Manifolds and graphs with slow heat kernel decay

Martin Barlow[*]
University of British Columbia
Vancouver, B.C., V6T 1Z2
Canada
barlow@math.ubc.ca

Thierry Coulhon[†]
Université de Cergy-Pontoise
95302 Pontoise
France
Thierry.Coulhon@math.u-cergy.fr

Alexander Grigor'yan[‡]
Imperial College
London SW7 2BZ
United Kingdom
a.grigoryan@ic.ac.uk

December 2000

Abstract

We give upper estimates on the long time behavior of the heat kernel on non-compact Riemannian manifolds and infinite graphs, which only depend on a lower bound of the volume growth. We also show that these estimates are sharp.

Contents

[*]Research partially supported by a NSERC (Canada) grant.

[†]Research partially supported by an EPSRC (United Kingdom) Visiting Fellowship and by the European Commission (European TMR Network "Harmonic Analysis" 1998-2001, Contract ERBFMRX-CT97-0159).

[‡]Research was supported by the EPSRC (United Kingdom) Research Fellowship B/94/AF/1782.

1 Introduction

Let M be a non-compact geodesically complete connected Riemannian manifold. Let $p_t(x,y)$ be the heat kernel on M, that is the smallest positive fundamental solution to the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

on $\mathbb{R}_+ \times M$. A lot of work has recently been devoted to connecting the large time behavior of the heat kernel with the geometry at infinity of M; see the surveys [31], [18], [32]. One may summarize a large part of these results by saying that the behavior of $\sup_{x\in M} p_t(x,x)$ as a function of $t \to +\infty$ is governed from above and below by the L^2 isoperimetric profile of M (see [17] for a precise definition).

However, this leaves open the question of the direct relationship between the volume growth of the manifold and the rate of decay of its heat kernel. Denote by $B(x,r)$ the geodesic ball of radius r centered at $x \in M$, and by $V(x,r)$ its Riemannian volume. Intuitively, one may expect that, the faster $V(x,r)$ grows to $+\infty$ as $r \to +\infty$, the faster $p_t(x,x)$ should decay to 0 as $t \to +\infty$.

Indeed, let M be a non-compact manifold with bounded geometry (the latter means that M has a positive injectivity radius, and its Ricci curvature is bounded from below). Then the volume growth on M is at least linear, and this reflects in the upper bound of the heat kernel, since one knows ([38], [7], [30]) that, for all such manifolds,

$$\sup_{x\in M} p_t(x,x) \le Ct^{-1/2}, \quad \forall t \ge 1. \tag{1.1}$$

This estimate is obviously sharp for $M = \mathbb{R}^1$ and for cylinders $M = \mathbb{R}^1 \times K$ where K is a compact manifold.

On the other hand, a lower bound of the volume function gives fairly poor information as far as the heat kernel decay is concerned. Indeed, even the maximal rate of volume growth for a manifold with bounded geometry, namely the exponential one

$$V(x,r) \ge ce^{cr}, \quad \forall\, x \in M,\ r \ge 1,$$

does not prevent the manifold from being recurrent (see [39, p.271], [26]). Since the recurrence of M means that

$$\int^{+\infty} p_t(x,x)\, dt = +\infty,$$

no estimate like

$$p_t(x,x) \le Ct^{-1-\varepsilon}, \quad \forall t \ge 1 \tag{1.2}$$

(where $\varepsilon > 0$) can hold on such a manifold. However, there is still a gap between t^{-1} and $t^{-1/2}$ that deserves to be explored.

So far, the emphasis in the study of the large time behavior of the heat kernel has been rather put on "nice" geometric situations (e.g. Lie groups, manifolds with non-negative curvature, covering manifolds) and on fast decays of the heat kernel. Here we would like to address the following question:

Given the volume growth, how *slow* can the heat kernel decay be?

Our first main result shows that, as soon as the volume growth is uniformly faster than linear, one can improve (1.1). In the following statement, we assume that M has bounded geometry. In fact, it is enough to assume that M has *weak bounded geometry* as will be explained in Section 3.

Theorem 1.1 *Let M be a geodesically complete non-compact Riemannian manifold with bounded geometry, and $r_0 > 0$ be its injectivity radius. Suppose that, for all points $x \in M$ and all $r \geq r_0$,*

$$V(x,r) \geq v(r) \tag{1.3}$$

where v is a continuous positive strictly increasing function on $[r_0, +\infty)$. Then, for all $t \geq t_0 := r_0^2$,

$$\sup_{x\in M} p_t(x,x) \leq \frac{C}{\gamma(ct)}, \tag{1.4}$$

where γ is defined by

$$t - t_0 = \int_{v_0}^{\gamma(t)} v^{-1}(s)ds. \tag{1.5}$$

Here $v_0 = v(r_0)$, v^{-1} is the inverse function, and C, c are positive constants.

For example if, for all $x \in M$ and r large enough,

$$V(x,r) \geq cr^D, \tag{1.6}$$

then, taking $v(r) = cr^D$, we obtain from (1.5) $\gamma(t) \simeq t^{\frac{D}{D+1}}$ and

$$\sup_{x\in M} p_t(x,x) \leq Ct^{-\frac{D}{D+1}}, \tag{1.7}$$

for t large enough, which is better than (1.1) as soon as $D > 1$ (the sign $\simeq$ means that the ratio of the left-hand side and of the right-hand side remains bounded between two positive constants as the argument tends to ∞).

If $v(r) = \exp(cr^\alpha)$, $0 < \alpha \leq 1$, then $\gamma(t) \simeq \frac{t}{(\log t)^{1/\alpha}}$ and

$$\sup_{x\in M} p_t(x,x) \leq C\frac{(\log t)^{1/\alpha}}{t}, \tag{1.8}$$

for t large enough, which is not very far from the transience threshold. Recall that the case $\alpha = 1$ corresponds to the maximal rate of volume growth for manifolds with bounded geometry.

These examples are summarized in the table below (neglecting constant multiples)

$V(x,r) \geq$	$\exp(r)$	$\exp(r^\alpha)$	r^D	r
$p_t(x,x) \leq$	$\frac{\log t}{t}$	$\frac{(\log t)^{1/\alpha}}{t}$	$t^{-\frac{D}{D+1}}$	$t^{-1/2}$

which shows how the upper bound of the heat kernel evolves between t^{-1} and $t^{-1/2}$ depending on the lower bound of the volume growth function. Note that, for $t < t_0$, the heat kernel on manifolds with bounded geometry satisfies

$$p_t(x,x) \simeq t^{-d/2}$$

where $d = \dim M$ (see [11], [10], [9], [28], [36]).

A natural question is whether the upper bound (1.4) is sharp under the condition (1.3). The second main result of this paper is a positive answer to this question. For a sequence of numbers D going to $+\infty$, we construct a manifold with bounded geometry having a uniform volume growth

$$V(x,r) \simeq r^D \tag{1.9}$$

and such that

$$p_t(x,x) \simeq t^{-\frac{D}{D+1}} \tag{1.10}$$

(Theorems 4.1 and 6.3). The construction is rather non-trivial and is motivated by recent developments in analysis on fractals (see [2], [3]). We first construct a graph with the required properties (Section 4) and then obtain a manifold by thickening the edges of the graph (Section 6).

For a more general function $v(r)$, we can still construct a manifold with bounded geometry satisfying $V(x,r) \geq v(r)$ and for which the upper bound of the heat kernel (1.4) is sharp up to at most a logarithmic factor. However, the volume growth function $V(x,r)$ does not satisfy in these examples a matching upper bound (see Theorems 5.1, 5.2 in Section 5 and Theorems 6.4, 6.5 in Section 6).

Let us give some historical comments on the exponent $D/(D+1)$ in (1.7). Denote by β_D the exponent corresponding to the slowest possible decay of the heat kernel of a manifold of bounded geometry having polynomial volume growth of exponent D, that is, satisfying (1.9). The above discussion shows that $\beta_D = \frac{D}{D+1}$. However, a priori it was not clear even whether $\beta_D > 0$. The first result in this direction was the estimate (1.1) proved in [38], which showed that $\beta_D \geq 1/2$. It was later proved in [24, Théorème 8] that $\beta_D \geq \frac{D}{D+1}$. Note that $D \geq 1$ and, hence, $\frac{D}{D+1} \geq 1/2$.

Upper estimates of β_D require construction of manifolds satisfying (1.9) but having slow heat kernel decay. The examples of recurrent manifolds with large volume growth ([39], [26]) suggested that $\beta_D \leq 1$ although none of those examples satisfied (1.9). The fact that $\beta_D < 1$ followed from the construction in [3], where a manifold was built satisfying (1.9), with $1 < D < 2$, and

$$p_t(x,x) \simeq t^{-\alpha},$$

with some α such that $\frac{D}{D+1} < \alpha < \frac{D}{2}$ (see also [4], [33] for the graph case).

Finally, in the present paper, we have a complete description of the range of possible heat kernel behavior for manifolds with polynomial volume growth (see Fig. 1).

Corollary 1.2 *Assume that the Riemannian manifold M has bounded geometry and that, for all $x \in M$ and $r \geq 1$,*

$$cr^D \leq V(x,r) \leq Cr^D. \tag{1.11}$$

Then, for all $t \geq 1$,

$$c' t^{-\frac{D}{2}} \leq \sup_{x \in M} p_t(x,x) \leq C' t^{-\frac{D}{D+1}}. \tag{1.12}$$

Moreover, both upper and lower bounds in (1.12) are sharp.

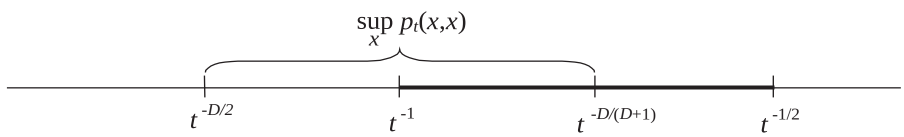

Figure 1 The possible range of the heat kernel decay under the volume growth $V(x,r) \simeq r^D$.

The upper bound in (1.12) and its optimality have been discussed above. The proof of the lower bound is contained in [20, Theorem 2.7.]. The optimality of the lower bound is clear from the example of $\mathbb{R}^D$.

Let us also observe that the difference between the two exponents $D/2$ and $D/(D+1)$ in (1.12) vanishes if $D=1$ (which is the minimal possible value of D) and increases as D increases, thus allowing a wider range for heat kernel long time decay. On the other hand, $\beta_D = D/(D+1)$ converges to 1 as $D \to +\infty$, which means that the slowest possible decay of the heat kernel approaches t^{-1}. Therefore, all the range between $t^{-1/2}$ and t^{-1} is really attained within the polynomial scale. Recall that up to a logarithmic factor, the function t^{-1} is also the slowest possible heat kernel decay in the case of the exponential volume growth (see (1.8)).

The proof of Theorem 1.1 consists of two ingredients. The crucial observation is that the volume growth hypothesis (1.3) implies a certain *Faber-Krahn inequality*:

$$\lambda_1(\Omega) \geq \Lambda(|\Omega|) \tag{1.13}$$

where Ω is a large enough open precompact subset of M, with volume $|\Omega|$, and $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of the Laplace operator in Ω. The function Λ is determined by the volume growth function $v(r)$ (see Proposition 2.5). In particular, if $V(x,r) \geq cr^D$ then we obtain (1.13) with

$$\Lambda(v) = cv^{-\frac{D+1}{D}}. \tag{1.14}$$

Let us note for comparison that in $\mathbb{R}^D$ we have $V(x,r) = cr^D$ whereas the stronger classical Faber-Krahn inequality holds with the function

$$\Lambda(v) = cv^{-2/D}.$$

The second ingredient of the proof is the general equivalence between the Faber-Krahn inequality (1.13) and heat kernel upper estimate (1.4) where the function γ is determined by Λ (see [29], [30] as well as Proposition 2.3 below).

In fact, the actual proof is more complicated since it involves an additional step of discretization: we first obtain a *discrete* version of (1.13) on a graph which is an ε-net of M, and then transfer it to M by using the discretization techniques from [34], [13], [25], [16] (see Section 3).

As we have already mentioned, the estimate (1.7) was obtained in [24, Théorème 8] under the much stronger assumption (1.9) (see also [37, Theorem 3.5] for a slightly different version). More generally, an estimate similar to (1.4) was obtained in [16, Corollary VI.2], but again assuming two-sided estimates of the volume growth function. The point of view adopted in these three papers was to use the upper and the lower bound of the volume to obtain so-called relaxed pseudo-Poincaré inequalities (see also [17] for more explanations). These inequalities, together with the volume lower bound, yield a Nash inequality which is equivalent to the Faber-Krahn inequality (1.13). In contrast to that, in the present paper we obtain the Faber-Krahn inequality in an entirely different way, directly from the uniform lower bound on the volume (see Lemma 2.4 and Proposition 2.5).

To prove the heat kernel lower bounds in the examples of Sections 4 and 5, we use the technique of *anti-Faber-Krahn inequalities* developed in [20]. An anti-Faber-Krahn inequality is nothing but the optimality at all scales of a Faber-Krahn inequality. In particular, we will prove that, on a certain manifold satisfying (1.11), the Faber-Krahn inequality (1.13) with the function (1.14) is sharp up to a constant factor.

The structure of the paper is as follows. In Section 2 we prove Theorem 2.1, which is the analogue of Theorem 1.1 for graphs. In Section 3, we obtain the heat kernel upper bounds for manifolds (i.e. Theorem 1.1) from those for graphs by using some discretization techniques. In Sections 4 and 5, we construct examples of graphs showing that the heat kernel upper bounds obtained by Theorem 2.1 are sharp. In Section 6, we show how to transfer these examples to the manifold setting.

Acknowledgments

The second author benefited of the support of an EPSRC grant. This work was done during a series of his visits at Imperial College (London) whose hospitality he gratefully acknowledges.

The work was finished during the third author's visit at the University of Cergy-Pontoise. He gratefully acknowledges financial support and hospitality of this university.

Notation

1. We denote everywhere by C, c positive constants which may change with occurrences. Normally C refers to a large constant whereas c is a small one.

2. The relation $f(s) \simeq g(s)$ means

$$cf(s) \leq g(s) \leq Cf(s),$$

for all s large enough. Here s is either a discrete or a continuous positive variable.

2 Faber-Krahn inequality and heat kernels on graphs

Let Γ be an infinite (countable) connected graph. If two points $x, y \in \Gamma$ are neighbors, i.e. are connected by an edge, we will write $x \sim y$ and denote the edge by $\overline{xy}$. We always assume that Γ is non-oriented (i.e. $x \sim y$ implies $y \sim x$), connected (i.e. any two points can be joined by a path in Γ) and locally finite (i.e. each point has a finite number of neighbors). Each edge $\overline{xy}$ will be equipped with a *weight* $\mu_{xy} = \mu_{yx} > 0$. We extend μ_{xy} to a function on all pairs $x, y \in \Gamma$ by setting $\mu_{xy} = 0$ if x and y are not neighbors.

The weight μ_{xy} induces also a weight μ on vertices defined by

$$\mu(x) = \sum_{y \sim x} \mu_{xy}$$

which extends to a measure on Γ by

$$\mu(\Omega) = \sum_{x \in \Omega} \mu(x)$$

for all finite subsets $\Omega \subset \Gamma$.

The pair (Γ, μ) is called a *weighted graph.* For example, if we set $\mu_{xy} = 1$ for all neighboring x and y, then $\mu(x)$ is equal to the degree of the vertex x. This weight μ is referred to as *the standard weight* on Γ.

The graph structure induces *the graph distance* $d(x, y)$ which is the minimal number of edges in any path connecting the vertices x and y. Denote balls related to this distance and their measures by

$$B(x, r) = \{y \in \Gamma : d(x, y) < r\} \quad \text{and} \quad V(x, r) = \mu(B(x, r)).$$

There is a natural random walk X_k on Γ associated with the weight μ. It is determined by the transition probability

$$P(x, y) := \frac{\mu_{xy}}{\mu(x)}$$

and is obviously reversible with respect to the measure μ, that is

$$P(x, y)\mu(x) = P(y, x)\mu(y).$$

Denote by $P_k(x, y)$ the transition function after k steps, that is

$$P_k(x, y) = \mathbb{P}\left(X_k = y \,|\, X_0 = x\right),$$

and by $p_k(x, y)$ the heat kernel defined by

$$p_k(x, y) = \frac{P_k(x, y)}{\mu(y)}.$$

The reversibility of X_k implies $p_k(x, y) = p_k(y, x)$. If μ is the standard weight then X_k is referred to as *the simple random walk* on Γ.

Our main result for graphs is the following theorem.

Theorem 2.1 *Let the weighted graph* (Γ, μ) *satisfy the hypothesis*

$$\inf_{x \sim y} \mu_{xy} > 0. \tag{2.1}$$

Suppose that, for all points $x \in \Gamma$ *and all* $r \geq 1$,

$$V(x, r) \geq v(r),$$

where v *is a continuous positive strictly increasing function on* $[1, +\infty)$. *Then, for all* $x \in \Gamma$ *and for all* $k \in \mathbb{N}$,

$$p_k(x, x) \leq \frac{C}{\gamma(ck)}, \tag{2.2}$$

where γ *is defined by*

$$t = \int_{v_0}^{\gamma(t)} v^{-1}(s)ds \tag{2.3}$$

and $v_0 = v(1)$.

Observe that the hypothesis (2.1) is automatically satisfied for the standard random walk.

Example 2.1 Assume that $v(r) = cr^D$. Then (2.3) and (2.2) yield

$$p_k(x,x) \le Ck^{-\frac{D}{D+1}}. \tag{2.4}$$

Example 2.2 Assume that $v(r) = c\exp(c'r^\alpha)$ where $\alpha \in (0,1]$. Then Theorem 2.1 implies, for all $k \ge 2$ and $x \in \Gamma$,

$$p_k(x,x) \le \frac{C(\log k)^{1/\alpha}}{k}. \tag{2.5}$$

Remark 2.1 Under the stronger assumption $V(x,r) \simeq r^D$, the estimate (2.4) was obtained in [24, Theorem 8]. For a general volume growth, an estimate similar to (2.2) was obtained in [16, Proposition V.5], however, under the much stronger assumption

$$v(r) \le V(x,r) \le Cv(r).$$

Remark 2.2 It useful to notice that the identity (2.3) defining the function γ can be rewritten as follows

$$\gamma^{-1}(u) = \int_{v_0}^{u} v^{-1}(s)ds. \tag{2.6}$$

If $v(r)$ is convex (which is typical for applications), then v^{-1} is concave whence

$$\frac{1}{2}(u - v_0)v^{-1}(u) \le \int_{v_0}^{u} v^{-1}(s)ds \le uv^{-1}(u).$$

Therefore, one gets from (2.6)

$$\gamma^{-1}(s) \simeq sv^{-1}(s). \tag{2.7}$$

Corollary 2.2 ([38], [5], [27], [7], [30]) *For any graph (Γ,μ) satisfying the hypothesis (2.1), we have*

$$p_k(x,x) \le Ck^{-\frac{1}{2}}, \tag{2.8}$$

for all $x \in \Gamma$ and for all positive integers k.

Proof. Indeed, the fact that Γ is infinite and connected together with (2.1) implies easily that $V(x,r) \ge cr$. Therefore, applying (2.4) with $D = 1$ we obtain (2.8). ■

The proof of Theorem 2.1 uses eigenvalues of the discrete Laplace operator. The Laplace operator on (Γ,μ) is defined by

$$\Delta f(x) = \sum_{y\sim x} p(x,y)f(y) - f(x) = \frac{1}{\mu(x)} \sum_{y\in\Gamma} (\nabla_{xy} f)\, \mu_{xy}$$

where $\nabla_{xy} f = f(y) - f(x)$.

For any non-empty finite set Ω, denote by $c_0(\Omega)$ the set of all real-valued functions on Ω extended by 0 outside Ω. Denote by Δ_Ω the restriction of Δ to $c_0(\Omega)$, that is,

$$\Delta_\Omega f(x) = \begin{cases} \Delta f(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

The space $c_0(\Omega)$ can be identified with $L^2(\Omega,\mu)$. Then the operator $-\Delta_\Omega$ acting in $L^2(\Omega,\mu)$ is self-adjoint, positive definite and has discrete positive spectrum. Denote by $\lambda_1(\Omega)$ its smallest eigenvalue. It is possible to show that $\lambda_1(\Omega) \in (0,1]$ (see, for example, [21]). The first eigenvalue admits the following variational characterization:

$$\lambda_1(\Omega) = \inf_{f \in c_0(\Omega)} \frac{E(f)}{\sum_{x\in\Gamma} f(x)^2 \mu(x)} \tag{2.9}$$

where the energy $E(f)$ is defined by

$$E(f) = \frac{1}{2} \sum_{x,y\in\Gamma} \left(\nabla_{xy} f\right)^2 \mu_{xy}. \tag{2.10}$$

(the $\frac{1}{2}$ compensates for the double counting of the edges $\overline{xy}$).

For certain graphs, $\lambda_1(\Omega)$ can be related to the measure $\mu(\Omega)$. For example, consider $\mathbb{Z}^D$ with the standard weight μ. Then it is possible to show that, for any non-empty finite set $\Omega \subset \mathbb{Z}^D$,

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-2/D}.$$

On the other hand, for any ball $B(x,r)$ in $\mathbb{Z}^D$, we have $V(x,r) \simeq r^D$ and

$$\lambda_1(B(x,r)) \simeq \frac{1}{r^2} \simeq V(x,r)^{-2/D}.$$

It turns out that the heat kernel long time behavior is closely related to *the Faber-Krahn inequalities* - lower estimates of $\lambda_1(\Omega)$ via the volume $\mu(\Omega)$. This was established in the case of manifolds in [6] and [29], and for the case of graphs in [16, Proposition V.1].

Proposition 2.3 *Assume that*

$$v_0 := \inf_{x\in\Gamma} \mu(x) > 0. \tag{2.11}$$

Suppose also that, for all non-empty finite sets Ω*, one has*

$$\lambda_1(\Omega) \geq \Lambda(\mu(\Omega)), \tag{2.12}$$

where Λ *is a positive decreasing function on* $[v_0, +\infty)$*. Then, for all* $x \in \Gamma$ *and for all* $k \in \mathbb{N}$*, one has*

$$p_k(x,x) \leq \frac{C}{\gamma(ck)}, \tag{2.13}$$

where γ *is defined by*

$$t = \int_{v_0}^{\gamma(t)} \frac{ds}{s\Lambda(s)}. \tag{2.14}$$

Remark 2.3 Note that v_0 can also be any positive number smaller than $\inf_{x\in\Gamma} \mu(x)$ since diminishing v_0 in (2.14) results in diminishing $\gamma(t)$.

Remark 2.4 The converse of Proposition 2.3 holds as well if one assumes in addition that the function γ satisfies a certain regularity condition (see [16], §V).

Example 2.3 Suppose that, under the hypotheses of Proposition 2.3, $\Lambda(s) \simeq s^{-1/\nu}$, that is,

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-1/\nu},$$

where $\nu > 0$. Then one gets from (2.14) $\gamma(t) \simeq t^{\nu}$ and

$$p_k(x,x) \leq Ck^{-\nu}.$$

Example 2.4 Suppose that, under the hypotheses of Proposition 2.3, $\Lambda(s) \simeq \frac{1}{s(\log s)^{1/\alpha}}$, that is,

$$\lambda_1(\Omega) \geq \frac{c}{\mu(\Omega)\log^{1/\alpha}\mu(\Omega)},$$

where $\alpha \in]0,1]$. Then one gets from (2.14) $\gamma(t) \simeq \frac{t}{(\log t)^{1/\alpha}}$ and

$$p_k(x,x) \leq \frac{C(\log k)^{1/\alpha}}{k}.$$

Hence, it order to prove a heat kernel upper bound, it suffices to prove a Faber-Krahn inequality. This motivates the following statement.

For any set $\Omega \subset \Gamma$, denote by $r(\Omega)$ its *inradius*, that is,

$$r(\Omega) = \max\left\{r \in \mathbb{N} : \exists x \in \Omega \text{ such that } B(x,r) \subset \Omega\right\}.$$

Lemma 2.4 *Let (Γ, μ) be a weighted graph satisfying the hypothesis (2.1). Then, for any non-empty finite set $\Omega \subset \Gamma$,*

$$\lambda_1(\Omega) \geq \frac{c}{r(\Omega)\mu(\Omega)}. \tag{2.15}$$

Proof. Let f be any function from $c_0(\Omega)$ normalized so that $\max|f| = 1$. Then we have

$$\sum_x f^2(x)\mu(x) \leq \mu(\Omega). \tag{2.16}$$

To estimate the energy

$$E(f) = \frac{1}{2}\sum_{x,y}\left(\nabla_{xy}f\right)^2 \mu_{xy}, \tag{2.17}$$

consider a point x_0 such that $|f(x_0)| = 1$ and the largest integer n such that the ball $B(x_0,n)$ is in Ω. Clearly, $n \leq r(\Omega)$. Then there exists a sequence of points

$$x_0 \sim x_1 \sim x_2 \sim \ldots \sim x_n$$

starting from $x_0 \in \Omega$ and terminating at a point $x_n \notin \Omega$ (see Fig, 2).

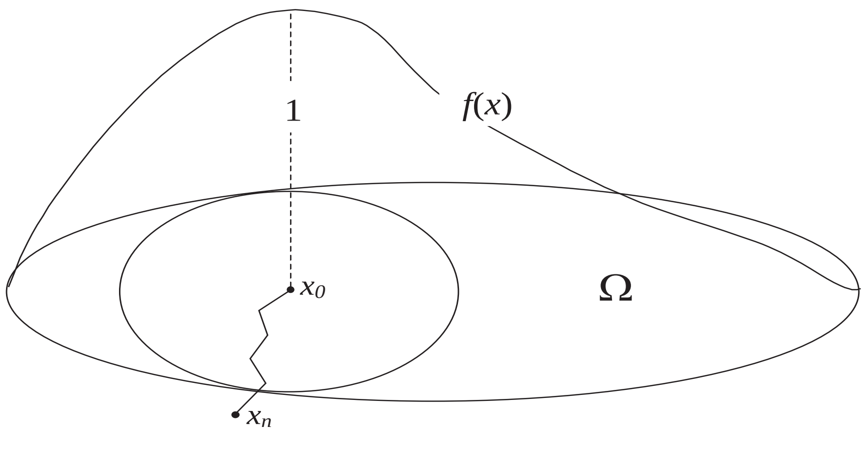

Figure 2 The chain $x_0 \sim ... \sim x_n$

Therefore,

$$\begin{aligned} E(f) &\geq \sum_{i=0}^{n-1} (f(x_i) - f(x_{i+1}))^2 \mu_{x_i x_{i+1}} \\ &\geq \frac{c}{n} \left(\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \right)^2 \geq \frac{c}{n}, \end{aligned} \tag{2.18}$$

where we have used the notation $c = \inf_{x \sim y} \mu_{xy}$ and the inequality

$$\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \geq |f(x_0) - f(x_n)| = 1.$$

Finally, we obtain

$$\frac{E(f)}{\sum_x f(x)^2 \mu(x)} \geq \frac{c}{n\mu(\Omega)} \geq \frac{c}{r(\Omega)\mu(\Omega)},$$

and (2.15) follows by (2.9). ■

Next we have

Proposition 2.5 *Let the weighted graph* (Γ, μ) *satisfy the hypothesis (2.1). Suppose that, for all points* $x \in \Gamma$ *and all* $r \geq 1$, *we have*

$$V(x, r) \geq v(r) \tag{2.19}$$

where v *is a continuous positive strictly increasing function on* $[1, +\infty)$. *Then, for any non-empty finite set* $\Omega \subset \Gamma$,

$$\lambda_1(\Omega) \geq \frac{c}{v^{-1}(\mu(\Omega))\mu(\Omega)}. \tag{2.20}$$

Example 2.5 If $V(x, r) \geq cr^D$ then (2.20) gives

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\frac{D+1}{D}}.$$

Example 2.6 If $V(x, r) \geq c\exp(c'r^\alpha)$ where $\alpha \in (0, 1]$ then (2.20) gives

$$\lambda_1(\Omega) \geq \frac{c}{\mu(\Omega) \log^{1/\alpha}(C\mu(\Omega))}.$$

Proof of Proposition 2.5. Denote $r = r(\Omega)$. Then, for some point $x \in \Omega$, we have $B(x, r) \subset \Omega$ whence the hypothesis (2.19) implies $v(r) \leq \mu(\Omega)$ and $r \leq v^{-1}(\mu(\Omega))$. Hence, (2.20) follows by Lemma 2.4 from (2.15). ■

Finally, we can prove Theorem 2.1.

Proof of Theorem 2.1. By Proposition 2.5, we have the Faber-Krahn inequality (2.12) of Proposition 2.3 with the function

$$\Lambda(s) = \frac{C}{v^{-1}(s)s}.$$

By Proposition 2.3 and by $v(1) \leq \mu(x)$, we obtain the upper bound (2.2) where $\gamma(t)$ is defined by (2.3). ■

In the rest of this section, we consider some further applications of the idea of the proof of Lemma 2.4. Denote by $\|f\|_q$ the L^q norm of a function f on Γ with respect to the measure μ. Denote also

$$\|\nabla f\|_q := \left(\sum_{y\sim x} |\nabla_{xy} f|^q \mu_{xy}\right)^{1/q}.$$

The following statement is an L^q extension of the inequality (2.18).

Proposition 2.6 *Under the hypotheses of Lemma 2.4, we have, for any $q \in [1, +\infty]$ and $f \in c_0(\Omega)$,*

$$\|f\|_\infty \leq C_q (r(\Omega))^{1-\frac{1}{q}} \|\nabla f\|_q. \tag{2.21}$$

Proof. Indeed, using the notation of Lemma 2.4, we have

$$f(x_0)^q \leq \left(\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|\right)^q \leq C n^{q-1} \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|^q \mu_{x_i x_{i+1}}$$

whence

$$\|f\|_\infty \leq C n^{1-\frac{1}{q}} \left(\sum_{x\sim y} |\nabla_{xy} f|^q \mu_{xy}\right)^{1/q},$$

and (2.21) follows, provided $q < +\infty$. The case $q = +\infty$ can be treated similarly (see [15, p.89]). ■

As in the proof of Proposition 2.5, the assumption

$$V(x, r) \geq v(r) \tag{2.22}$$

implies

$$\|f\|_\infty \leq C_q \left(v^{-1}(\mu(\Omega))\right)^{1-\frac{1}{q}} \|\nabla f\|_q. \tag{2.23}$$

This inequality gets stronger with increasing of q. As was noticed in [14], §III, there is no loss of information in (2.23) for $q = \infty$ since it is equivalent to (2.22). At the other end of the scale, for $q = 1$, (2.23) becomes

$$\|f\|_\infty \leq C \|\nabla f\|_1 \tag{2.24}$$

which contains very little information. Indeed, in isoperimetric terms, (2.24) simply means that the boundary of every set is non-empty, which follows from the fact that Γ is infinite and connected. Note that this information is enough to recover Corollary 2.2 (see [19, Prop. 3.1]). The point of Theorem 2.1 is that the inequality (2.23) for $q = 2$ contains quite sharp information as far as the heat kernel decay is concerned.

Let us consider another example of application of our approach for obtaining Faber-Krahn inequalities. Let us call a *capacitor* any couple (A, B) of sets in Γ such that $A \subset B$ and A is finite. Define the capacity of the capacitor (A, B) as

$$\operatorname{cap}(A, B) = \inf_{f \in c_0(B), f|_A = 1} E(f),$$

where the energy $E(f)$ is defined by (2.17). If $B = \Gamma$ then we write $\operatorname{cap}(A)$. One knows that the random walk X_k is transient if and only if the capacity $\operatorname{cap}(x)$ of any vertex $x \in \Gamma$ is positive (see, for example, [40]). We say that X_k is *uniformly transient* if

$$\inf_{x \in \Gamma} \operatorname{cap}(x) > 0.$$

Proposition 2.7 *Suppose that the random walk X_k is uniformly transient. Then, for any non-empty finite set $\Omega \subset \Gamma$,*

$$\lambda_1(\Omega) \geq \frac{c}{\mu(\Omega)}. \tag{2.25}$$

Consequently, for all $x \in \Gamma$ and for all positive integers k,

$$p_k(x,x) \leq C k^{-1}. \tag{2.26}$$

Proof. Let f be again a function as in the proof of Lemma 2.4. By changing its sign, we may assume $f(x_0) = 1$. Then f can be considered as a test function for the capacitor $(\{x_0\}, \Omega)$ whence

$$E(f) \geq \operatorname{cap}(x_0, \Omega) \geq \operatorname{cap}(x_0) \geq c.$$

Combining with (2.16), we obtain (2.25), whence (2.26) follows by Proposition 2.3 (cf. Example 2.3). ■

Remark 2.5 In terms of Sobolev type inequalities, the conclusion of Proposition 2.7 corresponds to

$$\|f\|_\infty \leq C \|\nabla f\|_2,$$

whereas (2.8) corresponds to

$$\|f\|_\infty^2 \leq C \|f\|_2 \|\nabla f\|_2.$$

Remark 2.6 If the random walk X_k is transient (but not necessarily uniformly transient) then one still gets, for any $x \in \Gamma$,

$$p_k(x,x) = O(k^{-1}), \quad k \to \infty$$

(see [8]).

Let us finish this section by pointing out that one can always build a graph with a given volume growth but such that

$$\sup_{x \in \Gamma} p_k(x,x) \geq c k^{-1}. \tag{2.27}$$

Indeed, using Proposition 4.2 below, one sees that (2.27) holds as soon as, for every k, the graph contains a set of vertices whose cardinality is of order k and which has only a finite set of boundary points. Sections 4 and 5 below are devoted to the construction of more sophisticated examples.

3 Heat kernels on manifolds with bounded geometry

Our main underlying space in this section is a *weighted manifold,* which is a slightly more general object than a Riemannian manifold, because the measure and the metric are independent. Let M be a smooth connected non-compact and geodesically complete Riemannian manifold. Denote by $d(x,y)$ the geodesic distance between $x,y\in M$ and by $B(x,r)$ the open geodesic ball of radius $r>0$ centered at $x\in M$.

Suppose that M is equipped with a Borel measure μ having a smooth positive density σ with respect to the Riemannian measure. We will take all integrals against the measure μ and denote

$$\|\varphi\|_p=\left(\int_M|\varphi|^p\,d\mu\right)^{1/p}\quad\text{and}\quad\mathcal{E}(\varphi)=\int_M|\nabla\varphi|^2\,d\mu.$$

The measure of geodesic balls will be denoted as follows:

$$V(x,r)=\mu(B(x,r)).$$

There is a natural Laplace operator Δ associated with the weighted manifold (M,μ) which is defined by

$$\Delta=\sigma^{-1}\nabla(\sigma\operatorname{div})$$

where ∇ and div are the Riemannian gradient and divergence, respectively. The operator Δ with domain $C_0^\infty(M)$ is essentially self-adjoint in $L^2(M,\mu)$ and non-positive definite. The associated heat semigroup $e^{t\Delta}$ has a smooth density $p_t(x,y)$ with respect to μ, which is the heat kernel of Δ. Equivalently, the heat kernel is the transition density of the diffusion process X_t on M generated by Δ.

Denote by $\lambda_1(\Omega)$ the bottom of the spectrum of the operator $-\Delta$ with domain $C_0^\infty(\Omega)$, where Ω is an open subset of M. If Ω is precompact then $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet boundary value problem for Δ in Ω. Alternatively, $\lambda_1(\Omega)$ is defined by the variational principle

$$\lambda_1(\Omega)=\inf_{\varphi\in Lip_0(\Omega)}\frac{\mathcal{E}(\varphi)}{\|\varphi\|_2^2},$$

where $Lip_0(\Omega)$ is the class of Lipschitz functions on M with support in Ω.

Given a geodesically complete weighted manifold (M,μ), we say that (M,μ) has *weak bounded geometry* if there exist $r_0>0$ (called a *radius of bounded geometry*) and a constant C such that

(D_0) *(the local doubling property)* for any $x\in M$ and $r\in(0,r_0)$,

$$V(x,2r)\le CV(x,r)\ ;$$

(P_0) *(the local Poincaré inequality)* for any $x\in M$, $r\in(0,r_0)$ and for any $\varphi\in C^\infty(B(x,2r))$,

$$\inf_{s\in\mathbb{R}}\int_{B(x,r)}(\varphi-s)^2d\mu\le Cr^2\int_{B(x,2r)}|\nabla\varphi|^2d\mu\,.$$

Our main result in this section is the following theorem containing Theorem 1.1 as a special case.

Theorem 3.1 *Let (M,μ) be a geodesically complete non-compact weighted manifold with weak bounded geometry. Suppose that, for all points $x\in M$ and all $r\geq r_0$ (where r_0 is a radius of bounded geometry),*

$$V(x,r)\geq v(r), \tag{3.1}$$

where v is a continuous positive strictly increasing function on $[r_0,+\infty)$. Then, for all $t\geq t_0:=r_0^2$,

$$\sup_{x\in M} p_t(x,x)\leq \frac{C}{\gamma(ct)}, \tag{3.2}$$

where γ is defined by

$$t-t_0=\int_{v_0}^{\gamma(t)} v^{-1}(s)ds \tag{3.3}$$

and $v_0=v(r_0)$ (here v^{-1} is the inverse function).

Proof. We shall deduce Theorem 3.1 from Proposition 2.5 and [16, Proposition III.2] (the latter being a continuous analogue of Proposition 2.3) by using well-known discretization techniques ([34], [13], [25], [16]). The ingredients we need are contained in the proof of [16, Theorem VI.1]; only trivial modifications are required in order to take into account the presence of the weight μ.

Fix some $\varepsilon\in(0,r_0/10)$ and denote by Γ an ε-net on M, that is a countable set of points of M such that the balls $B(x,2\varepsilon)$, $x\in\Gamma$, cover M whereas the balls $B(x,\varepsilon)$ do not intersect. We say that two points $x,y\in\Gamma$ are connected by an edge and write $x\sim y$ if $d(x,y)<2\varepsilon$. As follows from (D_0), Γ is a locally uniformly finite graph, i.e. the number of neighbors of any point in Γ is uniformly bounded. Denote by $\tilde d$ the graph distance on Γ and by $\tilde B(x,r)$ balls on Γ.

The weight $\tilde\mu_{xy}$ on Γ is defined by

$$\tilde\mu_{xy}=V(x,\varepsilon)+V(y,\varepsilon).$$

As follows from (3.1) and (D_0), $\tilde\mu_{xy}$ satisfies the hypothesis (2.1) necessary for Proposition 2.5.

Denote also by $\tilde\mu$ the induced measure on Γ and set $\tilde V(x,r)=\tilde\mu(\tilde B(x,r))$, for $x\in\Gamma$ and $r>0$. It follows from [25, Prop. 2.2 and Section 6], that (3.1) implies

$$\tilde V(x,r)\geq cv(cr),\quad \forall\, r\geq 1,$$

for some fixed $c>0$. Now Proposition 2.5 implies

$$\tilde\lambda_1(\Omega)\geq \frac{c}{v^{-1}(C\tilde\mu(\Omega))\tilde\mu(\Omega)}, \tag{3.4}$$

for every non-empty finite subset Ω of Γ, where $\tilde\lambda_1$ refers to the first eigenvalues on the graph $(\Gamma,\tilde\mu)$. Denote for simplicity

$$\Lambda(s)=\frac{c}{v^{-1}(Cs)s}$$

so that (3.4) can be written as

$$\tilde\lambda_1(\Omega)\geq\Lambda(\mu(\Omega)). \tag{3.5}$$

By [29, Lemma 2.1] (see also [1] and [12, Lemma 7.2]), one derives from (3.5) the following Nash inequality on Γ:

$$E(f) \geq \frac{1}{2} \|f\|_2^2 \, \Lambda(4 \frac{\|f\|_1^2}{\|f\|_2^2}),$$

for all finitely supported functions f on Γ. Following the second part of the proof of [16, Theorem VI.1], one finds that, for every fixed $a > 0$,

$$\mathcal{E}(\varphi) \geq c \, \|\varphi\|_2^2 \, \Lambda(C \frac{\|\varphi\|_1^2}{\|\varphi\|_2^2}), \tag{3.6}$$

for all non-negative functions $\varphi \in C_0^\infty(M)$ such that

$$\|\varphi\|_2 \leq a \, \|\varphi\|_1 .$$

Note that (D_0) and (P_0) imply by [36, Theorem 4.2], for all $t \leq t_0 := r_0^2$ and $x \in M$,

$$p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})}.$$

Together with $V(x, r_0) \geq v_0 = v(r_0)$ (which follows from the hypothesis (3.1)), this yields

$$p_{t_0}(x, x) \leq \frac{C}{v_0}. \tag{3.7}$$

Then (3.6) and (3.7) imply, by [16, Proposition III.2], the upper bound (3.2). ■

There is another proof of Theorem 3.1 which does not use the discretization techniques and which follows the same lines as the proof of Theorem 2.1 for graphs. However, the proof of a continuous analogue of the key Lemma 2.4 is more complicated: instead of considering a chain of vertices $x_0, x_1, ..., x_n$ one considers a chain of *balls* of some radius r centered at x_i and uses the local Poincaré inequality (P_0) to estimate the difference $f_r(x_i) - f_r(x_{i+1})$, where $f_r(x)$ is the mean value of f in the ball $B(x, r)$.

4 An example with pinched polynomial growth

The aim of this section is to prove the following result.

Theorem 4.1 *For arbitrarily large values of D, there exists a locally uniformly finite graph Γ such that, for the standard weight μ on Γ and for all $x \in \Gamma$ and $r \geq 1$,*

$$c \, r^D \leq V(x, r) \leq C \, r^D,$$

whereas, for even k,

$$\sup_{x \in M} p_k(x, x) \simeq k^{-\frac{D}{D+1}}. \tag{4.1}$$

The upper bound in (4.1) follows by Theorem 2.1 so we will have only to prove the lower bound.

Our example is motivated by sets studied in the 'diffusions on fractals' literature, and is based on an N-dimensional version of the Vicsek set (see [2]). We first explain why it is natural to look at this example. Let F be a fractal set with Hausdorff dimension $d_f \in [1, +\infty)$. Given sufficient regularity (and in particular if F is in one of the families of these sets with good behavior, such as nested fractals or generalized Sierpinski carpets [35], [3]), then the standard continuous-time heat kernel on F (see [2] for a definition) satisfies, for some $d_s > 0$ and for all $x \in F$, $t \in (0,1)$

$$p_t(x,x) \simeq t^{-d_s/2}. \tag{4.2}$$

It is convenient to define a third quantity d_w by $d_w = 2d_f/d_s$. One finds that $d_w \geq 2$, so that $d_s \leq d_f$. Under certain regularity conditions, and provided that the shortest path metric on F is comparable to Euclidean distance, one has in addition that

$$d_w \leq 1 + d_f, \tag{4.3}$$

which implies $d_s \geq 2d_f/(1+d_f)$. For a precise statement see [2, Theorem 3.20] – a proof has not yet been written up. The underlying point is that if $\zeta = d_f - d_w$ then the effective resistance between points at distance $r \in (0,1)$ apart is roughly r^ζ; if F is connected then one must have $\zeta \leq 1$.

Note that d_f is directly related to the volume growth on F, namely, $V(x,r) \simeq r^{d_f}$, for $r \in (0,1)$. Hence, constructing the fractal with the minimal possible spectral dimension $d_s = 2d_s/(1+d_f)$ means to have $\zeta = 1$. The latter can be achieved if F is a tree (so contains no loops) which explains why our example is based on the Vicsek set.

Given any suitably regular fractal F then one can construct an infinite 'pre-fractal' graph Γ_F such that the large scale structure of Γ_F mimics the small scale structure of F. In particular, the graph Γ_F has a volume growth of order r^{d_f} and the heat kernel associated with the simple random walk on Γ_F satisfies, for all x and even k,

$$p_k(x,x) \simeq k^{-d_s/2}. \tag{4.4}$$

If $d_s/2 = d_f/(1+d_f)$ then Γ_F is exactly the required example.

We remark that while it would probably be possible to piece together what we need from the 'fractals' literature, the results we need are sometimes not given very explicitly, or require a good deal of technical apparatus to explain. So we prefer to give here a direct proof, using the method of anti-Faber-Krahn inequalities developed in [20] and [22], which is based in the following statement.

Proposition 4.2 ([20, Propositions 2.3, 4.4], [22, Proposition 4.2]) *For any graph (Γ, μ), for any non-empty finite set $\Omega \subset M$ and for any even*[1] *integer k, the following inequality holds*

$$\sup_{x\in\Gamma} p_k(x,x) \geq \frac{e^{-2\lambda_1(\Omega)k}}{\mu(\Omega)}, \tag{4.5}$$

provided $\lambda_1(\Omega) \leq 1/2$.

[1] Note that, in general, one cannot claim any non-trivial lower bound for $p_k(x,x)$ if k is odd. Indeed, on any bipartite graph (including $\mathbb{Z}^N$), one has $p_k(x,x) = 0$ for all odd k.

Proof of Theorem 4.1. Given $D > 1$, let us set $q = 3^D$ and suppose that we can find in Γ a family of finite subsets $\{\Omega_n\}_{n\in\mathbb{N}}$ such that

$$\mu(\Omega_n) \le Cq^n \quad \text{and} \quad \lambda_1(\Omega_n) \le C(3q)^{-n}\,. \tag{4.6}$$

Then if k is even and large enough, choose n such that

$$k \simeq (3q)^n.$$

From (4.5) and (4.6), we deduce

$$\sup_{x\in\Gamma} p_k(x,x) \ge \frac{c'}{\mu(\Omega_n)} \ge ck^{-\log q/\log(3q)} = ck^{-D/(D+1)}.$$

If at the same time the volume growth function of Γ satisfies

$$cr^D \le V(x,r) \le Cr^D, \tag{4.7}$$

then Γ is the required example.

Next we show how to construct Γ and $\{\Omega_n\}$ satisfying (4.6) and (4.7). We begin by defining a fractal set F as a motivation for Γ. Let $F_0 = [0,1]^N$ be the unit cube on $\mathbb{R}^N$. Let $x_1, \dots, x_{2^N}$ be the corners of F_0, and let $x_0 = (\frac{1}{2}, \dots, \frac{1}{2})$ be the center of F_0. Define the contraction ψ_i centered at x_i by

$$\psi_i(x) = x_i + \frac{1}{3}(x - x_i), \quad 0 \le i \le 2^N,$$

and, for compact sets $K \subset \mathbb{R}^N$, set

$$\Psi(K) = \bigcup_{i=0}^{2^N} \psi_i(K).$$

In other words, $\Psi(K)$ is a union of $q := 2^N + 1$ copies of K scaled by the factor $\frac{1}{3}$ with the centers x_i.

Define inductively $F_n = \Psi(F_{n-1})$. It is easily seen that $\{F_n\}_{n\ge1}$ is a decreasing sequence of non-empty compact connected sets. The set $F = \cap_{i=1}^{\infty} F_n$ is the N-dimensional Vicsek set (see Fig. 3).

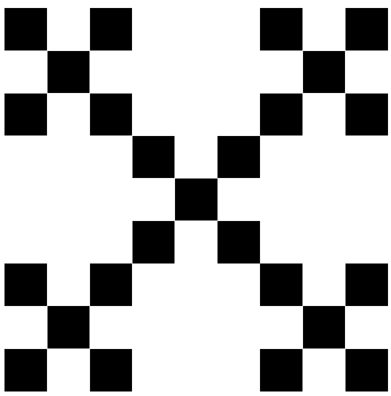

Figure 3 The set $F_2 = \Psi(\Psi(F_0))$ consists of 25 black squares with the sides 1/9 (case $N = 2$)

Now we explain how to construct a graph Γ which has at the large scale the same structure as F at the small scale. Let us consider the sets

$$H_n = 3^n F_n.$$

Note that $F_n \subset [0,1]^N$ consists of q^n cubes each of side 3^{-n}, and that cubes touch only at their corners. So $H_n \subset [0,3^n]^N$, and consists of q^n copies of the unit cube, arranged the same way as the little cubes in F_n. It is not hard to check that if $m > n$ then $H_m \cap [0,3^n]^N = H_n$. Set

$$H = \cup_{n=0}^{\infty} H_n.$$

Then H is connected and consists of a countable union of cubes of side 1, all of which have corners in $\mathbb{Z}^N$ and edges parallel to the axes. Write $\mathcal{C}$ for this collection of cubes.

The vertex set of the required graph Γ is the set of corners and centers of the cubes in $\mathcal{C}$. The edges of Γ connect the center of any cube from $\mathcal{C}$ to each corner of that cube. Since H is connected, Γ is also connected. As in Section 2, denote by d the graph distance in Γ and by $B(x,r)$ the combinatorial ball, that is,

$$B(x,r) = \{y : d(x,y) < r\}.$$

Let μ be the standard weight on Γ. For each vertex $x \in \Gamma$, we have then $1 \le \mu(x) \le 2^N$, which, in particular, implies, for any vertex set Ω, $\mu(\Omega) \simeq |\Omega|$ where $|\Omega|$ is the cardinality of Ω.

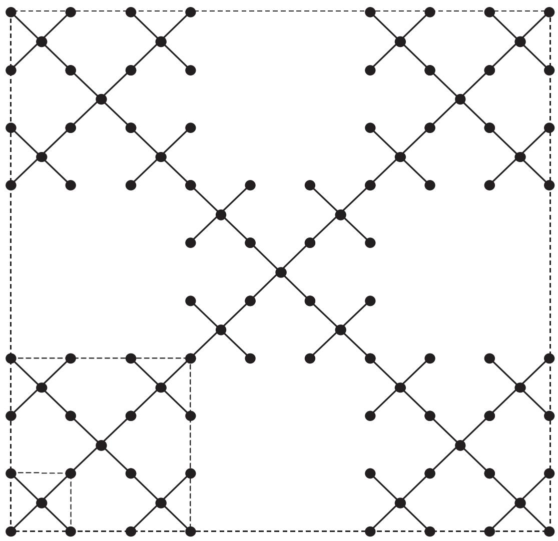

Figure 4 A fragment of the graph Γ. The sets Ω_0, Ω_1 and Ω_2 are bounded by the dashed squares. This picture contains one 2-block, five 1-blocks and twenty five 0-blocks.

The sets $\{\Omega_n\}$ are defined by

$$\Omega_n = \Gamma \cap [0,3^n]^N. \tag{4.8}$$

We will regard Ω_n both as a set of vertices and as a subgraph of Γ. We call a *n-block* any subgraph of Γ isomorphic to Ω_n (see Fig. 4). Note that Γ is a union of n-blocks, for any integer $n \ge 0$.

We claim that

$$|\Omega_n| = 1 + 2^N q^n, \tag{4.9}$$

which implies

$$\mu(\Omega_n) \simeq q^n. \tag{4.10}$$

Indeed, Ω_{n+1} is the union of q n-blocks, which are disjoint except for the 2^N vertices at which the outer n-blocks meet the central one. Therefore,

$$|\Omega_{n+1}| = q\,|\Omega_n| - 2^N.$$

Noting that $|\Omega_0| = q$, we obtain (4.9) by induction.

We will also use the following structure property of n-blocks. The boundary of Ω_n (that is, those vertices in Ω_n which are connected by edges to vertices outside Ω_n) consists only of one of its corners (other n-blocks have up to 2^N boundary points). The distance between any two corners is equal to $2 \cdot 3^n$ which is also the diameter of Ω_n. The distance from the center of Ω_n to its corners is equal to 3^n.

Let us estimate $\lambda_1(\Omega_n)$. By the variational definition (2.9), we have

$$\lambda_1(\Omega_n) \le \frac{E(f)}{\sum_x f(x)^2 \mu(x)}, \tag{4.11}$$

for any function $f \in c_0(\Omega_n)$, where

$$E(f) = \frac{1}{2} \sum_{x,y} |\nabla_{xy} f|^2 \mu_{xy}.$$

Denote by z_0 the center of Ω_n and by z_i, $i \ge 1$, its corners. Define f as follows: $f(z_0) = 1$, $f(z_i) = 0$, $i \ge 1$, and extend f as a harmonic function in the rest of Ω_n. Then f is linear on each of the paths of length 3^n, which connect z_0 with the corners z_i, and is constant elsewhere (see Fig, 5).

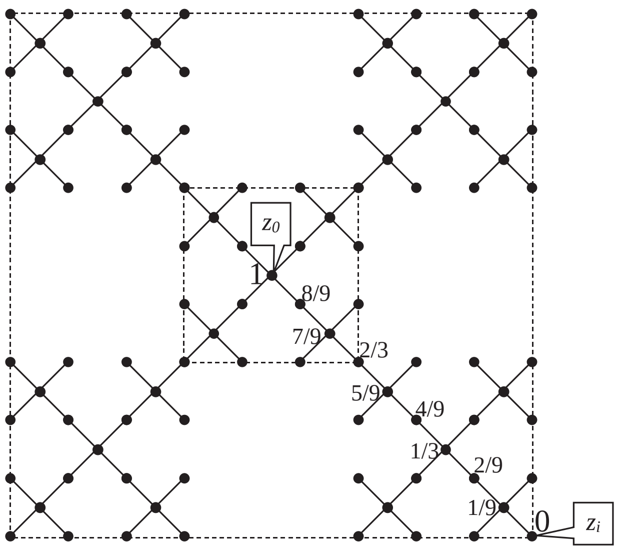

Figure 5 The values of the function f on the diagonal $z_0 z_i$. The function remains constant on all paths transversal to the diagonal (except for the other diagonals).

Since $f(y) \geq \frac{2}{3}$ for all y in the $(n-1)$-block with the center z_0, we have, by (4.10),

$$\sum_x f^2(x)\mu(x) \geq \frac{4}{9}\mu(\Omega_{n-1}) \simeq q^n. \tag{4.12}$$

Also, since $|\nabla_{xy} f| = 3^{-n}$ for any two neighboring points x, y on each of the diagonals connecting z_0 and z_i, and $|\nabla_{xy} f| = 0$ otherwise, we obtain

$$E(f) = \sum_{i=1}^{2^N} 3^{-2n} d(z_0, z_i) = 2^N 3^{-n}. \tag{4.13}$$

Thus, by (4.11), (4.12) and (4.13), $\lambda_1(\Omega_n) \leq C(3q)^{-n}$, as desired. Hence, both inequalities (4.6) are proved.

Let us finally verify that

$$cr^D \leq V(x,r) \leq Cr^D, \tag{4.14}$$

for all $x \in \Gamma$ and $r \geq 1$, where $D = \log q / \log 3$. If $r \leq 3$ then (4.14) is trivial. Otherwise, find an integer $n \geq 1$ so that

$$3^n \leq r < 3^{n+1}.$$

Since the vertex x belongs to some $(n-1)$-block and the diameter of this block is $2 \cdot 3^{n-1} < 3^n$, this block is contained in the ball $B(x,r)$ (see Fig. 6). Hence,

$$V(x,r) \geq \mu(\Omega_{n-1}) \simeq q^n \simeq r^D.$$

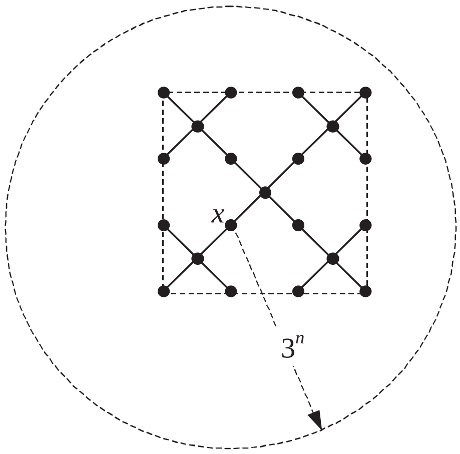

Figure 6 The ball $B(x, 3^n)$ contains a $(n-1)$-block.

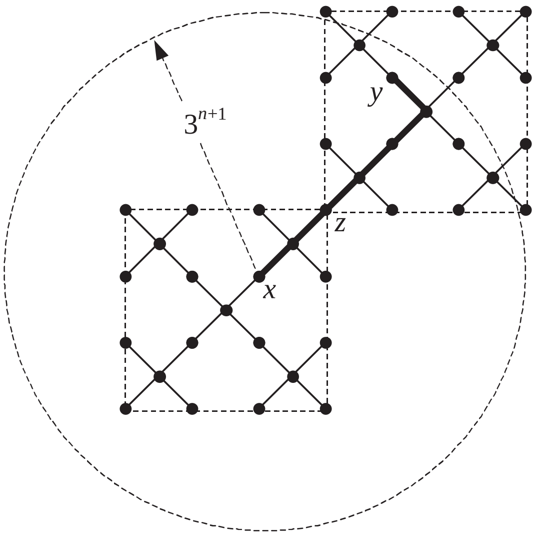

Figure 7 The ball $B(x,r)$ is contained in the union of the $(n+1)$-block A and the neighboring $(n+1)$-blocks.

The vertex x belongs also to some $(n+1)$-block; denote it by A. If a point $y \in B(x,r)$ is not in A then there is a shortest path of length $< 3^{n+1}$ connecting x and y and going through one of the corners of A; denote this corner by z (see Fig. 7). Then the segment zy of the path xy must lie in another $(n+1)$-block to which z is a corner. Therefore, $B(x,r)$ lies in the union of A and of at most 2^N the other $(n+1)$-blocks neighboring to A. Hence,

$$V(x,r) \leq q\mu\left(\Omega_{n+1}\right) \simeq r^D,$$

finishing the proof of (4.14). ■

We remark that while we have proved, for even k,

$$\sup_{x\in\Gamma} p_k(x,x) \simeq ck^{-D/(D+1)},$$

it is clear from the fractal literature that

$$p_k(x,x) \simeq p_k(y,y)$$

for all $x, y \in \Gamma$, which is due to the homogeneity of the graph Γ. So our example actually gives the stronger result that there exists a graph with the volume growth (4.14) and for which

$$ck^{-D/(D+1)} \leq p_k(x,x) \leq Ck^{-D/(D+1)}, \tag{4.15}$$

for all $x \in \Gamma$ and positive even k.

Note that the values of D valid for the above construction are given by

$$D = \frac{\log(1+2^N)}{\log 3}, \quad N = 2, 3, 4, ...$$

It is possible to show, by using a more sophisticated version of the construction, that one can in fact get all real values $D > 1$.

In the next section, we will provide a simpler example showing that Theorem 2.1 is sharp for all $D > 1$. However, that example will *not* possess the matching upper bound for the volume growth function.

In the rest of this section, let us discuss isoperimetric properties of the graph (Γ, μ) constructed in Theorem 4.1. By Proposition 2.5 and by $V(x,r) \geq cr^D$, we have the Faber-Krahn inequality

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\frac{D+1}{D}}, \tag{4.16}$$

for all non-empty finite sets $\Omega \subset \Gamma$. On the other hand, for the sets Ω_n, we have also the anti-Faber-Krahn inequality (cf. (4.6) and (4.10))

$$\lambda_1(\Omega_n) \leq C\mu(\Omega_n)^{-\frac{D+1}{D}},$$

so that (4.16) is sharp at all scales of the volume. Let us compare (4.16) with the isoperimetric profile for Γ. For any set $\Omega \subset \Gamma$, let us denote by $\partial\Omega$ its boundary, that is

$$\partial\Omega = \{x \in \Omega : \exists y \sim x,\ y \notin \Omega\}.$$

We say that the graph (Γ, μ) satisfies *the isoperimetric inequality* with a function $I(s)$ if, for all non-empty finite sets $\Omega \subset \Gamma$,

$$\mu(\partial\Omega) \geq I(\mu(\Omega)). \tag{4.17}$$

A discrete version of Cheeger's inequality says that if $I(v)/v$ is decreasing then (4.17) implies the following Faber-Krahn inequality

$$\lambda_1(\Omega) \geq \frac{1}{4}\left(\frac{I(\mu(\Omega))}{\mu(\Omega)}\right)^2. \tag{4.18}$$

Obviously, (4.17) holds with the trivial function $I(s) = 1$, for any infinite connected graph with the standard weight. However, for the graph Γ, this trivial isoperimetric inequality is *optimal*. Indeed, each set Ω_n has only 1 boundary point so that $\mu(\partial\Omega_n) \leq C$.

For the constant function I, (4.17) and (4.18) imply

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-2},$$

which is weaker than (4.16). The conclusion is that the isoperimetric inequality (4.17) contains less information than the Faber-Krahn inequality (4.16). There are other situations where this phenomenon can be observed - see for example [23] and [6] (cf. the discussion after Proposition 2.6).

5 Examples with volume bounded from below

In this section, we construct examples showing the sharpness of Theorem 2.1 for a regular enough volume growth function $v(r)$. We separate the polynomial function $v(r)$ from the general case because the results for the polynomial case are more precise.

Theorem 5.1 *For every $D \geq 1$, there exists a locally uniformly finite graph Γ such that, for the standard weight μ on Γ and for all $x \in \Gamma$ and $r \geq 1$,*

$$V(x,r) \geq cr^D, \tag{5.1}$$

whereas, for even k,

$$\sup_{x\in M} p_k(x,x) \simeq k^{-\frac{D}{D+1}}. \tag{5.2}$$

Unlike the example in the previous section, we do not have here the matching upper bound for the volume $V(x,r)$ nor do we know whether the *pointwise* heat kernel estimate (4.15) holds. On the other hand, this example does show that Theorem 2.1 is sharp, and it works for any $D \geq 1$. Moreover, the construction is much easier. Note that the value $D = 1$ is trivially covered by $\Gamma = \mathbb{Z}$ so we may assume in the sequel $D > 1$.

Proof. The approach to proving the lower bound (5.2) will be the same as in the previous section (note that the upper bound in (5.2) is a consequence of Theorem 2.1). Suppose that there exists a sequence $\{\Omega_n\}_{n\geq 1}$ of non-empty finite sets in Γ such that

$$\mu(\Omega_n) \leq Cn^D \tag{5.3}$$

and

$$\lambda_1(\Omega_n) \leq Cn^{-(D+1)}. \tag{5.4}$$

Given a large enough even integer k, find an integer n so that

$$k \simeq n^{D+1}. \tag{5.5}$$

Then, by (4.5), (5.4) and (5.3),

$$\sup_{x\in\Gamma} p_k(x,x) \geq \frac{c'}{\mu(\Omega_n)} \geq cn^{-D} \simeq k^{-D/(D+1)}, \tag{5.6}$$

as required.

For a given $D > 1$, first fix an integer N such that

$$N > D + 1. \tag{5.7}$$

Our graph Γ will be a subgraph of $\mathbb{Z}^N$, with the standard weight μ. Denote for simplicity

$$z_n = (n, 0, ..., 0) \in \mathbb{Z}^N.$$

First, Γ contains all nodes z_n for $n = 0, 1, 2, ...$, connected by edges as in $\mathbb{Z}^N$. Then, at each node z_n, we assign a plate P_n defined by

$$P_n = \left\{x \in \mathbb{Z}^N : x_1 = n \quad \text{and} \quad |x_i| \leq n^\alpha, \quad i = 2, 3, ..., N\right\}. \tag{5.8}$$

The points on P_n are connected by edges as in $\mathbb{Z}^N$. The exponent α is determined by

$$\alpha = \frac{D-1}{N-1} < 1. \tag{5.9}$$

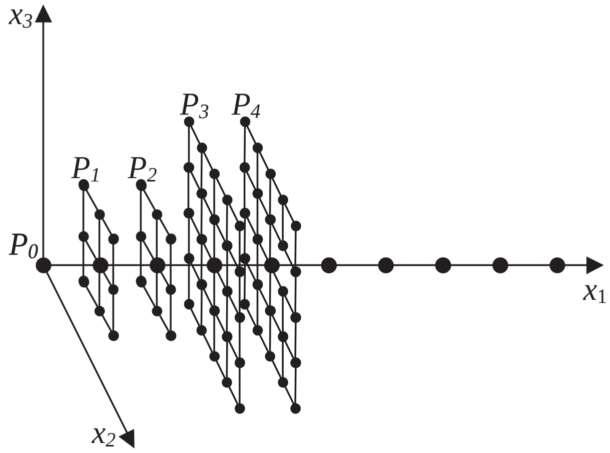

Figure 8 A fragment of the graph Γ in $\mathbb{Z}^3$ (with $\alpha \approx 0.7$). The set Ω_4 is the union of all the plates from P_0 to P_4.

The set Ω_n is defined as the union of all plates P_i for $i = 0, 1, ..., n$ (see Fig. 8). Since the plate P_i contains $\simeq (i^\alpha)^{N-1}$ vertices, we obtain

$$\mu(\Omega_n) \simeq \sum_{i=1}^{n} i^{\alpha(N-1)} \simeq n^{\alpha(N-1)+1} \simeq n^D, \tag{5.10}$$

and (5.3) follows.

In order to estimate $\lambda_1(\Omega_n)$, we use the variational definition (4.11) of λ_1. Choose a test function $f \in c_0(\Omega_n)$ as follows:

$$f(x) = (n - x_1)_+ \tag{5.11}$$

where x_1 is the first coordinate of x in $\mathbb{R}^N$. The function f is constant on each plate P_i so that $\nabla_{xy} f = 0$ if $x, y \in P_i$. If x and y are consecutive nodes z_i, z_{i+1} then $|\nabla_{xy} f| = 1$. Therefore,

$$E(f) = \frac{1}{2} \sum_{x,y} |\nabla_{xy} f|^2 \mu_{xy} = \sum_{i=0}^{n-1} \left|\nabla_{z_i z_{i+1}} f\right|^2 = n. \tag{5.12}$$

The function f restricted to $\Omega_{\lfloor n/2 \rfloor}$ (where $\lfloor \cdot \rfloor$ is the integer part) is bounded below by $n/2$. Hence,

$$\sum_x f^2(x)\mu(x) \geq \frac{n^2}{4} \mu(\Omega_{\lfloor n/2 \rfloor}) \simeq n^{2+D}. \tag{5.13}$$

From (5.12), (5.13) and (4.11), we obtain (5.4).

We are left to prove the volume estimate (5.1). It suffices to assume that the radius r is large enough because, for a bounded r, (5.1) is trivial. Consider the following cases.

Case 1. Let $x = z_0$. Then, for any integer $n \in (0, r)$, the ball $B(z_0, r)$ contains the ball $B(z_n, r - n)$ (see Fig. 9).

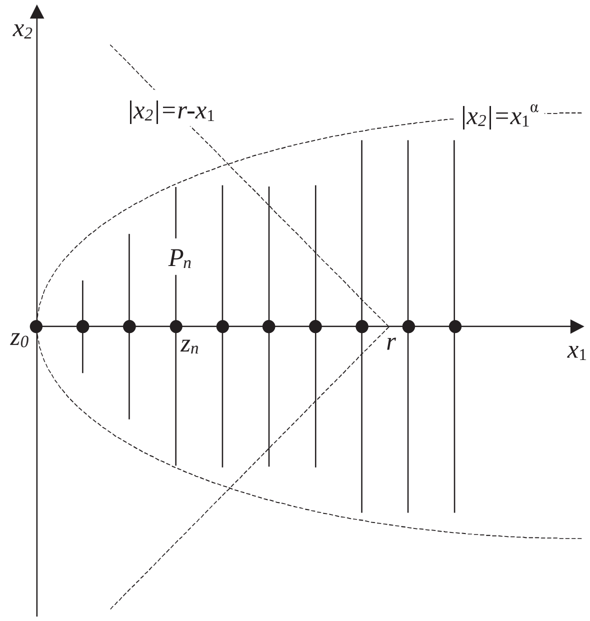

Figure 9 The ball $B(z_0, r)$ lies in the region bounded by the two dashed lines.

The distance from z_n to any other point on P_n does not exceed $(N-1)n^\alpha$. If $n < r/N$ then $(N-1)n^\alpha < r - n$, which implies that

$$P_n \subset B(z_n, r-n) \subset B(z_0, r).$$

Taking the maximal n with this property (which is of the order r) we obtain

$$V(z_0, r) \geq \sum_{i=0}^{n} \mu(P_i) = \mu(\Omega_n) \simeq n^D \simeq r^D.$$

Case 2. Let $x = z_n$. Then $V(z_n, r) \geq V(z_0, r)$ because the translate of Γ by the vector $(n, 0, 0, ...0)$ is a subgraph of Γ, and the image of $B(z_0, r)$ is contained in $B(z_n, r)$.

Case 3. Now, consider the general case. Suppose $x \in P_n$. If $B(x, r)$ does not cover all of P_n then the intersection $B(x, r) \cap P_n$ is a ball in P_n with volume $\simeq r^{N-1} \geq r^D$ whence (5.1) follows.

Let $B(x, r)$ contain P_n. Obviously, $B(x, r)$ contains also the ball $B(z_{n+1}, R)$ (see Fig. 10) where

$$R = (r - (N-1)n^\alpha - 1)_+ \ . \tag{5.14}$$

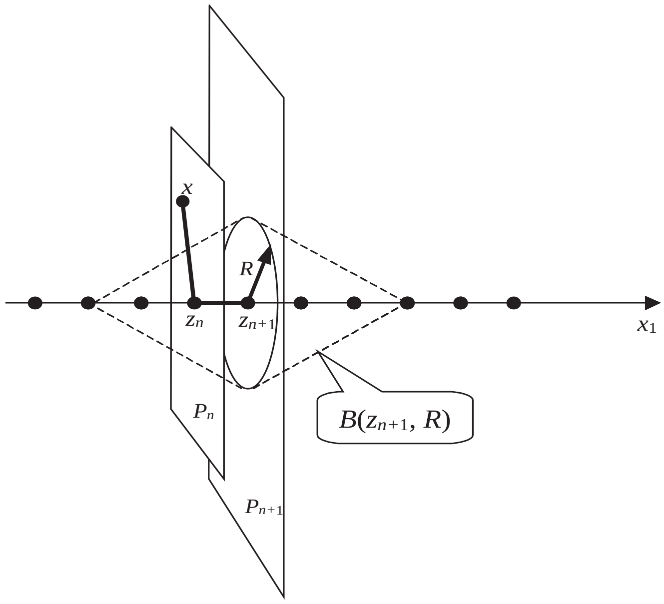

Figure 10 The ball $B(x,r)$ contains P_n and $B(z_{n+1},R)$

Therefore, using the estimate of $V(z_{n+1},R)$ from the previous case, we obtain

$$V(x,r) \geq \max\left(\mu(P_n), V(z_{n+1},R)\right) \geq c(n^{\alpha(N-1)} + R^D).$$

Finally, let us show that

$$n^{\alpha(N-1)} + R^D \geq cr^D.$$

Indeed, by (5.7) $N-1 > D$. Therefore, we have

$$n^{\alpha(N-1)} + R^D \geq (n^\alpha)^D + R^D \geq c\left[(N-1)\,n^\alpha + R\right]^D \geq c' r^D,$$

where in the last inequality we have applied (5.14). ■

Since Γ is a subgraph of $\mathbb{Z}^N$, we have the following upper bound of the volume

$$V(x,r) \leq Cr^N, \tag{5.15}$$

which is optimal if $x = z_n$ with $n >> r$.

A modification of the above construction allows to treat also superpolynomial functions $v(r)$.

Theorem 5.2 *Let $v \in C^1(0,+\infty)$ be a positive strictly monotone increasing function such that $v = 0$. Assume that v satisfies the following conditions:*

(i) v' is positive and monotone increasing;

(ii) $\frac{v'}{v}$ is monotone decreasing.

Then there exists a locally uniformly finite graph (Γ,μ) (with the standard weight μ on Γ) such that, for all $x \in \Gamma$ and $r \geq 1$,

$$V(x,r) \geq cv(cr) \tag{5.16}$$

and, for all positive even integers k,

$$\sup_{x\in\Gamma} p_k(x,x) \geq \frac{c}{\gamma(Ck)} \tag{5.17}$$

where the function γ is defined by

$$\gamma^{-1}(s) = s^2 \frac{d}{ds} v^{-1}(s). \tag{5.18}$$

Remark 5.1 The hypothesis (i) means that $v(r)$ grows at least linearly, whereas the hypothesis (ii) implies that $v(r)$ grows at most exponentially. So, the hypotheses (i) and (ii) of Theorem 5.2 restrict the rate of growth of v only within the natural limits of the volume growth for manifolds of bounded geometry.

Example 5.1 If $v(r) = r^D$ then we obtain from (5.18) $\gamma^{-1}(s) \simeq s^{1+1/D}$ and from (5.17)

$$\sup_{x\in\Gamma} p_k(x,x) \geq ck^{-\frac{D}{D+1}}$$

as in Theorem 5.1. However, we have preferred to treat the polynomial case separately because of the upper bound (5.15), which is not present for the graph of Theorem 5.2.

Example 5.2 Let $v(r) = \exp(r^\alpha)$ for $\alpha \in (0,1]$. Then Theorem 5.2 yields

$$\gamma^{-1}(s) \simeq s\,(\log s)^{1/\alpha-1}$$

and

$$\sup_{x\in\Gamma} p_k(x,x) \geq \frac{c\,(\log k)^{1/\alpha-1}}{k}. \tag{5.19}$$

For comparison, let us recall that Theorem 2.1 implies (cf. Example 2.2)

$$\sup_{x\in\Gamma} p_k(x,x) \leq \frac{C(\log k)^{1/\alpha}}{k}.$$

Hence, for such a superpolynomial volume growth, there is a logarithmic gap between the upper bound of Theorem 2.1 and the lower bound of Theorem 5.2 (cf. Remark 6.1).

The proof of Theorem 5.2 is similar to the previous one, and we use the same notation. Now the plate P_n will be a copy of the ball of radius $\rho(n)$ on the binary tree (see Fig. 11). The graph Γ is the union of P_n's connected by the edges $\overline{z_n z_{n+1}}$. It is no longer a subgraph of $\mathbb{Z}^N$ but still can be considered as a subset of $\mathbb{R}^N$ so that P_n projects onto the point $z_n = (n, 0, ..., 0)$ on the axis x_1.

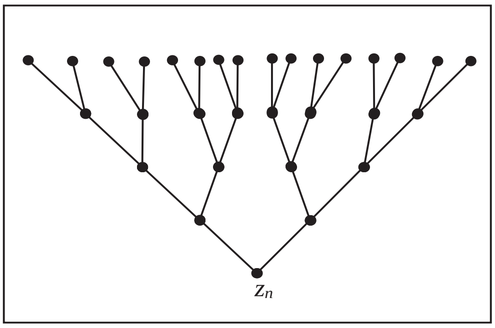

Figure 11 The plate P_n is a ball of radius $\rho(n)$ on the binary tree attached to the point $z_n \in \mathbb{R}^N$. On this picture, $4 < \rho(n) < 5$.

The radius function ρ is taken as follows

$$\rho(s) = C + \log_2(v(s+1) - v(s)). \tag{5.20}$$

The hypothesis (i) implies that

$$v(s+1) - v(s) \geq v(1) - v(0) > 0.$$

Hence, if C is large enough then $\rho(s)$ is positive (and increasing).

As before, let Ω_n be the union of the plates P_i for $i = 0, 1, ..., n$. Let μ be the standard weight on Γ. We have then

$$\mu(\Omega_n) \simeq |\Omega_n| \simeq \sum_{i=0}^{n} 2^{\rho(i)} = 2^C v(n+1).$$

Note that

$$v(n+1) \simeq v(n). \tag{5.21}$$

Indeed, the fact that v'/v is decreasing implies, for large r,

$$v'(r)/v(r) \leq C. \tag{5.22}$$

By integrating this inequality from n to $n+1$, we obtain $v(n+1) \leq Cv(n)$, whence (5.21) follows. Therefore, we have

$$\mu(\Omega_n) \simeq v(n). \tag{5.23}$$

The proof of the volume estimate (5.16) is similar to that for Theorem 5.1. It suffices to assume that r is large enough. Let us first observe that (5.22) implies

$$v(r) \leq \exp(Cr). \tag{5.24}$$

Consider the following cases.

Case 1. Let $x = z_0$. Then, for any positive integer n such that

$$n + \rho(n) < r, \tag{5.25}$$

the ball $B(z_0, r)$ contains the entire plate P_n. The maximal n satisfying (5.25) is of order r, as follows from (5.20) and (5.24). Therefore, we obtain

$$V(z_0, r) \geq \sum_{i=0}^{n} \mu(P_i) = \mu(\Omega_n) \geq cv(n) \geq cv(cr).$$

Case 2. Let $x = z_n$. Then $V(z_n, r) \geq V(z_0, r)$ because the translate of Γ by the vector $(n, 0, 0, ...0)$ is a subgraph of Γ, and the image of $B(z_0, r)$ is contained in $B(z_n, r)$.

Case 3. Let $x \in P_n$. If $B(x, r)$ does not cover all of P_n then the intersection $B(x, r) \cap P_n$ is a ball in P_n with volume $\geq 2^{r/2} \geq cv(cr)$ whence (5.16) follows.

Let $B(x, r)$ contain P_n. Obviously, $B(x, r)$ contains also the ball $B(z_{n+1}, R)$ where

$$R = (r - \rho(n) - 1)_+ \, . \tag{5.26}$$

Therefore, using the estimate of $V(z_{n+1}, R)$ from the previous case, we obtain

$$V(x, r) \geq \max\left(\mu(P_n), V(z_{n+1}, R)\right) \geq c(2^{\rho(n)} + v(cR)).$$

By the hypothesis (i), the function v is convex so that it satisfies the inequality

$$\frac{v(a)+v(b)}{2} \geq v(\frac{a+b}{2}).$$

Therefore, using also (5.24) and (5.26), we have

$$2^{\rho(n)} + v(cR) \geq cv(c\rho(n)) + v(cR) \geq cv(c\rho(n)+cR) \geq cv(cr)$$

whence (5.16) follows.

Next we will estimate $\lambda_1(\Omega_n)$. For any $r>0$, define r^* by

$$v(r^*) = \frac{1}{2}v(r). \tag{5.27}$$

Since $v(0)=0$, r^* is defined for all $r>0$ and $0<r^*<r$. Since $v(\infty)=\infty$, we have $r^*\to\infty$ as $r\to\infty$. Moreover, the following is true.

Lemma 5.3 *The function $r\mapsto r-r^*$ is monotone increasing.*

Proof. We have by (5.27)

$$\log 2 = \log v(r) - \log v(r^*) = \int_{r^*}^{r} \frac{v'(s)}{v(s)}ds. \tag{5.28}$$

Since $\frac{v'}{v}$ is decreasing, the length of the interval $[r^*, r]$ should be increasing in r to preserve the constant value of the integral in (5.28). ■

To estimate $\lambda_1(\Omega_n)$, consider the following continuous function $\varphi(s)$ on $\mathbb{R}_+$ (see Fig. 12)

$$\varphi(s) = \begin{cases} 0, & s\geq n \\ 1, & s\leq n^* \\ \text{linear}, & n^*<s<n \end{cases}$$

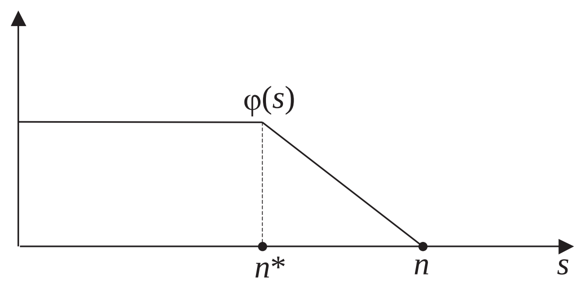

Figure 12 The function $\varphi(s)$

Define a function $f(x)$ on Γ by $f(x)=\varphi(x_1)$. It is clear that $f\in c_0(\Omega_n)$. Since $f(x)$ is constant on any plate P_i, we have the following estimate of the energy of f

$$E(f) = \sum_{i=0}^{n-1} \left|\nabla_{z_i z_{i+1}} f\right|^2 = \sum_{i=\lfloor n^*\rfloor}^{n-1} |\varphi(i)-\varphi(i+1)|^2 \leq \frac{n-\lfloor n^*\rfloor}{(n-n^*)^2} \simeq \frac{1}{n-n^*}.$$

On the other hand, since $f \geq 1$ on $\Omega_{\lfloor n^* \rfloor}$, we have, by (5.23) and (5.27),

$$\sum_{x \in \Gamma} f^2(x)\mu(x) \geq \mu(\Omega_{\lfloor n^* \rfloor}) \simeq v(\lfloor n^* \rfloor) \simeq v(n^*) = \frac{1}{2}v(n).$$

Hence, by the variational property (2.9) of the eigenvalues,

$$\lambda_1(\Omega_n) \leq \frac{C}{(n - n^*)v(n)}. \tag{5.29}$$

Next we need the following lemmas.

Lemma 5.4 *If a function $v(r)$ satisfies the hypotheses of Theorem 5.2 then*

$$\frac{1}{2}(r - r^*)v(r) \leq \int_0^r v(s)ds \leq 2(r - r^*)v(r) \tag{5.30}$$

Proof. The lower bound in (5.30) follows by the monotonicity of $v(s)$ and $v(r^*) = \frac{1}{2}v(r)$. In order to obtain the upper bound for the integral, consider the decreasing sequence $\{r_i\}_{i \geq 0}$ defined by

$$r_0 = r, \quad r_{i+1} = (r_i)^*.$$

Clearly, $r_i \downarrow 0$ as $i \to \infty$ and

$$\int_0^r v(s)ds \leq \sum_{i=0}^{\infty}(r_i - r_{i+1})v(r_i).$$

The sequence $\{v(r_i)\}$ is a decreasing geometric series with ratio 1/2. As follows from Lemma 5.3, the sequence $\{(r_i - r_{i+1})\}$ is decreasing. Therefore, $(r_i - r_{i+1})v(r_i)$ decays at least as fast as the geometric series whence

$$\sum_{i=0}^{\infty}(r_i - r_{i+1})v(r_i) \leq 2(r_0 - r_1)v(r_0) = 2(r - r^*)v(r)$$

which was to be proved. ∎

Lemma 5.5 *If a function $v(r)$ satisfies the hypotheses of Theorem 5.2 then*

$$\int_0^r v(s)ds \simeq \frac{v^2(r)}{v'(r)}. \tag{5.31}$$

Proof. By Lemma 5.4, it suffices to verify that

$$r - r^* \simeq \frac{v(r)}{v'(r)}. \tag{5.32}$$

As follows from (5.28), for some $\xi \in (r^*, r)$,

$$\log 2 = (r - r^*)\frac{v'(\xi)}{v(\xi)}.$$

To prove the upper bound in (5.32), we use the hypothesis (ii) which implies

$$r - r^* = (\log 2)\frac{v(\xi)}{v'(\xi)} \le (\log 2)\frac{v(r)}{v'(r)}.$$

Since the functions v and v' are monotone increasing, we obtain

$$\frac{v(\xi)}{v'(\xi)} \ge \frac{v(r^*)}{v'(r)} = \frac{1}{2}\frac{v(r)}{v'(r)},$$

which yields the lower bound in (5.32). ■

Hence, (5.29) together with Lemmas 5.4 and 5.5 implies

$$\lambda_1(\Omega_n) \le C\frac{v'(n)}{v^2(n)}. \tag{5.33}$$

Given a large enough k, choose $n \in \mathbb{N}$ so that

$$k \simeq \frac{v^2(n)}{v'(n)}. \tag{5.34}$$

It is always possible to find such n because $\frac{v^2(n)}{v'(n)}$ grows with n at most as a geometric series, due to (5.21) and to the fact that v' is increasing. Then, by (5.33), $\lambda_1(\Omega_n)k \le C$. In particular, $\lambda_1(\Omega_n) \le 1/2$ provided k is large enough, which enables us to apply Proposition 4.2, and to obtain

$$\sup_{x\in\Gamma} p_k(x,x) \ge \frac{e^{-2\lambda_1(\Omega_n)k}}{\mu(\Omega_n)} \ge \frac{c}{\mu(\Omega_n)} \ge \frac{c}{v(n)}, \tag{5.35}$$

for large enough even k.

From (5.18) and (5.34), we obtain

$$\gamma^{-1}(v(n)) = v^2(n)\left(\frac{d}{ds}v^{-1}\right)(v(n)) = \frac{v^2(n)}{v'(n)} \simeq k,$$

whence $v(n) \le \gamma(Ck)$. Together with (5.35), this implies (5.17), for k large enough. Since $p_k(x,x)$ is monotone decreasing in k, the lower bound (5.17) is true also for all positive even integers k, by adjusting the constants c, C.

6 How to transfer the examples to the manifold setting

Let (M,μ) be a weighted manifold. We use the notation of Section 3, and we denote by $Lip_0(M)$ the set of all Lipschitz functions on M with compact support. Then we have the following functional analogue of Proposition 4.2.

Proposition 6.1 ([20, Proposition 2.1]) *For any function $\varphi \in Lip_0(M)\setminus\{0\}$ and for any $t>0$, we have*

$$\sup_{x\in M} p_t(x,x) \ge \frac{\|\varphi\|_2^2}{\|\varphi\|_1^2}\exp\left(-\frac{\mathcal{E}(\varphi)}{\|\varphi\|_2^2}t\right). \tag{6.1}$$

This proposition shows that, in order to obtain a lower bound for the heat kernel, it suffices to construct, for any t, a "good" test function φ to use in (6.1). Such test functions can be built from finite sets of vertices in a discretization of M as in the proof of the next statement.

Proposition 6.2 *Let (M,μ) be a geodesically complete non-compact weighted manifold with weak bounded geometry, and let $(\Gamma,\tilde{\mu})$ be a discretization of M in the sense of Section 3. Then the heat kernel on M admits the following lower bound,*

$$\sup_{x\in M} p_t(x,x) \geq \frac{\exp\left(-C\tilde{\lambda}_1(\Omega)t\right)}{C\tilde{\mu}(\Omega)}, \tag{6.2}$$

for any $t>0$ and every non-empty finite subset Ω of Γ.

Proof. Let Ω be a non-empty finite subset of Γ. In the variational definition (2.9) of the first eigenvalue, the test function f can be taken non-negative. Hence, there exists a non-negative non-trivial function $f\in c_0(\Omega)$ such that

$$\frac{E(f)}{\|f\|_2^2} \leq 2\tilde{\lambda}_1(\Omega).$$

By the Cauchy-Schwarz inequality, we have also

$$\frac{\|f\|_1^2}{\|f\|_2^2} \leq \tilde{\mu}(\Omega).$$

By [25, Lemme 6.2 and Lemme 6.4], one concludes that there exists a function $\varphi\in Lip_0(M)$ such that

$$\frac{\mathcal{E}(\varphi)}{\|\varphi\|_2^2} \leq C\tilde{\lambda}_1(\Omega) \quad \text{and} \quad \frac{\|\varphi\|_1^2}{\|\varphi\|_2^2} \leq C\tilde{\mu}(\Omega). \tag{6.3}$$

Therefore, (6.2) follows by Proposition 6.1. ∎

Let Γ be any locally uniformly finite connected graph and μ be the standard weight on Γ. It is clear that one can turn Γ into a manifold M with bounded geometry just by replacing the edges by tubes of length 1, and by gluing them smoothly at the vertices. Let $\hat{\mu}$ be the Riemannian measure on M. It is also clear that Γ can be identified with an ε-net on M (with ε close to $\frac{1}{2}$). Then Γ is endowed with a new measure $\tilde{\mu}$ defined as in Section 3. Obviously, we have $\mu\simeq\tilde{\mu}$ so that all L^p norms in Γ with respect to the measures μ and $\tilde{\mu}$ are uniformly equivalent.

By using the thickening procedure, we will now transfer the graph examples of Theorems 4.1, 5.1 and 5.2 to the manifold setting.

Theorem 6.3 *For arbitrarily large values of D, there exists a complete non-compact Riemannian manifold M with bounded geometry such that, for all $x\in M$ and $r\geq r_0>0$,*

$$cr^D \leq V(x,r) \leq C\,r^D, \tag{6.4}$$

and, for all $t\geq t_0>0$,

$$ct^{-\frac{D}{D+1}} \leq \sup_{x\in M} p_t(x,x) \leq Ct^{-\frac{D}{D+1}}. \tag{6.5}$$

Proof. Let (Γ, μ) be the graph constructed in Theorem 4.1 and let M be a manifold obtained by thickening of edges of Γ. The volume growth function $V(x, r)$ on M (with the Riemannian measure) is the same as on Γ, up to constant multiples, so (6.4) follows from the same property for Γ.

The sequence of the sets $\{\Omega_n\} \subset \Gamma$ constructed in the proof of Theorems 4.1, satisfies (4.6), that is

$$\mu(\Omega_n) \leq Cq^n \quad \text{and} \quad \lambda_1(\Omega_n) \leq C(3q)^{-n}.$$

Taking in (6.2) $\Omega = \Omega_n$ and choosing n from $t = (3q)^n$, we obtain the lower bound in (6.5). The upper bound follows by Theorem 3.1. ■

In the same way, one modifies the proofs of Theorems 5.1 and 5.2 to obtain their continuous analogues.

Theorem 6.4 *For every $D \geq 1$, there exists a complete non-compact Riemannian manifold M with bounded geometry such that, for all $x \in M$ and $r \geq r_0 > 0$,*

$$V(x, r) \geq cr^D$$

and, for all $t \geq t_0 > 0$,

$$ct^{-\frac{D}{D+1}} \leq \sup_{x \in M} p_t(x, x) \leq Ct^{-\frac{D}{D+1}}.$$

Theorem 6.5 *For any function v as in Theorem 5.2, there exists a complete non-compact Riemannian manifold M with bounded geometry such that, for all $x \in M$ and $r \geq r_0 > 0$,*

$$V(x, r) \geq cv(cr)$$

and, for all $t \geq t_0 > 0$,

$$\frac{c}{\gamma_2(Ct)} \leq \sup_{x \in M} p_t(x, x) \leq \frac{C}{\gamma_1(ct)}, \tag{6.6}$$

where γ_1 is defined by

$$\gamma_1^{-1}(s) = sv^{-1}(s) \tag{6.7}$$

and γ_2 is defined by

$$\gamma_2^{-1}(s) = s^2 \frac{d}{ds} v^{-1}(s). \tag{6.8}$$

Proof. The upper bound in (6.6) comes from Theorem 3.1 and from estimate (2.7) of the function γ (see Remark 2.2). The lower bound is proved in the same way as in Theorem 5.2, using Proposition 6.2 as in the previous proof. ■

Remark 6.1 If the function v^{-1} is polynomial then one typically has

$$\frac{d}{ds} v^{-1}(s) \simeq \frac{v^{-1}(s)}{s}$$

so that we obtain from (6.7) and (6.8) $\gamma_1^{-1} \simeq \gamma_2^{-1}$. Hence, for a polynomial volume growth, there is no gap between the upper and lower bounds in (6.6).

If v^{-1} is a logarithmic function then one typically has

$$\frac{d}{ds} v^{-1}(s) \simeq \frac{v^{-1}(s)}{s \log s}$$

whence $\gamma_1^{-1} \simeq \frac{1}{\log s} \gamma_2^{-1}$, which accounts for a logarithmic gap between the upper and lower bounds in (6.6) in the case of an exponential volume growth (cf. Example 5.2).

Note added in proof. The slow heat kernel decay on fractal-like manifolds is related to the slow heat propagation on such manifolds. A similar phenomenon takes place for solutions to the wave equation – cf. S. Kusuoka, X.Y. Zhou, Waves on fractal-like manifolds and effective energy propagation, P.T.R.F. **110** (1998) p.473-495.

References

[1] **Bakry D., Coulhon T., Ledoux M., Saloff-Coste L.**, Sobolev inequalities in disguise, *Indiana Univ. Math. J.*, **124** (1995) no.4, 1033-1074.

[2] **Barlow M.T.**, Diffusions on fractals, *in:* "Lectures on Probability Theory and Statistics, Ecole d'été de Probabilités de Saint-Flour XXV - 1995", Lecture Notes Math. 1690, Springer, 1998. 1-121.

[3] **Barlow M.T., Bass R.F.**, Brownian motion and harmonic analysis on Sierpinski carpets, *Canad. J. Math.*, **54** (1999) 673-744.

[4] **Barlow M.T., Bass R.F.**, Random walks on graphical Sierpinski carpets, *in:* "Random walks and discrete potential theory (Cortona, Italy, 1997)", ed. M. Picardello, W. Woess, Symposia Math. 39, Cambridge Univ. Press, Cambridge, 1999. 26-55.

[5] **Carlen E.A., Kusuoka S., Stroock D.W.**, Upper bounds for symmetric Markov transition functions, *Ann. Inst. H. Poincaré, Prob. et Stat.*, (1987) suppl. au no.2, 245-287.

[6] **Carron G.**, Inégalités isopérimétriques de Faber-Krahn et conséquences, *in:* "Actes de la table ronde de géométrie différentielle (Luminy, 1992)", Collection SMF Séminaires et Congrès 1, 1996. 205–232.

[7] **Chavel I., Feldman E.A.**, Modified isoperimetric constants, and large time heat diffusion in Riemannian manifolds, *Duke Math. J.*, **64** (1991) no.3, 473-499.

[8] **Chavel I., Feldman E.A., Karp L.**, Large time decay of the heat kernel of λ-transient Riemannian manifolds, unpublished manuscript, 1989.

[9] **Cheeger J., Gromov M., Taylor M.**, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, *J. Diff. Geom.*, **17** (1982) 15-53.

[10] **Cheeger J., Yau S.-T.**, A lower bound for the heat kernel, *Comm. Pure Appl. Math.*, **34** (1981) 465-480.

[11] **Cheng S.Y., Li P., Yau S.-T.**, On the upper estimate of the heat kernel of a complete Riemannian manifold, *Amer. J. Math.*, **103** (1981) no.5, 1021-1063.

[12] **Chung F.R.K., Grigor'yan A., Yau S.-T.**, Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs, *Comm. Anal. Geom*, **8** (2000) no.5, 969-1026.

[13] **Coulhon T.**, Noyau de la chaleur et discrétisation d'une variété riemannienne, *Israël J. Math.*, **80** (1992) 289-300.

[14] **Coulhon T.**, Dimensions at infinity for Riemannian manifolds, *Potential Anal.*, **4** (1995) no.5, 335-344.

[15] **Coulhon T.**, Espaces de Lipschitz et inégalités de Poincaré, *J. Funct. Anal.*, **136** (1996) no.1, 81-113.

[16] **Coulhon T.**, Ultracontractivity and Nash type inequalities, *J. Funct. Anal.*, **141** (1996) no.2, 510-539.

[17] **Coulhon T.**, Large time behaviour of heat kernels on Riemannian manifolds: fast and slow decays, *in:* "Journées équations aux dérivées partielles, St-Jean-de-Monts", (1998) II,1-II,12.

[18] **Coulhon T.**, Heat kernels on non-compact Riemannian manifolds: a partial survey, *Séminaire de théorie spectrale et géométrie, 1996-1997, Institut Fourier, Grenoble*, **15** (1998) 167-187.

[19] **Coulhon T.**, Analysis on infinite graphs with regular volume growth, *in:* "Random walks and discrete potential theory", ed. M. Picardello, W. Woess, Symposia Math. XXXIX, Cambridge University Press, 1999. 165-187.

[20] **Coulhon T., Grigor'yan A.**, On-diagonal lower bounds for heat kernels on non-compact manifolds and Markov chains, *Duke Math. J.*, **89** (1997) no.1, 133-199.

[21] **Coulhon T., Grigor'yan A.**, Random walks on graphs with regular volume growth, *Geom. and Funct. Analysis*, **8** (1998) 656-701.

[22] **Coulhon T., Grigor'yan A., Pittet Ch.**, A geometric approach to on-diagonal heat kernel lower bounds on groups, *Ann. Inst. Fourier, Grenoble*, **51** (2001) no.6, 1763-1827.

[23] **Coulhon T., Ledoux M.**, Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz: un contre-exemple, *Ark. Mat.*, **32** (1994) 63-77.

[24] **Coulhon T., Saloff-Coste L.**, Isopérimétrie pour les groupes et les variétés, *Revista Matemática Iberoamericana*, **9** (1993) no.2, 293-314.

[25] **Coulhon T., Saloff-Coste L.**, Variétés riemanniennes isométriques à l'infini, *Revista Matemática Iberoamericana*, **11** (1995) no.3, 687-726.

[26] **Fernández J., Rodríguez J.M.**, Area growth and Green's function on Riemann surfaces, *Arkiv Mat.*, **30** (1992) no.1, 83-92.

[27] **Grigor'yan A.**, On solutions of the heat equation on non-compact Riemannian manifolds, (in Russian) *Uspekhi Matem. Nauk*, **42** (1987) no.4, 144-145.

[28] **Grigor'yan A.**, On the fundamental solution of the heat equation on an arbitrary Riemannian manifold, (in Russian) *Mat. Zametki*, **41** (1987) no.3, 687-692. Engl. transl. *Math. Notes*, **41** (1987) no.5-6, 386-389.

[29] **Grigor'yan A.**, Heat kernel upper bounds on a complete non-compact manifold, *Revista Matemática Iberoamericana*, **10** (1994) no.2, 395-452.

[30] **Grigor'yan A.**, Heat kernel on a manifold with a local Harnack inequality, *Comm. Anal. Geom.*, **2** (1994) no.1, 111-138.

[31] **Grigor'yan A.**, Heat kernel on a non-compact Riemannian manifold, *in:* "1993 Summer Research Institute on Stochastic Analysis", ed. M.Pinsky et al., *Proceedings of Symposia in Pure Mathematics*, **57** (1995) 239-263.

[32] **Grigor'yan A.**, Estimates of heat kernels on Riemannian manifolds, *in:* "Spectral Theory and Geometry. ICMS Instructional Conference, Edinburgh 1998", ed. B.Davies and Yu.Safarov, London Math. Soc. Lecture Note Series 273, Cambridge Univ. Press, 1999. 140-225.

[33] **Jones O.D.**, Transition probabilities for the simple random walk on the Sierpinski graph, *Stoch. Proc. Appl.*, **61** (1996) 45-69.

[34] **Kanai M.**, Analytic inequalities, and rough isometries between non-compact Riemannian manifolds, Lecture Notes Math. 1201, Springer, 1986. 122-137.

[35] **Lindstrøm T.**, Brownian motion on nested fractals, *Mem. Amer. Math. Soc.*, **83** (1990) no.420, 128pp.

[36] **Saloff-Coste L.**, A note on Poincaré, Sobolev, and Harnack inequalities, *Duke Math J.*, **65** no.3, *Internat. Math. Res. Notices*, **2** (1992) 27-38.

[37] **Saloff-Coste L.**, Isoperimetric inequalities and decay of iterated kernels for almost-transitive Markov chains, *Combinatorics, Prob. Comput.*, **4** (1995) 419-442.

[38] **Varopoulos N.Th.**, Brownian motion and random walks on manifolds, *Ann. Inst. Fourier*, **34** (1984) 243-269.

[39] **Varopoulos N.Th.**, Small time Gaussian estimates of heat diffusion kernel. I. The semigroup technique, *Bull. Sci. Math.(2)*, **113** (1989) no.3, 253-277.

[40] **Woess W.**, Random walks on infinite graphs and groups - a survey on selected topics, *Bull. LMS*, **26** (1994) 1-60.