon\|_{\Lambda^{m,s}_{p,q}} \simeq \|g_\epsilon - f_\epsilon\|_{B^{m\beta/2+s}_{p,q}} < C\epsilon.$$

In this way, for any $\epsilon > 0$, we find an $f_\epsilon \in \mathrm{Dom}(L^{m^*/2}) \cap B^{m\beta/2+s}_{p,q}$ such that

$$\|f - f_\epsilon\|_{\widetilde{\Lambda}^{m,s}_{p,q}} \le \|f - g_\epsilon\|_{\widetilde{\Lambda}^{m,s}_{p,q}} + \|g_\epsilon - f_\epsilon\|_{\widetilde{\Lambda}^{m,s}_{p,q}} < (C+1)\epsilon.$$

This finishes the proof of the lemma. □

Proof of Theorem 1.5(b). Let $m \in [0, \infty)$, $s \in (0, \Theta \wedge (\beta/2))$ and $r = m\beta/2 + s$. Fix $m^* \in [m, \infty) \cap (0, \infty)$. By Lemmas 3.1 and 3.2, the spaces $B^r_{p,q}$ and $\widetilde{\Lambda}^{m,s}_{p,q}$ have a common dense subset $\mathrm{Dom}(L^{m^*/2}) \cap B^r_{p,q}$. For functions f in this common dense subset, by (3.12) and Theorem 1.5(a), one has

$$\|f\|_{B^r_{p,q}} \simeq \|f\|_{\Lambda^{m,s}_{p,q}}. \tag{3.13}$$

From this, one deduces that $B^r_{p,q} = \widetilde{\Lambda}^{m,s}_{p,q}$ with equivalent norms. Being more precise, for any $f \in B^r_{p,q}$, since $\mathrm{Dom}(L^{m^*/2}) \cap B^r_{p,q}$ is dense in $B^r_{p,q}$, we can find

$$\{f_j\}_{j=1}^\infty \subset \mathrm{Dom}(L^{m^*/2}) \cap B^r_{p,q}$$

such that $f_j \to f$ in $B^r_{p,q}$, and hence in L^p. Notice that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $B^r_{p,q}$, which is also a Cauchy sequence in $\widetilde{\Lambda}^{m,s}_{p,q}$ because of (3.13). Thus $\{f_j\}_{j=1}^\infty$ converges to some element $\widetilde{f}$ in $\widetilde{\Lambda}^{m,s}_{p,q}$, so that it also converges to $\widetilde{f}$ in L^p. This forces $f = \widetilde{f}$ in L^p and almost everywhere. Hence, any f in $B^r_{p,q}$ also belongs to $\widetilde{\Lambda}^{m,s}_{p,q}$ and

$$\|f\|_{\widetilde{\Lambda}^{m,s}_{p,q}} = \lim_{j\to\infty} \|f_j\|_{\Lambda^{m,s}_{p,q}} \le C \lim_{j\to\infty} \|f_j\|_{B^r_{p,q}} = C\|f\|_{B^r_{p,q}}.$$

The converse part follows in a similar way; we omit the details. □

3.3 Proof of Theorem 1.5(c)

Lemma 3.3. *Assume that (P2) holds for all time $t \in (0,\infty)$.*

(i) *Let $k \in \mathbb{N}$ and $\nu \in [0,1)$. Then, for any $\gamma \in (0,\beta)$, there exists a constant $C > 0$ such that for all $t \in (0,\infty)$ and $x, y \in M$,*

$$|q_{k+\nu,t}(x,y)| \le C\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{3.14}$$

If $k = 0$ and $\nu \in (0,1)$, then there exists a constant $C > 0$ such that for all $t \in (0,\infty)$ and $x, y \in M$,

$$|q_{\nu,t}(x,y)| \le C\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\nu\beta)}. \tag{3.15}$$

(ii) *Let $p \in [1,\infty]$, $k \in \mathbb{Z}_+$ and $\nu \in [0,1)$. Then there exists a constant $C > 0$ such that for all $f \in L^p$ and $t \in (0,\infty)$,*

$$\|(tL)^{k+\nu}e^{-tL}f\|_{L^p} \le C\|f\|_{L^p}. \tag{3.16}$$

Proof. Observe that (ii) follows from (i) and the same argument as in the proof of Proposition 2.4(i). Thus it suffices to show (i).

For $k \in \mathbb{N}$, since now (P2) holds for all time $t \in (0,\infty)$, it follows from the proof of Proposition 2.3(i) that for all $t \in (0,\infty)$ and $x, y \in M$,

$$|q_{k,t}(x,y)| \le C\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{3.17}$$

This proves (3.14) for the case $k \in \mathbb{N}$ and $\nu = 0$.

Now we let $k \in \mathbb{Z}_+$ and $\nu \in (0,1)$. Choose $\gamma \in (\nu\beta, \beta)$. It follows from (2.15) that

$$q_{k+\nu,t}(x,y) = C_\nu \int_0^\infty \frac{t^{k+\nu}\xi^{1-\nu}}{(t+\xi)^{k+2}}\, q_{k+2,\,t+\xi}(x,y)\,d\xi,$$

so by (3.17) one has

$$|q_{k+\nu,t}(x,y)| \le C\int_0^\infty \frac{t^{k+\nu}\xi^{1-\nu}}{(t+\xi)^{k+2+\alpha/\beta}}\left[1+\frac{\rho(x,y)}{(t+\xi)^{1/\beta}}\right]^{-(\alpha+\gamma)} d\xi.$$

We split the last integral into integrals over the intervals $(0,t]$ and $[t,\infty)$, and denote those integrals by J_1 and J_2, respectively. In J_1, we have $t+\xi \simeq t$ and thus

$$J_1 \simeq \int_0^t \frac{t^{k+\nu}\xi^{1-\nu}}{t^{k+2+\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)} d\xi \le C\frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{3.18}$$

In J_2, we have $t+\xi \simeq \xi$ and

$$J_2 \simeq \int_t^\infty \frac{t^{k+\nu}}{\xi^{k+\nu+1+\alpha/\beta}}\left[1+\frac{\rho(x,y)}{\xi^{1/\beta}}\right]^{-(\alpha+\gamma)} d\xi.$$

If $\rho(x,y) \le 2t^{1/\beta}$, then

$$J_2 \le C\int_t^\infty \frac{t^{k+\nu}}{\xi^{k+\nu+1+\alpha/\beta}}\,d\xi \le C\frac{1}{t^{\alpha/\beta}} \simeq \frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}. \tag{3.19}$$

If $\rho(x,y) > 2t^{1/\beta}$, then by a change of variables $\tau = \frac{\rho(x,y)}{\xi^{1/\beta}}$, we obtain

$$J_2 \simeq \frac{t^{k+\nu}}{\rho(x,y)^{(k+\nu)\beta+\alpha}}\int_0^{\frac{\rho(x,y)}{t^{1/\beta}}} \tau^{(k+\nu)\beta+\alpha}(1+\tau)^{-(\alpha+\gamma)}\,\frac{d\tau}{\tau}$$

When $k \ge 1$, by $(k+\nu)\beta+\alpha \ge \beta+\alpha > \alpha+\gamma$, we conclude that

$$J_2 \lesssim \frac{t^{k+\nu}}{\rho(x,y)^{(k+\nu)\beta+\alpha}}\left(\frac{\rho(x,y)}{t^{1/\beta}}\right)^{(k+\nu)\beta-\gamma} \simeq \frac{1}{t^{\alpha/\beta}}\left(\frac{\rho(x,y)}{t^{1/\beta}}\right)^{-\alpha-\gamma} \simeq \frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\gamma)}.$$

Combining this with (3.18) and (3.19) implies (3.14) for the case $k \in \mathbb{N}$ and $\nu \in (0,1)$.

When $k = 0$ and $\nu \in (0,1)$, applying $\nu\beta < \gamma$ yields that

$$\int_0^{\frac{\rho(x,y)}{t^{1/\beta}}} \tau^{\nu\beta+\alpha}(1+\tau)^{-(\alpha+\gamma)}\,\frac{d\tau}{\tau} \le C < \infty,$$

and thus when $\rho(x,y) > 2t^{1/\beta}$ one has

$$J_2 \le C\frac{t^{\nu}}{\rho(x,y)^{\nu\beta+\alpha}} \simeq \frac{1}{t^{\alpha/\beta}}\left(\frac{\rho(x,y)}{t^{1/\beta}}\right)^{-\alpha-\nu\beta} \simeq \frac{1}{t^{\alpha/\beta}}\left[1+\frac{\rho(x,y)}{t^{1/\beta}}\right]^{-(\alpha+\nu\beta)}.$$

Combining this with (3.18) and (3.19) yields (3.15). This finishes the proof of (i). □

Proof of Theorem 1.5(c). Given any $k \in \mathbb{Z}_+$ and $\nu \in [0,1)$, by (3.16), we see that the operator $(tL)^{k+\nu}e^{-tL}$ is bounded on L^1 uniformly in $t \in (0,\infty)$. This will guarantee that Proposition 2.9 remains valid when $p = 1$. Consequently, the arguments in the proof of Theorem 1.5(a) also works for $p = 1$. Hence

$$\|f\|_{\Lambda^{m,s}_{1,q}} \simeq \|f\|_{B^{m\beta/2+s}_{1,q}}, \qquad f \in \mathrm{Dom}(L^{m/2}) \cap L^p.$$

The density lemmas in Subsection 3.2 are also valid for $p = 1$, so that $B^{m\beta/2+s}_{1,q}$ and $\widetilde{\Lambda}^{m,s}_{1,q}$ have a common dense subset $\mathrm{Dom}(L^{m^*/2}) \cap B^{m\beta/2+s}_{1,q}$, where $m^* \in [m,\infty) \cap (0,\infty)$. The rest of the proof follows exactly the same as in the proof of Theorem 1.5(b), and we conclude that $B^{m\beta/2+s}_{1,q} = \widetilde{\Lambda}^{m,s}_{1,q}$ with equivalent norms. □

3.4 Proof of Theorem 1.6

Proof of Theorem 1.6. Since p_t is non-negative, then following the proof of [21, Propositions 7.27 and 7.28], we deduce that for all $t \in (0,\infty)$ and $x, y \in M$,

$$p_t(x,y) \ge \frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{\rho(x,y)}{t^{1/\beta}}\right)$$

where the function Φ_1 is given by

$$\Phi_1(\tau) := \begin{cases} C, & if\ \tau \le \eta \\ 0, & if\ \tau > \eta, \end{cases}$$

with C and η being some positive constants. This combined with (P3) implies that p_t has two sides estimates: for all $t \in (0,\infty)$ and $x, y \in M$,

$$\frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{\rho(x,y)}{t^{1/\beta}}\right) \le p_t(x,y) \le \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{\rho(x,y)}{t^{1/\beta}}\right), \tag{3.20}$$

where Φ_2 is a non-negative monotone decreasing function on $[0,\infty)$ such that

$$\int^\infty \tau^{\alpha+\delta\beta}\Phi_2(\tau)\,\frac{d\tau}{\tau} < \infty$$

for all $\delta \in (0,1)$. It follows from (3.20) and [11, Theorem 3.1] that $\mu(B(x,r)) \simeq r^\alpha$ for all $x \in M$ and $r > 0$. In particular $(V_\alpha)_\le$ holds. Notice that (P2) follows from $p_t \ge 0$ and (P1). So far, all conditions of Theorem 1.5 are verified. Thus, if $s \in (0, \Theta \wedge (\beta/2))$, then (1.13) is known from Theorem 1.5(b).

To prove (1.13) for $s \in [\Theta, \beta/2)$, we shall use the following result in [11, Corollary 5.5]: if (3.20) holds, then for any $f \in \mathrm{Dom}(L^\delta)$,

$$\|f\|^2_{\dot\Lambda^{\delta\beta/2}_{2,2}} \simeq (L^\delta f,\ f). \tag{3.21}$$

With $s \in [\Theta, \beta/2)$, we let $m \in [0,\infty)$ and $r = m\beta/2 + s$. Choose $s^* \in (0,\Theta)$ and $m^* \in (0,\infty)$ such that $m^*\beta/2 + s^* = r$. Observe that $m^* > m$. We claim that for all $f \in \mathrm{Dom}(L^{m^*/2+1})$,

$$\|f\|_{B^r_{2,2}} \simeq \|f\|_{\Lambda^{m,s}_{2,2}}. \tag{3.22}$$

Assuming (3.22) for the moment, we prove (1.13). By Lemma 3.1, $\mathrm{Dom}(L^{m^*/2+1}) \cap B^r_{2,2}$ is a dense subset of $B^r_{2,2}$. Applying (3.22) and $m^* > m$, we proceed as in the proof in Lemma 3.2 and obtain that $\mathrm{Dom}(L^{m^*/2+1}) \cap B^r_{2,2}$ is also a dense subset of $\widetilde\Lambda^{m,s}_{2,2}$. This tells us that $B^r_{2,2}$ and $\Lambda^{m,s}_{2,2}$ have equivalence norms on their common dense subset $\mathrm{Dom}(L^{m^*/2+1}) \cap B^r_{2,2}$. Following the arguments in the proof of Theorem 1.5(b) yields that $B^r_{2,2} = \widetilde\Lambda^{m,s}_{2,2}$ with equivalent norms. This proves (1.13).

Finally, we let $f \in \mathrm{Dom}(L^{m^*/2+1})$ and prove (3.22). By $m^*/2 + 1 > 2s^*/\beta + m^*/2 = 2s/\beta + m/2$ and (3.12), we see that

$$f \in \mathrm{Dom}(L^{2s^*/\beta+m^*/2}) \subset \mathrm{Dom}(L^{m^*/2}) \quad and \quad f \in \mathrm{Dom}(L^{2s/\beta+m/2}) \subset \mathrm{Dom}(L^{m/2}).$$

Then, from (3.21) and $2s^*/\beta + m^*/2 = 2r/\beta = 2s/\beta + m/2$, it follows that

$$\|L^{m^*/2}f\|^2_{\dot\Lambda^{s^*}_{2,2}} \simeq (L^{2s^*/\beta+m^*/2}f,\ L^{m^*/2}f) = (L^{2r/\beta}f,\ f) = (L^{2s/\beta+m/2}f,\ L^{m/2}f) \simeq \|L^{m/2}f\|^2_{\dot\Lambda^{s}_{2,2}}.$$

Since $s^* < \Theta$, applying Theorem 1.5(a) implies that

$$\|f\|_{B^r_{2,2}} \simeq \|f\|_{\Lambda^{m^*,s^*}_{2,2}} = \|f\|_{L^2} + \|L^{m^*/2}f\|_{\dot\Lambda^{s^*}_{2,2}} \simeq \|f\|_{L^2} + \|L^{m/2}f\|_{\dot\Lambda^{s}_{2,2}} = \|f\|_{\Lambda^{m,s}_{2,2}},$$

which proves (3.22). This completes the proof of the theorem. □

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