COMPARISON INEQUALITIES FOR HEAT SEMIGROUPS AND HEAT KERNELS ON METRIC MEASURE SPACES

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ABSTRACT. We prove a certain inequality for a subsolution of the heat equation associated with a regular Dirichlet form. As a consequence of this inequality, we obtain various interesting comparison inequalities for heat semigroups and heat kernels, which can be used for obtaining pointwise estimates of heat kernels. As an example of application, we present a new method of deducing sub-Gaussian upper bounds of the heat kernel from on-diagonal bounds and tail estimates.

CONTENTS

1. INTRODUCTION

In this paper, we are concerned with certain inequalities involving heat kernels on arbitrary metric measure spaces. The motivation comes from the following three results.

1. Let M be a Riemannian manifold and $p_t(x, y)$ be the heat kernel on M associated with the Laplace-Beltrami operator Δ. Let $\{X_t\}_{t \geq 0}$ be the diffusion process generated by Δ. For any open set Ω, denote by $\psi_\Omega(t, x)$ the probability that X_t exits from Ω before the time t, provided $X_0 = x$. It was proved in [8] that, for any

two disjoint open subsets U and V of M and for all $x \in U$, $y \in V$, $t, s > 0$,

$$p_{t+s}(x,y) \leq \psi_U(t,x) \sup_{\substack{s \leq t' \leq t+s \\ u \in \partial U}} p_{t'}(u,y) + \psi_V(s,y) \sup_{\substack{t \leq t' \leq t+s \\ v \in \partial V}} p_{t'}(v,x) \qquad (1.1)$$
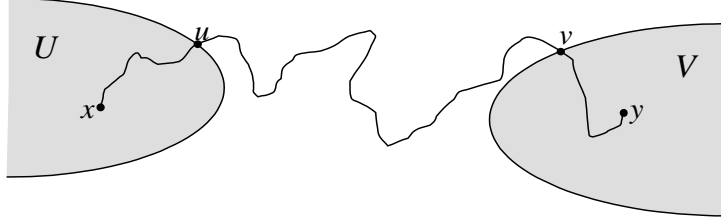
(see Fig. 1).

FIGURE 1. Any sample path, connecting x and y, either exits from the set U before time t when starting at x, or exits from the set V before time s when starting at y.

Similarly, if $U \subset V$ then, for all $x \in U$ and $y \in V$,

$$p_{t+s}(x,y) \leq p_{t+s}^V(x,y) + \psi_U(t,x) \sup_{\substack{s \leq t' \leq t+s \\ u \in \partial U}} p_{t'}(u,y) + \psi_V(s,y) \sup_{\substack{t \leq t' \leq t+s \\ v \in \partial V}} p_{t'}(v,x), \quad (1.2)$$

where $p_t^V(x,y)$ is the heat kernel in V with the Dirichlet boundary condition in ∂V (see Fig. 2).
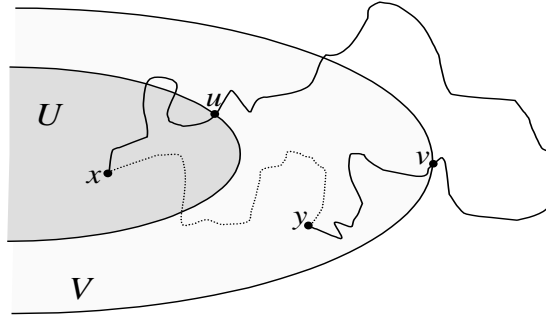
FIGURE 2. Any sample path, connecting x and y, either stays in V, or exits from the set U before time t when starting at x, or exits from the set V before time s when starting at y.

The estimates (1.1) and (1.2) were used in [8] to obtain heat kernel bounds on manifolds with ends.

2. Let now $\{X_t\}_{t \geq 0}$ be a diffusion process on a metric measure space (M, d, μ), and assume that $\{X_t\}$ possesses a continuous transition density $p_t(x,y)$ that will be called the heat kernel. It was proved in [11] that, for any open set $V \subset M$ and for all $x \in V$, $t > 0$,

$$p_{2t}(x,x) \leq p_{2t}^V(x,x) + 2\psi_V(t,x) \sup_{v \in V} p_t(v,v). \qquad (1.3)$$

In the setting of manifolds, one sees that (1.3) is a particular case of (1.2) where $U = V$ and $x = y$ since

$$\sup_{\substack{t \le t' \le 2t \\ v \in \partial V}} p_{t'}(v, v) \le \sup_{v \in V} p_t(v, v).$$

Kigami used (1.3) in [11] to develop a technique for obtaining an upper bound of $p_t(x, x)$, given a certain estimate of the Dirichlet heat kernel $p_t^V(x, x)$. He then applied this technique to obtain heat kernel estimates on post-critically finite self-similar fractals.

3. In the previous setting, but without the continuity of the heat kernel, the authors proved in [6] the following inequality:

$$\operatorname*{esup}_{y \in V} p_{t+s}(x, y) \le \operatorname*{esup}_{y \in V} p_t^V(x, y) + \psi_V(t, x) \operatorname*{esup}_{y, z \in V} p_s(y, z) \qquad (1.4)$$

for all $t, s > 0$ and almost all $x \in V$, where esup stands for the essential supremum.

We refer to the estimates of types (1.1), (1.3), (1.4) as to *comparison inequalities* for heat kernels. The purpose of this paper is to prove such inequalities in the most general setting where the heat semigroups are determined by regular Dirichlet forms and under minimal a priori assumptions about the underlying space and the Dirichlet form. Our method applies to local as well as to non-local regular Dirichlet forms, that is, the associated Hunt process can be a diffusion or not. We prove the comparison inequalities for the heat semigroups without assuming the existence of the heat kernels. If the heat kernels do exist, then we obtain the comparison inequalities for the heat kernels without assuming their continuity. We hope that this level of generality for comparison inequalities will find applications in diverse settings of both diffusion and jump processes on abstract metric measure spaces.

Despite the probabilistic motivation, all the proofs in this paper are entirely analytic and are based on the version of the parabolic maximum principle, developed by the authors [5], [7] in the abstract setting. Our basic result is the inequality (3.3) of Theorem 3.1, which holds true for a weak subsolution of the heat equation associated with any regular Dirichlet form. A refinement of Theorem 3.1 for quasi-local Dirichlet forms is given in Theorem 4.3. It turns out that this basic inequality (3.3) (and its version (4.4) for quasi-local forms) is a source of various interesting comparison inequalities for heat semigroups and heat kernels.

For example, the inequality (5.13) of Theorem 5.1 contains (1.1), and the inequality (5.12) contains (1.2) and (1.3). General comparison estimates for heat semigroups are given by Proposition 4.1 for arbitrary regular Dirichlet forms and by Corollary 4.8 for quasi-local Dirichlet forms.

The structure of this paper is as follows. In Section 2 we give the preliminaries on Dirichlet forms and weak solutions of the associated heat equation. In Section 3, we prove the basic Theorem 3.1. The consequences of Theorem 3.1 – various comparison inequalities, are proved in Section 4 for the heat semigroups and in Section 5 for the heat kernels. Finally, in Section 6, we give an example of application of the comparison inequalities deducing the off-diagonal sub-Gaussian upper bound of the heat kernel from the on-diagonal bound and the tail estimate.

2. Preliminaries on Dirichlet forms

In this section, we first recall some terminology from the theory of Dirichlet form (cf. [4]) and prove some further properties of Dirichlet forms, which are of independent interest for their own right.

Let (M, d, μ) be a *metric measure space*, that is, the couple (M, d) is a locally compact separable metric space and μ is a Radon measure on M with a full support, that is, $\mu(\Omega) > 0$ for any non-empty open subset Ω of M. Let $(\mathcal{E}, \mathcal{F})$ be a *Dirichlet form* in $L^2 := L^2(M, \mu)$, that is, $\mathcal{F}$ is a dense subspace of L^2 and $\mathcal{E}(f, g)$ is a bilinear, symmetric, non-negative definite, closed, and Markovian functional on $\mathcal{F} \times \mathcal{F}$. The closedness of $(\mathcal{E}, \mathcal{F})$ means that $\mathcal{F}$ is a Hilbert space with the norm $\left(\|f\|_2^2 + \mathcal{E}(f) \right)^{1/2}$, where $\|\cdot\|_2$ is the norm of $L^2(M, \mu)$ and $\mathcal{E}(f) := \mathcal{E}(f, f)$. The Markovian property means that $f \in \mathcal{F}$ implies $\widetilde{f} := (f \vee 0) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\widetilde{f}) \leq \mathcal{E}(f)$.

Let Δ be the generator of $(\mathcal{E}, \mathcal{F})$, that is, an operator in L^2 with the maximal domain $\mathrm{dom}\,(\Delta) \subset \mathcal{F}$ such that

$$\mathcal{E}(f, g) = -(\Delta f, g) \text{ for all } f \in \mathrm{dom}\,(\Delta), g \in \mathcal{F}.$$

Then Δ is a non-positive definite self-adjoint operator in L^2. Let $\{P_t\}_{\{t \geq 0\}}$ be the *heat semigroup* associated with the form $(\mathcal{E}, \mathcal{F})$, that is, $P_t = \exp(t\Delta)$. It follows that, for any $t \geq 0$, P_t is a bounded self-adjoint operator in L^2. The relation between P_t and Δ is given also by the identity

$$\Delta f = L^2\text{-}\lim_{t \to 0} \frac{1}{t}(P_t f - f),$$

where the limit exists if and only if $f \in \mathrm{dom}(\Delta)$. A similar relation takes place between P_t and $\mathcal{E}$:

$$\mathcal{E}(f, g) = \lim_{t \to 0} \frac{1}{t}(f - P_t f, g),$$

for all $f, g \in \mathcal{F}$. The heat semigroup $\{P_t\}$ of a Dirichlet form is always *Markovian*, that is, for any $0 \leq f \leq 1$ a.e. in M, we have that $0 \leq P_t f \leq 1$ a.e. in M for any $t > 0$.

A family $\{p_t\}_{t>0}$ of $\mu \times \mu$-measurable functions on $M \times M$ is called the *heat kernel* of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ if p_t is the integral kernel of the operator P_t, that is, for any $t > 0$ and for any $f \in L^2(M, \mu)$,

$$P_t f(x) = \int_M p_t(x, y) f(y) \, d\mu(y) \tag{2.1}$$

for μ-almost all $x \in M$.

The form $(\mathcal{E}, \mathcal{F})$ is *regular* if the space $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ and in $C_0(M)$, where $C_0(M)$ is the space of all real-valued continuous functions in M with compact support. For any two subsets U, Ω of M, a *cut-off function* ϕ for the couple (U, Ω) is a function in $\mathcal{F} \cap C_0(M)$ such that $0 \leq \phi \leq 1$ in M, $\phi = 1$ in an open neighborhood of $\overline{U}$, and $\mathrm{supp}\,(\phi) \subset \Omega$. If $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form, then a cut-off function exists for any couple (U, Ω) provided that Ω is open and $\overline{U}$ is a compact subset of Ω (cf. [4, p.27]).

Let Ω be a non-empty open subset of M. We identify the space $L^2(\Omega)$ as a subspace of $L^2(M)$ by extending any function $f \in L^2(\Omega)$ to M by setting $f = 0$ outside Ω. Denote by $\mathcal{F}(\Omega)$ the closure of $\mathcal{F} \cap C_0(\Omega)$ in $\mathcal{F}$-norm. It is known that

if $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega)$ (cf. [4]). We refer to $(\mathcal{E}, \mathcal{F}(\Omega))$ as a *restricted* Dirichlet form. Denote by $\left\{P_t^\Omega\right\}_{\{t \geq 0\}}$ the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$. It is known that, for any two open subsets $\Omega_1 \subset \Omega_2$ of M, for any $0 \leq f \in L^2$, and for any $t > 0$,

$$P_t^{\Omega_1} f \leq P_t^{\Omega_2} f \quad \text{a.e. in } M.$$

Also, if $\{\Omega_k\}_{k=1}^\infty$ is an increasing sequence of open sets and $\Omega = \bigcup_{k=1}^\infty \Omega_k$ then, for any $t > 0$,

$$P_t^{\Omega_k} f \to P_t^\Omega f \quad \text{a.e.in } M \ \text{ as } k \to \infty$$

(see [5, Lemma 4.17]).

The form $(\mathcal{E}, \mathcal{F})$ is called *local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with disjoint compact supports in M, and is *strongly local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with compact supports in M such that f is constant in a neighborhood of $\text{supp}(g)$.

For some $0 \leq \rho < \infty$, the form $(\mathcal{E}, \mathcal{F})$ is said to be *ρ-local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with compact supports in M and such that

$$\text{dist}(\text{supp}(f), \text{supp}(g)) > \rho.$$

In particular, if $\rho = 0$ then ρ-local is the same as local. We say that the form $(\mathcal{E}, \mathcal{F})$ is *quasi-local* if it is ρ-local for some $\rho \geq 0$.

Let Ω be an open subset of M and I be an open interval in $\mathbb{R}$. A path $u : I \to L^2(\Omega)$ is said to be *weakly differentiable* at $t \in I$ if, for any $\varphi \in L^2(\Omega)$, the function $(u(\cdot), \varphi)$ is differentiable at t, that is, the limit

$$\lim_{\varepsilon \to 0} \left(\frac{u(t + \varepsilon) - u(t)}{\varepsilon}, \varphi \right)$$

exists. If this is the case then it follows from the principle of uniform boundedness that there is a (unique) function $w \in L^2(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \left(\frac{u(t + \varepsilon) - u(t)}{\varepsilon}, \varphi \right) = (w, \varphi),$$

for all $\varphi \in L^2(\Omega)$. We refer to the function w as the *weak derivative* of u at t and write $w = \frac{\partial u}{\partial t}$.

A path $u : I \to \mathcal{F}$ is called a *weak subsolution* of the *heat equation* in $I \times \Omega$, if the following two conditions are fulfilled:

- the path $t \mapsto u(t)|_\Omega$ is weakly differentiable in $L^2(\Omega)$ at any $t \in I$;
- for any non-negative $\varphi \in \mathcal{F}(\Omega)$, we have

$$\left(\frac{\partial u}{\partial t}, \varphi \right) + \mathcal{E}(u, \varphi) \leq 0. \tag{2.2}$$

Similarly one can define the notions of weak *supersolution* and *weak* solution of the heat equation.

Remark 2.1. Note that, for any $f \in L^2(\Omega)$, the function $P_t^\Omega f$ is a weak solution in $(0, \infty) \times \Omega$ (cf. [5, Example 4.10]) and, hence, in $(0, +\infty) \times U$ for any open subset $U \subset \Omega$.

We use the following notation:

$$f_+ := f \vee 0 \text{ and } f_- = -(f \wedge 0).$$

Denote by the sign $\overset{\mathcal{H}}{\rightharpoonup}$ a weak convergence in a Hilbert space $\mathcal{H}$ and by $\overset{\mathcal{H}}{\to}$ the strong (norm) convergence in $\mathcal{H}$. The following statements will be used in this paper.

Proposition 2.2 ([6, Proposition 4.9]). *Let $\{u_k\}$ be a sequence of functions in $\mathcal{F}$ such that $u_k \overset{L^2}{\rightharpoonup} u \in \mathcal{F}$ as $k \to \infty$. If in addition the sequence $\{\mathcal{E}(u_k)\}$ is bounded, then $u_k \overset{\mathcal{F}}{\rightharpoonup} u$ as $k \to \infty$.*

Proposition 2.3 ([4, Theorem 1.4.2]). *Any Dirichlet form $(\mathcal{E}, \mathcal{F})$ possesses the following properties:*

- *If $u, v \in \mathcal{F}$ then also the following functions $u \wedge v$, $u \vee v$, $u \wedge 1$, u_+, u_-, $|u|$ also belong to $\mathcal{F}$.*
- *If $u, v \in \mathcal{F} \cap L^\infty(M)$ then $uv \in \mathcal{F}$.*
- *If $0 \le u \in \mathcal{F}$ then $u \wedge n \overset{\mathcal{F}}{\to} u$ as $n \to \infty$.*
- *Let $\phi(s)$ be a Lipschitz function on $\mathbb{R}$ such that $\phi(0) = 0$. Then, for any $u \in \mathcal{F}$, also $\phi(u) \in \mathcal{F}$. Moreover, if $\{u_n\}_{n=1}^\infty$ is a sequence of functions from $\mathcal{F}$ and $u_n \overset{\mathcal{F}}{\to} u \in \mathcal{F}$ as $n \to \infty$ then $\phi(u_n) \overset{\mathcal{F}}{\rightharpoonup} \phi(u)$. Furthermore, if $\phi(u) = u$ then $\phi(u_n) \overset{\mathcal{F}}{\to} \phi(u)$.*

Proposition 2.4 ([5, Lemma 4.4]). *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form, and let $u \in \mathcal{F}$ and Ω be an open subset of M. Then the following are equivalent:*

(1). *$u_+ \in \mathcal{F}(\Omega)$.*
(2). *$u \le v$ in M for some function $v \in \mathcal{F}(\Omega)$.*

Proposition 2.5 (parabolic maximum principle [7, Proposition 5.2]). *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in L^2. For $T \in (0, +\infty]$ and for an open subset Ω of M, let u be a weak subsolution of the heat equation in $(0, T) \times \Omega$ satisfying the following boundary and initial conditions:*

- *$u_+(t, \cdot) \in \mathcal{F}(\Omega)$ for any $t \in (0, T)$;*
- *$u_+(t, \cdot) \overset{L^2(\Omega)}{\longrightarrow} 0 \quad$ as $t \to 0$.*

Then $u(t, x) \le 0$ for any $t \in (0, T)$ and μ-almost all $x \in \Omega$.

Next we prove further general results on Dirichlet forms that will be used later on and that are of independent interest.

Proposition 2.6. *Let Ω be a non-empty open subset of M. Then, for any non-negative $f \in L^2(\Omega)$, the path $u(t) = P_t^\Omega f$ is a weak subsolution of the heat equation in $(0, \infty) \times M$.*

Proof. We know that $u(t)$ is weakly differentiable in t in $L^2(\Omega)$. Let us show that $u(t)$ is weakly differentiable also in $L^2(M)$. Indeed, for any function $\varphi \in L^2(M)$, we have

$$\left(\frac{u(t+s) - u(t)}{s}, \varphi \right) = \left(\frac{u(t+s) - u(t)}{s}, \varphi \mathbf{1}_\Omega \right) + \left(\frac{u(t+s) - u(t)}{s}, \varphi \mathbf{1}_{\Omega^c} \right).$$

$$(2.3)$$

Since $\varphi \mathbf{1}_\Omega \in L^2(\Omega)$, the first term in the right hand side of (2.3) converges to $\left(\frac{\partial u}{\partial t}, \varphi \mathbf{1}_\Omega\right)$ where $\frac{\partial u}{\partial t}$ is the weak derivative in $L^2(\Omega)$. The second term is obviously 0 whence the convergence of the whole sum to $\left(\frac{\partial u}{\partial t}, \varphi\right)$ follows.

Next, let us show that, for any non-negative $\psi \in \mathcal{F}$,

$$\left(\frac{\partial u}{\partial t}, \psi\right) + \mathcal{E}(u, \psi) \leq 0 \text{ for any } t > 0. \tag{2.4}$$

Indeed, noting that $P_s u(t) \geq P_s^\Omega u(t) = u(t+s)$, we obtain as $s \to 0+$ that

$$\begin{aligned}
\mathcal{E}_s(u, \psi) &= \frac{1}{s}(u - P_s u, \psi) \\
&\leq \frac{1}{s}\left(u - P_s^\Omega u, \psi\right) \\
&= \frac{1}{s}(u(t) - u(t+s), \psi) \\
&\to \left(-\frac{\partial u}{\partial t}, \psi\right).
\end{aligned}$$

Since also $\mathcal{E}_s(u, \psi) \to \mathcal{E}(u, \psi)$ as $s \to 0$, the desired inequality (2.4) follows. ∎

The following proposition will be used to prove Proposition 2.9.

Proposition 2.7. *Let Ω_1, Ω_2 be two non-empty open subsets of M. Then*

$$\mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2) = \mathcal{F}(\Omega_1 \cap \Omega_2). \tag{2.5}$$

Proof. Since $\mathcal{F}(\Omega_1 \cap \Omega_2) \subset \mathcal{F}(\Omega_i)$ for $i = 1, 2$, we see that

$$\mathcal{F}(\Omega_1 \cap \Omega_2) \subset \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2).$$

To prove the opposite inclusion, we need to verify that $f \in \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2)$ implies $f \in \mathcal{F}(\Omega_1 \cap \Omega_2)$. Assume first that $f \geq 0$. Let $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ be two sequence from $\mathcal{F} \cap C_0(\Omega_1)$ and $\mathcal{F} \cap C_0(\Omega_2)$, respectively, that both converge to f in $\mathcal{F}$-norm. As $f \geq 0$ and, hence, $f_+ = f$, it follows from Proposition 2.3 that

$$(f_k)_+ \xrightarrow{\mathcal{F}} f \text{ and } (g_k)_+ \xrightarrow{\mathcal{F}} f \text{ as } k \to \infty. \tag{2.6}$$

Since $(f_k)_+ \in \mathcal{F} \cap C_0(\Omega_1)$ and $(g_k)_+ \in \mathcal{F} \cap C_0(\Omega_2)$, we see that

$$h_k := (f_k)_+ \wedge (g_k)_+ \in \mathcal{F} \cap C_0(\Omega_1 \cap \Omega_2) \subset \mathcal{F}(\Omega_1 \cap \Omega_2).$$

Setting $u_k = (f_k)_+ - (g_k)_+$ and noticing that $u_k \xrightarrow{\mathcal{F}} 0$ as $k \to \infty$, we obtain by Proposition 2.3 that $|u_k| \xrightarrow{\mathcal{F}} 0$ as $k \to \infty$. It follows that

$$h_k = \frac{1}{2}\left[(f_k)_+ + (g_k)_+ - \left|(f_k)_+ - (g_k)_+\right|\right] \xrightarrow{\mathcal{F}} f \text{ as } k \to \infty.$$

Since $\mathcal{F}(\Omega_1 \cap \Omega_2)$ is a closed and, hence, weakly closed subspace of $\mathcal{F}$, we conclude that $f \in \mathcal{F}(\Omega_1 \cap \Omega_2)$.

For a signed function $f \in \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2)$, we have $f_+, f_- \in \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2)$, whence, by the first part of the proof, $f_+, f_- \in \mathcal{F}(\Omega_1 \cap \Omega_2)$ and $f = f_+ - f_- \in \mathcal{F}(\Omega_1 \cap \Omega_2)$, which finishes the proof. ∎

Proposition 2.8. *Let U be a non-empty open subset of M, and let $u \in \mathcal{F}$ such that* $\text{supp}(u)$ *is compact and* $\text{supp}(u) \subset U$. *Then $u \in \mathcal{F}(U)$.*

Proof. We can assume that $u \geq 0$ because a signed u all follows from the decomposition $u = u_+ - u_-$. Next, we can assume that u is bounded because otherwise consider a sequence $u_k := u \wedge k$ that tends to u in $\mathcal{F}$-norm as $k \to \infty$ by Proposition 2.3; if we already know that $u_k \in \mathcal{F}(U)$ then we can conclude that also $u \in \mathcal{F}(U)$.. Hence, we can assume in the sequel that u is non-negative and bounded in M, say $0 \leq u \leq 1$.

Let φ be a cut-off function for the couple $(\mathrm{supp}(u), U)$. Let $\{u_k\}_{k=1}^{\infty}$ be a sequence from $\mathcal{F} \cap C_0(M)$ such that $u_k \xrightarrow{\mathcal{F}} u$ as $k \to \infty$. As $u \geq 0$, we have by [4, Theorem 1.4.2(v),p.26] that $(u_k)_+ \xrightarrow{\mathcal{F}} u$ as $k \to \infty$ and $\left|(u_k)_+ - \varphi\right| \xrightarrow{\mathcal{F}} |u - \varphi|$ as $k \to \infty$. It follows that

$$
\begin{aligned}
(u_k)_+ \wedge \varphi &= \frac{1}{2}\left[(u_k)_+ + \varphi - \left|(u_k)_+ - \varphi\right|\right] \\
&\xrightarrow{\mathcal{F}} \frac{1}{2}\left[u + \varphi - |u - \varphi|\right] \\
&= u \wedge \varphi = u \text{ as } k \to \infty.
\end{aligned}
$$

Since $(u_k)_+ \wedge \varphi \in \mathcal{F} \cap C_0(U)$, we conclude that $u \in \mathcal{F}(U)$, which finishes the proof. ∎

Proposition 2.9. *Let Ω be a precompact open subset of M and U be an open subset of M, and let K be a closed subset of M such that $K \subset U$ (see Fig. 3). Let $u \in \mathcal{F}$ be a function such that $u_+ \in \mathcal{F}(\Omega)$ and $u \leq \psi$ in $\Omega \setminus K$ for some $0 \leq \psi \in \mathcal{F}$. Then*
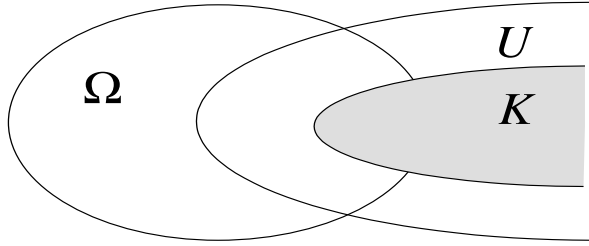
$$(u - \psi)_+ \in \mathcal{F}(\Omega \cap U). \tag{2.7}$$

FIGURE 3. Domains Ω, U and K.

Proof. Since $u - \psi \leq u_+ \in \mathcal{F}(\Omega)$, it follows by Proposition 2.4 that $(u - \psi)_+ \in \mathcal{F}(\Omega)$. Let us verify that

$$(u - \psi)_+ \in \mathcal{F}(U), \tag{2.8}$$

which will then imply (2.7) by Proposition 2.7. Indeed, noticing that $(u - \psi)_+ = 0$ in $\Omega \setminus K$ and in Ω^c, we see that

$$\mathrm{supp}\left((u - \psi)_+\right) \subset \overline{K \cap \Omega} \subset K \cap \overline{\Omega}.$$

On the other hand, the set $K \cap \overline{\Omega}$ is compact and is contained in U, so that (2.8) follows from Proposition 2.8. ∎

Remark 2.10. The statement of Proposition 2.9 was proved in [6, Proposition 4.10] under additional condition that $u \in L^{\infty}(M) \cap \mathcal{F}(\Omega)$ and $\mathrm{supp}(\psi)$ is compact. The present proof is also shorter than the one from [6].

3. Basic comparison theorem

The next theorem is the basic technical result of this paper.

Theorem 3.1. *Let (M, d, μ) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Let $\Omega \subset M$ be a precompact open set and $U \subset M$ be an open such that $\mu(U) < \infty$. Let u be a weak subsolution of the heat equation in $(0, T_0) \times (\Omega \cap U)$ where $T_0 \in (0, +\infty]$, such that*

$$u_+ (t, \cdot) \in \mathcal{F}(\Omega) \text{ for any } t \in (0, T_0), \tag{3.1}$$

$$u_+ (t, \cdot) \overset{L^2(\Omega \cap U)}{\longrightarrow} 0 \text{ as } t \to 0. \tag{3.2}$$

Let K be a closed subset of M such that $K \subset U$. Then, for any $t \in (0, T_0)$ and for almost all $x \in M$,

$$u(t, x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \leq t} \|u_+ (s, \cdot)\|_{L^\infty(\Omega \setminus K)}, \tag{3.3}$$

provided that $\sup_{0 < s \leq t} \|u_+ (s, \cdot)\|_{L^\infty(\Omega \setminus K)} < \infty$.

Remark 3.2. If $\Omega \subset U$, then all the conditions of Proposition 2.5 are satisfied, so that we conclude $u \leq 0$ in $(0, T_0) \times \Omega$. Hence, in this case the inequality (3.3) is trivially satisfied.

Remark 3.3. If U, Ω are open domains in $\mathbb{R}^n$ with smooth boundaries, then one can rephrase the statement of Theorem 3.1 for strong solutions as follows: if u solves the heat equation in $(0, T_0) \times \Omega \cap U$ and satisfies the initial and boundary conditions

$$\begin{aligned} u &\leq & 0 \text{ on } \partial\Omega \cap U \text{ (instead of } u_+ \in \mathcal{F}(\Omega)), \\ u &\leq & m \text{ on } \partial U \cap \Omega \text{ for some } m \geq 0 \text{ (instead of } u \leq m \text{ on } \Omega \setminus K), \\ u(t, \cdot) &\to & 0 \text{ as } t \to 0 \text{ in } \Omega \cap U, \end{aligned}$$

then $u \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) m$ in $(0, T_0) \times (\Omega \cap U)$ (see Fig 4). Indeed, the function $v = \left(1 - P_t^U \mathbf{1}\right) m$ satisfies the heat equation in $(0, \infty) \times U$, the boundary conditions $v \geq 0$ on $\partial\Omega$, $v = m$ on ∂U, and the initial condition $v(t, \cdot) \to 0$ as $t \to 0$ in U. Applying the classical parabolic maximum principle in $\Omega \cap U$, we obtain $u \leq v$.
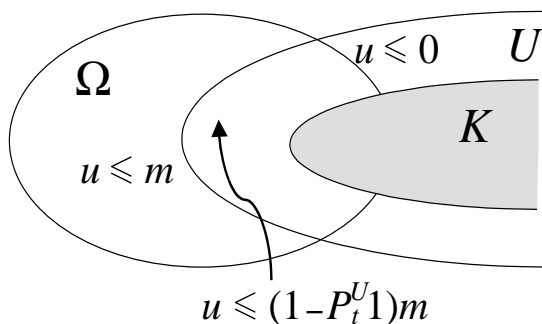

FIGURE 4. Illustration to Theorem 3.1 in the classical case.

Proof. Outside Ω the inequality (3.3) is trivial because $u \leq 0$ by (3.1). In $\Omega \setminus U$ (3.3) is also obvious because $P_t^U \mathbf{1}_U = 0$ and $K \subset U$. It remains to prove (3.3) in $\Omega \cap U$. Fix a number $T \in (0, T_0)$ and define m by

$$m = \sup_{0 < t \leq T} \|u_+(t, \cdot)\|_{L^\infty(\Omega \setminus K)}. \tag{3.4}$$

Let us first prove that, for any $t \in (0, T)$ and for μ-almost all $x \in \Omega \cap U$,

$$u(t, x) \leq m. \tag{3.5}$$

Let ϕ be a cut-off function for the couple (Ω, M) and consider the function

$$w = u - m\phi. \tag{3.6}$$

Then (3.5) will follow if we show that $w \leq 0$ in $(0, T) \times (\Omega \cap U)$. The latter will be proved by using the maximum principle of Proposition 2.5. We need to verify the following conditions.

- The function w is a weak subsolution of the heat equation in $(0, T) \times (\Omega \cap U)$. Indeed, the function ϕ, considered as a function of (t, x), is a weak supersolution of the heat equation in $(0, \infty) \times \Omega$, since for any non-negative function $\psi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(\phi, \psi) = \lim_{t \to 0} t^{-1} (\phi - P_t \phi, \psi) = \lim_{t \to 0} t^{-1} (1 - P_t \phi, \psi) \geq 0.$$

  Since u is a weak subsolution in $(0, T) \times (\Omega \cap U)$, we see from (3.6) that so is w.
- For any $t \in (0, T)$, we have $w_+(t, \cdot) \in \mathcal{F}(\Omega \cap U)$. Indeed, using the facts that $u_+(t, \cdot) \in \mathcal{F}(\Omega)$ and $u \leq m = m\phi$ in $\Omega \setminus K$ (which is true by (3.4)), we obtain from Proposition 2.9 that

$$w_+(t, \cdot) = (u(t, \cdot) - m\phi)_+ \in \mathcal{F}(\Omega \cap U).$$

- The initial condition $w_+(t, \cdot) \overset{L^2(\Omega \cap U)}{\to} 0$ as $t \to 0$ follows from $w_+(t, \cdot) \leq u_+(t, \cdot)$ and (3.2).

Therefore, by the parabolic maximum principle of Proposition 2.5, we conclude that $w \leq 0$ in $(0, T) \times (\Omega \cap U)$, thus proving (3.5).

We are now in a position to prove the following improvement of (3.5):

$$u \leq \left(1 - P_t^U \mathbf{1}_U\right) m \quad \text{in } (0, T) \times (\Omega \cap U) \tag{3.7}$$

(see Fig. 5 where the case $U \subset \Omega$ is shown). The path $t \mapsto u(t, \cdot)$ is weakly differentiable in $L^2(\Omega \cap U)$ and, hence, is strongly continuous in $L^2(\Omega \cap U)$ (see [7, Lemma 5.1]). The same applies to the path $t \to P_t^U \mathbf{1}_U$ so that the inequality (3.7) extends to $t = T$ by continuity. Hence, (3.7) implies (3.3). Consider the function

$$v = u - m\phi \left(1 - P_t^U \mathbf{1}_U\right), \tag{3.8}$$

where m and ϕ are the same as above. As $\mu(U) < \infty$, we have $\mathbf{1}_U \in L^2(U, \mu)$ and, hence, $P_t^U \mathbf{1}_U \in \mathcal{F}(U)$. We claim that v is a weak subsolution of the heat equation in $(0, T) \times (\Omega \cap U)$. Since u is a weak subsolution, it suffices to show that the function

$$f := \phi \left(1 - P_t^U \mathbf{1}_U\right)$$
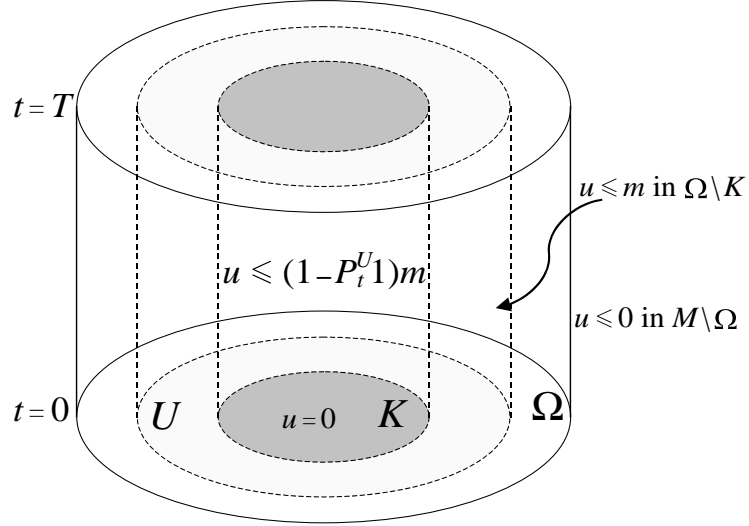
FIGURE 5. Illustration to the proof of (3.7) in the case $U \subset \Omega$.

is a weak supersolution in $(0, T) \times (\Omega \cap U)$. Since the both functions ϕ and $P_t^U \mathbf{1}_U$ belong to $L^\infty(M) \cap \mathcal{F}$, so does the product $\phi P_t^U \mathbf{1}_U$, whence

$$f = \phi - \phi P_t^U \mathbf{1}_U \in L^\infty(M) \cap \mathcal{F}.$$

For any $t, s \in (0, T)$, we have that in $\Omega \cap U$,

$$
\begin{aligned}
f - P_s f &= \phi \left(1 - P_t^U \mathbf{1}_U\right) - P_s \left(\phi \left(1 - P_t^U \mathbf{1}_U\right)\right) \\
&\geq \left(1 - P_t^U \mathbf{1}_U\right) - P_s \left(1 - P_t^U \mathbf{1}_U\right) \\
&= (1 - P_s 1) - P_t^U \mathbf{1}_U + P_s \left(P_t^U \mathbf{1}_U\right) \geq P_{t+s}^U \mathbf{1}_U - P_t^U \mathbf{1}_U,
\end{aligned}
$$

which yields that, for any $0 \leq \psi \in \mathcal{F}(\Omega \cap U)$,

$$
\begin{aligned}
\mathcal{E}(f, \psi) &= \lim_{s \to 0} \frac{1}{s} \left(f - P_s f, \psi\right) \\
&\geq \lim_{s \to 0} \frac{1}{s} \left(P_{t+s}^U \mathbf{1}_U - P_t^U \mathbf{1}_U, \psi\right) = \left(\frac{\partial}{\partial t} P_t^U \mathbf{1}_U, \psi\right).
\end{aligned}
$$

On the other hand,

$$
\left(\frac{\partial f}{\partial t}, \psi\right) = \left(-\phi \frac{\partial}{\partial t} P_t^U \mathbf{1}_U, \psi\right) = -\left(\frac{\partial}{\partial t} P_t^U \mathbf{1}_U, \psi\right).
$$

Therefore,

$$
\left(\frac{\partial f}{\partial t}, \psi\right) + \mathcal{E}(f, \psi) \geq 0,
$$

showing that f is a weak supersolution. Hence, we have proved that v is a weak subsolution.

Since $v \leq u$, it follows from (3.2) that

$$
v_+(t, \cdot) \overset{L^2(U \cap \Omega)}{\longrightarrow} 0 \text{ as } t \to 0.
$$

It remains to verify the boundary condition: $v_+(t, \cdot) \in \mathcal{F}(\Omega \cap U)$ for any $t \in (0, T)$. Observe that

$$u - m\phi \leq 0 \text{ in } M \qquad (3.9)$$

because

(1) $u - m\phi \leq 0$ in $M \setminus \Omega$ by (3.1),
(2) $u - m\phi = u - m \leq 0$ in $\Omega \setminus U$ by (3.4),
(3) $u - m\phi = u - m \leq 0$ in $\Omega \cap U$ by (3.5).

Using (3.9), we obtain that in M

$$v = u - m\phi \left(1 - P_t^U \mathbf{1}_U\right) \leq m\phi P_t^U \mathbf{1}_U \leq m P_t^U \mathbf{1}_U.$$

Since the function $P_t^U \mathbf{1}_U$ belongs to $\mathcal{F}(U)$, we conclude by using Proposition 2.4 that also $v_+ \in \mathcal{F}(U)$. On the other hand, we have

$$v = u - m\phi \left(1 - P_t^U \mathbf{1}_U\right) \leq u \leq u_+ \in \mathcal{F}(\Omega),$$

whence it follows that $v_+ \in \mathcal{F}(\Omega)$. Therefore, by 2.7 we obtain that $v_+ \in \mathcal{F}(U \cap \Omega)$, thus proving the boundary condition. Finally, we conclude by the maximum principle of Proposition 2.5 that $v \leq 0$ in $(0, T) \times (\Omega \cap U)$ whence (3.7) follows. $\blacksquare$

Remark 3.4. The boundary condition (3.1) in Theorem 3.1 can be relaxed as follows:

$$u_+(t, \cdot) \in \mathcal{F}(\Omega) \text{ for any } t \in (0, T_0) \cap \mathbb{Q}, \qquad (3.10)$$

provided one assumes in addition that

$$t \mapsto u(t, \cdot) \text{ is weakly continuous in } L^2(\Omega), \qquad (3.11)$$

$$t \mapsto \mathcal{E}(u(t, \cdot)) \text{ is locally bounded}, \qquad (3.12)$$

for $t \in (0, T_0)$. Under the hypotheses (3.10)-(3.12), the inequality (3.3) can be replaced by a stronger one:

$$u(t, x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{\substack{0 < s \leq t, \\ s \in \overline{\mathbb{Q}}}} \|u_+(s, \cdot)\|_{L^\infty(\Omega \setminus K)}. \qquad (3.13)$$

The proof goes exactly as above except that in the definition (3.4) of the constant m the supremum is taken only over *rational* $t \in (0, T]$. Then we need to verify that the functions w and v, defined by (3.6), (3.8), respectively, satisfy the boundary condition (3.1) for all *real* $t \in (0, T)$ in order to be able to use the maximum principle of Proposition 2.5. Indeed, for any $t \in (0, T)$, let $\{t_k\}_{k=1}^\infty$ be a sequence of rationals such that $t_k \to t$ as $k \to \infty$. By (3.6) and (3.11), we have

$$w(t_k, \cdot) - w(t, \cdot) = u(t_k, \cdot) - u(t, \cdot) \overset{L^2(\Omega)}{\longrightarrow} 0,$$

and thus

$$w_+(t_k, \cdot) \overset{L^2(\Omega)}{\longrightarrow} w_+(t, \cdot).$$

By (3.12), $\mathcal{E}(w(t_k, \cdot))$ is bounded as $k \to \infty$. Hence, we obtain by Proposition 2.2 that

$$w_+(t_k, \cdot) \overset{\mathcal{F}}{\longrightarrow} w_+(t, \cdot).$$

Since $w_+(t_k, \cdot) \in \mathcal{F}(\Omega)$ by (3.10), we conclude that $w_+(t, \cdot) \in \mathcal{F}(\Omega)$. Similarly, one has $v_+(t, \cdot) \in \mathcal{F}(\Omega)$ for all real $t \in (0, T)$, which finishes the proof.

The inequality (3.3) gives a rise to various interesting comparison inequalities for heat semigroups and heat kernels that will be presented in the next sections. Before that, let us state a useful particular case of Theorem 3.1 when $U \subset \Omega$ (cf. Fig. 5).

Corollary 3.5. *Let (M, d, μ) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Let $\Omega \subset M$ be a precompact open set and U be an open subset of Ω. Let u be a weak subsolution of the heat equation in $(0, T_0) \times U$ where $T_0 \in (0, +\infty]$, such that*

$$u_+(t, \cdot) \in \mathcal{F}(\Omega) \text{ for any } t \in (0, T_0), \tag{3.14}$$

$$u_+(t, \cdot) \xrightarrow{L^2(U)} 0 \text{ as } t \to 0.$$

Then the conclusion of Theorem 3.1 holds for any compact subset K of U, any $t \in (0, T_0)$ and almost all $x \in M$.

4. Comparison results for the heat semigroups

In this section, we give various applications of Theorem 3.1 to the semigroup solutions, including a specific case of quasi-local Dirichlet form.

4.1. **General regular Dirichlet forms.**

Proposition 4.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$, and let Ω, U be two non-empty open subsets of M such that $\mu(U) < \infty$. Let K be any closed subset of M such that $K \subset U$. Then, for any $0 \leq f \in L^2(\Omega)$ and for all $t > 0$ and almost all $x \in M$,*

$$P_t^\Omega f(x) - P_t^U f(x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(\Omega \setminus K)}. \tag{4.1}$$

Proof. Without loss of generality, assume that $0 \leq f \in L^\infty(\Omega)$ (otherwise, apply (4.1) to function $f_k = f \wedge k$ and then pass to the limit as $k \to \infty$). Let $\{\Omega_i\}$ be a sequence of precompact open subsets exhausting Ω. Consider the function

$$u(t, \cdot) := P_t^{\Omega_i} f - P_t^{\Omega_i \cap U} f$$

and verify that u satisfies all the hypothesis of Theorem 3.1 with the sets Ω_i and U. Indeed, u is a weak subsolution of the heat equation in $(0, \infty) \times (\Omega_i \cap U)$ because so are $P_t^{\Omega_i} f$ and $P_t^{\Omega_i \cap U} f$ (cf. Remark 2.1). Next, $u(t, \cdot) \in \mathcal{F}(\Omega_i)$ because both $P_t^{\Omega_i} f$ and $P_t^{\Omega_i \cap U} f$ belong to $\mathcal{F}(\Omega_i)$. Since both $P_t^{\Omega_i} f$ and $P_t^{\Omega_i \cap U} f$ converge to f as $t \to 0$ in $L^2(\Omega_i \cap U)$, we obtain $u(t, \cdot) \xrightarrow{L^2(\Omega_i \cap U)} 0$ as $t \to 0$. By Theorem 3.1, we obtain that

$$P_t^{\Omega_i} f - P_t^{\Omega_i \cap U} f \leq \left(1 - P_t^U \mathbf{1}_U\right) \sup_{0 < s \leq t} \|P_s^{\Omega_i} f - P_s^{\Omega_i \cap U} f\|_{L^\infty(\Omega_i \setminus K)}$$

$$\leq \left(1 - P_t^U \mathbf{1}_U\right) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(\Omega \setminus K)}.$$

Noticing that $P_t^{\Omega_i \cap U} f \leq P_t^U f$ and passing to the limit as $i \to \infty$, we obtain (4.1). ∎

Remark 4.2. Let us mention for comparison that the following inequality was proved in [6, Proposition 4.7]:

$$P_t^\Omega f(x) - P_t^U f(x) \le \sup_{0 < s \le t} \|P_s^\Omega f\|_{L^\infty(\Omega \setminus K)}. \tag{4.2}$$

Obviously, (4.1) is an improvement of (4.2). On the other hand, the estimate (4.2) was proved in [6] for arbitrary open set U without the hypotheses of the finiteness of its measure. For applications of (4.2) see [6, Theorem 5.12].

4.2. **Quasi-local Dirichlet forms.** Given an open set $U \subset M$ and non-negative real ρ, define the *ρ-neighborhood U_ρ* of U as follows:

$$
\begin{aligned}
U_\rho &= \{x \in M : d(x, U) < \rho\} \text{ if } \rho > 0, \\
U_\rho &= U \quad \text{if } \rho = 0,
\end{aligned}
$$

where $d(x, U) = \inf_{y \in U} d(x, y)$.

Theorem 4.3. *Assume that $(\mathcal{E}, \mathcal{F})$ is a ρ-local regular Dirichlet form in $L^2(M, \mu)$ where $\rho \ge 0$. Let U be an open subset of M such that U_ρ is precompact, and let u be a weak subsolution of the heat equation in $(0, T_0) \times U$ where $T_0 \in (0, +\infty]$. Assume that, for any $t \in (0, T_0)$, $u(t, \cdot) \in L^\infty(M)$ and*

$$u_+(t, \cdot) \xrightarrow{L^2(U)} 0 \text{ as } t \to 0. \tag{4.3}$$

Then, for any compact subset K of U, for all $t \in (0, T_0)$, and almost all $x \in U_\rho$,

$$u(t, x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \le t} \|u_+(s, \cdot)\|_{L^\infty(U_\rho \setminus K)}, \tag{4.4}$$

provided $\sup_{0 < s \le t} \|u_+(s, \cdot)\|_{L^\infty(U_\rho \setminus K)} < \infty$.

Proof. Since $P_t^U \mathbf{1}_U = 0$ outside U, the inequality (4.4) is trivially satisfied if $x \in U_\rho \setminus U$. Hence, it suffices to prove (4.4) if $x \in U$. Fix an open subset W of U such that $\overline{W} \subset U$. Then $\overline{W_\rho} \subset U_\rho$ so that W_ρ is precompact. Let ϕ be a cut-off function for the couple (W_ρ, U_ρ). Let us show that the function $w = u\phi$ satisfies all the hypothesis of Corollary 3.5 where the domains Ω, U are replaced by U_ρ and W respectively. Note that the function u does not satisfy the condition (3.14) so that we have to use w instead.

Let us first show that w is a weak subsolution of the heat equation in $(0, T_0) \times W$. Indeed, since $u(t, \cdot), \phi \in \mathcal{F} \cap L^\infty(M)$ for any $t \in (0, T_0) \times W$, it follows that $w(t, \cdot) \in \mathcal{F}$. Since u is a subsolution in $(0, T_0) \times W$ and $\phi \equiv 1$ in W, we have, for any non-negative function $\psi \in \mathcal{F}(W)$,

$$
\begin{aligned}
\left(\frac{\partial w}{\partial t}, \psi\right) = \left(\phi \frac{\partial u}{\partial t}, \psi\right) = \left(\frac{\partial u}{\partial t}, \psi\right) &\le -\mathcal{E}(u, \psi) \\
&= -\mathcal{E}(w, \psi) + \mathcal{E}((\phi - 1)u, \psi) = -\mathcal{E}(w, \psi), \tag{4.5}
\end{aligned}
$$

where we have used the fact that $\mathcal{E}((\phi - 1)u, \psi) = 0$ by the ρ-locality of $\mathcal{E}$, because $\operatorname{supp}(\psi) \subset \overline{W}$, and the function $(\phi - 1)u$ is compactly supported outside $\overline{W_\rho}$, so that the distance between the supports of ψ and $(\phi - 1)u$ is larger than ρ.

Since $\operatorname{supp} \varphi \subset U_\rho$ and, hence, $\operatorname{supp} w(t, \cdot) \subset U_\rho$, we see that $w(t \cdot) \in \mathcal{F}(U_\rho)$ and, hence, $w_+(t, \cdot) \in \mathcal{F}(U_\rho)$. Moreover, it follows from (4.3) that

$$w_+(t, \cdot) = \phi u_+(t, \cdot) \xrightarrow{L^2(W)} 0 \text{ as } t \to 0.$$

Hence, w satisfied the required boundary and initial conditions, and by Theorem 3.1 we obtain from (3.3) that in $(0, T_0) \times W$

$$u(t, x) = w(t, x) \leq \left(1 - P_t^W \mathbf{1}_W(x)\right) \sup_{0 < s \leq t} \|w_+(s, \cdot)\|_{L^\infty(U_\rho \setminus K)}$$

$$\leq \left(1 - P_t^W \mathbf{1}_W(x)\right) \sup_{0 < s \leq t} \|u_+(s, \cdot)\|_{L^\infty(U_\rho \setminus K)}.$$

Taking an exhaustion of U by sets like W and passing to the limit when $W \to U$, we obtain (4.4). ∎

Remark 4.4. If function u in Theorem 4.3 further satisfies (3.11) and (3.12) with $\Omega = U_\rho$, then we conclude from Remark 3.4 that the inequality (4.4) can be replaced by a stronger one:

$$u(t, x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{\substack{0 < s \leq t, \\ s \in \mathbb{Q}}} \|u_+(s, \cdot)\|_{L^\infty(U_\rho \setminus K)}. \tag{4.6}$$

For the case of local Dirichlet forms, we obtain the following improvement of Theorem 4.3 where the condition of the compactness of U_ρ is dropped.

Corollary 4.5. *Assume that $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form in $L^2(M, \mu)$. Let U be an open subset of M and let u be a weak subsolution of the heat equation in $(0, T_0) \times U$ where $T_0 \in (0, +\infty]$. Assume that, for any $t \in (0, T_0)$, $u(t, \cdot) \in L^\infty(M)$ and*

$$u_+(t, \cdot) \xrightarrow{L^2(U)} 0 \text{ as } t \to 0.$$

Then, for any compact subset K of U, for all $t \in (0, T_0)$, and almost all $x \in U$,

$$u(t, x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \leq t} \|u_+(s, \cdot)\|_{L^\infty(U \setminus K)}. \tag{4.7}$$

provided $\sup_{0 < s \leq t} \|u_+(s, \cdot)\|_{L^\infty(U \setminus K)} < \infty$.

Proof. Let $\{U_i\}_{i=1}^\infty$ be an exhaustion of U by precompact open sets U_i such that $K \subset U_i$ for all i. By Theorem 4.3, we obtain the estimate (4.7) for U_i instead of U, and then pass to the limit as $i \to \infty$. ∎

Remark 4.6. A particular case of the estimate (4.7) with $K = \emptyset$ was proved in [6, Lemma 4.3]. However, having an arbitrary K can be an advantage in certain situations. For example, if U is precompact and $u(t, \cdot)$ is continuous in $\overline{U}$ then taking exhaustion of U by compact sets $K \subset U$, one can replace the L^∞-norm in (4.7) by $\sup_{\partial U} u_+$.

Remark 4.7. If $(\mathcal{E}, \mathcal{F})$ is ρ-local with $\rho > 0$ and in addition all metric balls in M are precompact then the hypothesis of the compactness of U_ρ in Theorem 4.3 can be also dropped. Indeed, firstly, it suffices to assume that U is precompact, since it implies that U_ρ is precompact. Then one extends the result to all open sets U as in the proof of Corollary 4.5.

As another consequence of Theorem 4.3, we obtain the following useful comparison inequality for heat semigroups.

Corollary 4.8. *Assume that $(\mathcal{E}, \mathcal{F})$ is a ρ-local regular Dirichlet form in $L^2(M, \mu)$ where $\rho \geq 0$. Let U, Ω be two open subsets of M such that U_ρ is precompact and $U_\rho \subset \Omega$. Then, for any $0 \leq f \in L^\infty(M)$, for all $t > 0$ and almost all $x \in U_\rho$,*

$$P_t^\Omega f(x) - P_t^U f(x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(U_\rho \setminus K)} \qquad (4.8)$$

for any compact subset K of U.

Moreover, if $\rho = 0$, that is, $(\mathcal{E}, \mathcal{F})$ is local then the same is true without assuming that U_ρ is precompact. In this case, (4.8) becomes

$$P_t^\Omega f(x) - P_t^U f(x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(U \setminus K)}. \qquad (4.9)$$

Proof. Consider the function

$$u(t, \cdot) = P_t^\Omega f(\cdot) - P_t^U f(\cdot),$$

that is bounded on M for any $t > 0$, is a weak subsolution of the heat equation in $(0, \infty) \times U$, and satisfies the initial condition (4.3). Hence, if follows from (4.4) that, for all $t > 0$ and almost all $x \in U_\rho$,

$$P_t^\Omega f(x) - P_t^U f(x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \leq t} \|P_s^\Omega f - P_s^U f\|_{L^\infty(U_\rho \setminus K)},$$

whence (4.8) follows.

In the case of a local form, one passes from precompact U to arbitrary U as in the proof of Corollary 4.5. ∎

Remark 4.9. In fact, the inequality (4.8) can be improved as follows:

$$P_t^\Omega f(x) - P_t^U f(x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{\substack{0 < s \leq t, \\ s \in \mathbb{Q}}} \|P_s^\Omega f\|_{L^\infty(U_\rho \setminus K)}, \qquad (4.10)$$

because the function $u = P_t^\Omega f - P_t^U f$ automatically satisfies conditions (3.11) and (3.12). Indeed, since $U \subset \Omega$, it suffices to verify that the function $u = P_t^\Omega f$ satisfies (3.11) and (3.12). Then (3.11) follows from the strong continuity of the semigroup $\{P_t^\Omega\}$ in $L^2(\Omega)$ and (3.12) follows from the fact that $\mathcal{E}\left(P_t^\Omega f\right)$ is a decreasing function of t, the latter being a consequence of the identity

$$\mathcal{E}\left(P_t^\Omega f\right) = \int_0^\infty \lambda e^{-2\lambda t} d\left(E_\lambda f, f\right),$$

where $\{E_\lambda\}$ is the spectral resolution of the operator $\mathcal{L}^\Omega$.

Hence, (4.10) follows from (4.6).

Remark 4.10. The estimate (4.9) with $K = \emptyset$ was proved also in [6, (4.10) in Corollary 4.4]. Let us mention a useful particular case of (4.9) is when the function f vanishes in U. In this case, (4.8) becomes

$$P_t^\Omega f(x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0 < s \leq t} \|P_s^\Omega f\|_{L^\infty(U_\rho \setminus K)}. \qquad (4.11)$$

5. Comparison results for heat kernels

In this section we will prove a symmetric comparison inequality for the heat kernel of a ρ-local Dirichlet form. The motivation is as follows. Let $(\mathcal{E}, \mathcal{F})$ be an arbitrary regular Dirichlet form and $U, V \subset \Omega$ be three open subsets of M such that $U \cap V = \emptyset$. We claim that, for all $t, s > 0$ and μ-almost all $x \in U$, $y \in V$,

$$p_{t+s}^\Omega(x, y) \leq \left[1 - P_t^U \mathbf{1}_U(x)\right] \left\|p_s^\Omega(\cdot, y)\right\|_{L^\infty(\Omega \setminus U)}$$
$$+ \left[1 - P_s^V \mathbf{1}_V(y)\right] \left\|p_t^\Omega(\cdot, x)\right\|_{L^\infty(\Omega \setminus V)}. \tag{5.1}$$

Indeed, noticing that

$$\int\limits_{\Omega \setminus U} p_t^\Omega(x, z) d\mu(z) \leq 1 - P_t^\Omega \mathbf{1}_U(x) \leq 1 - P_t^U \mathbf{1}_U(x),$$

we obtain that

$$\int\limits_{\Omega \setminus U} p_t^\Omega(x, z) p_s^\Omega(z, y) d\mu(z) \leq \left\|p_s^\Omega(\cdot, y)\right\|_{L^\infty(\Omega \setminus U)} \int\limits_{\Omega \setminus U} p_t^\Omega(x, z) d\mu(z)$$
$$\leq \left[1 - P_t^U \mathbf{1}_U(x)\right] \left\|p_s^\Omega(\cdot, y)\right\|_{L^\infty(\Omega \setminus U)}. \tag{5.2}$$

In a similar way, we have

$$\int\limits_{\Omega \setminus V} p_t^\Omega(x, z) p_s^\Omega(z, y) d\mu(z) \leq \left[1 - P_s^V \mathbf{1}_V(y)\right] \left\|p_t^\Omega(\cdot, x)\right\|_{L^\infty(\Omega \setminus V)}. \tag{5.3}$$

Therefore, by the semigroup property,

$$p_{t+s}^\Omega(x, y) = \int\limits_\Omega p_t^\Omega(x, z) p_s^\Omega(z, y) d\mu(z)$$
$$\leq \int\limits_{\Omega \setminus U} p_t^\Omega(x, z) p_s^\Omega(z, y) d\mu(z) + \int\limits_{\Omega \setminus V} p_t^\Omega(x, z) p_s^\Omega(z, y) d\mu(z),$$

which together with (5.2) and (5.3) yields (5.1).

The purpose of the next theorem is to use the ρ-locality in order to replace in (5.1) the L^∞ norms in $\Omega \setminus U$, $\Omega \setminus V$ by those in smaller sets, which is frequently critical for applications.

Theorem 5.1. *Let $(\mathcal{E}, \mathcal{F})$ be a ρ-local regular Dirichlet form in $L^2(M, \mu)$ where $\rho \geq 0$, and let U, V, Ω be three open subsets of M such that U_ρ, V_ρ are precompact and $U_\rho, V_\rho \subset \Omega$. Assume that all the Dirichlet heat kernels p_t^U, p_t^V, p_t^Ω exist and that $p_t^\Omega(x, y)$ is locally bounded in $\mathbb{R}_+ \times \Omega \times \Omega$. Then, for all $t, s > 0$ and μ-almost all $x \in U$, $y \in V$,*

$$p_{t+s}^\Omega(x, y) \leq \int_\Omega p_t^U(x, z) p_s^V(z, y) d\mu(z) + \left[1 - P_t^U \mathbf{1}_U(x)\right] \sup_{s < t' \leq t+s} \left\|p_{t'}^\Omega(\cdot, y)\right\|_{L^\infty(U_\rho \setminus K_1)}$$
$$+ \left[1 - P_s^V \mathbf{1}_V(y)\right] \sup_{t < t' \leq t+s} \left\|p_{t'}^\Omega(\cdot, x)\right\|_{L^\infty(V_\rho \setminus K_2)}, \tag{5.4}$$

where K_1, K_2 are any compact subsets of U and V, respectively.

In the case $\rho = 0$, that is, when $(\mathcal{E}, \mathcal{F})$ is local, the assumption of the compactness of U_ρ, V_ρ can be dropped.

Proof. Let v be a non-negative function from $L^\infty \cap L^1(V)$. Setting $f = P_s^\Omega v$ and noticing that all the hypotheses of Corollary 4.8 are satisfied, we obtain by (4.10) that the following inequality is true in U for all $t > 0$:

$$
\begin{aligned}
P_{t+s}^\Omega v &\leq P_t^U \left(P_s^\Omega v\right) + \left[1 - P_t^U \mathbf{1}_U\right] \sup_{\substack{0 < t' \leq t, \\ t' \in \mathbb{Q}}} \|P_{t'+s}^\Omega v\|_{L^\infty(U_\rho \setminus K_1)} \\
&= P_t^U \left(P_s^\Omega v\right) + \left[1 - P_t^U \mathbf{1}_U\right] \sup_{\substack{s < t' \leq t+s, \\ t' \in \mathbb{Q}}} \|P_{t'}^\Omega v\|_{L^\infty(U_\rho \setminus K_1)} \quad (5.5)
\end{aligned}
$$

where we have used that $P_t^\Omega f = P_{t+s}^\Omega v$. Consider the function

$$
F(y) := \sup_{\substack{s < t' \leq t+s, \, z \in U_\rho \setminus K_1 \\ t' \in \mathbb{Q}}} \operatorname{esup} \, p_{t'}^\Omega(z, y),
$$

which is bounded in V. Note that $F(y)$ is measurable as the supremum of a countable family of measurable functions of y since

$$
y \mapsto \operatorname*{esup}_{z \in U_\rho \setminus K_1} p_{t'}^\Omega(z, y)
$$

is measurable by Proposition 7.1, and t' varies in $\mathbb{Q}$. We have then

$$
\sup_{\substack{s < t' \leq t+s, \\ t' \in \mathbb{Q}}} \|P_{t'}^\Omega v\|_{L^\infty(U_\rho \setminus K_1)} = \sup_{\substack{s < t' \leq t+s, \\ t' \in \mathbb{Q}}} \operatorname*{esup}_{z \in U_\rho \setminus K_1} \int_V p_{t'}^\Omega(z, y) \, v(y) \, d\mu(y)
$$

$$
\leq \int_V F(y) v(y) d\mu(y). \quad (5.6)
$$

Multiplying (5.5) by a non-negative function $u \in L^\infty \cap L^1(U)$ and integrating over U, we obtain

$$
\begin{aligned}
\left(P_{t+s}^\Omega v, u\right) &\leq \left(P_t^U \left(P_s^\Omega v\right), u\right) \\
&\quad + \int \int_{U \times V} \left[1 - P_t^U \mathbf{1}_U(x)\right] F(y) u(x) v(y) d\mu(x) d\mu(y). \quad (5.7)
\end{aligned}
$$

On the other hand, observe that

$$
\left(P_t^U \left(P_s^\Omega v\right), u\right) = \left(P_s^\Omega v, P_t^U u\right) = \left(v, P_s^\Omega P_t^U u\right). \quad (5.8)
$$

Using (4.10) again, now with $f = P_t^U u$ and with V in place of U, we obtain the following inequality in V:

$$
P_s^\Omega P_t^U u = P_s^\Omega f \leq P_s^V f + \left[1 - P_s^V \mathbf{1}_V\right] \sup_{\substack{0 < t' \leq s, \\ t' \in \mathbb{Q}}} \left\|P_{t'}^\Omega f\right\|_{L^\infty(V_\rho \setminus K_2)}. \quad (5.9)
$$

Observing that $P_t^U u \leq P_t^\Omega u$, we obtain that

$$
P_{t'}^\Omega f = P_{t'}^\Omega P_t^U u \leq P_{t'}^\Omega P_t^\Omega u = P_{t'+t}^\Omega u.
$$

Similarly to (5.6), we have

$$
\sup_{\substack{t < t' \leq t+s \\ t' \in \mathbb{Q}}} \left\|P_{t'}^\Omega u\right\|_{L^\infty(V_\rho \setminus K_2)} \leq \int_U G(x) u(x) d\mu(x)
$$

where
$$G(x) := \sup_{\substack{t<t'\leq t+s \\ t'\in\mathbb{Q}}} \operatorname*{esup}_{z\in V_\rho\setminus K_2} p_{t'}^\Omega(z,x)$$

is a bounded measurable function on U. Substituting into (5.9), we obtain that in V

$$P_s^\Omega P_t^U u \leq P_s^V\left(P_t^U u\right) + \left[1 - P_s^V \mathbf{1}_V\right]\int_U G(x)u(x)\,d\mu(x). \tag{5.10}$$

Multiplying (5.10) by v and integrating over V, we obtain

$$\left(v, P_s^\Omega P_t^U u\right) \leq \left(v, P_s^V\left(P_t^U u\right)\right) + \int\int_{U\times V}\left[1 - P_s^V \mathbf{1}_V(y)\right]G(x)u(x)v(y)d\mu(x)d\mu(y).$$

Combining this with (5.7) and (5.8), we obtain

$$\begin{aligned}\left(P_{t+s}^\Omega v, u\right) &\leq \left(v, P_s^V\left(P_t^U u\right)\right) \\ &+ \int\int_{U\times V}\left[1 - P_t^U \mathbf{1}_U(x)\right]F(y)u(x)v(y)d\mu(x)d\mu(y) \\ &+ \int\int_{U\times V}\left[1 - P_s^V \mathbf{1}_V(y)\right]G(x)u(x)v(y)d\mu(x)d\mu(y).\end{aligned}$$

Since

$$\left(v, P_s^V\left(P_t^U u\right)\right) = \int\int_{U\times V}\left(\int_\Omega p_t^U(x,z)p_s^V(z,y)d\mu(z)\right)u(x)v(y)d\mu(x)d\mu(y)$$

we can rewrite the previous inequality in the form

$$\int\int_{U\times V} p_{t+s}^\Omega(x,y)u(x)v(y)d\mu(x)d\mu(y) \leq \int\int_{U\times V}\Phi(x,y)\,u(x)v(y)d\mu(x)d\mu(y), \tag{5.11}$$

where

$$\Phi(x,y) = \int_{U\cap V} p_t^U(x,z)p_s^V(z,y)d\mu(z) + \left[1 - P_t^U \mathbf{1}_U(x)\right]F(y) + \left[1 - P_s^V \mathbf{1}_V(y)\right]G(x).$$

Obviously, $\Phi(x,y)$ is a bounded measurable function on $U\times V$. By [6, Lemma 3.4], the inequality (5.11) implies

$$p_{t+s}^\Omega(x,y) \leq \Phi(x,y)$$

for almost all $x\in U$ and $y\in V$, which proves (5.4).

In the case of a local form $(\mathcal{E},\mathcal{F})$, one obtains the claim for arbitrary open sets U,V by passing to the limit when exhausting U and V by precompact open sets. ∎

Remark 5.2. If $U\subset V$, it follows that

$$\int_M p_t^U(x,z)p_s^V(z,y)d\mu(z) \leq \int_M p_t^V(x,z)p_s^V(z,y)d\mu(z) = p_{s+t}^V(x,y).$$

Therefore, we obtain from (5.4) that

$$\begin{aligned}p_{t+s}^\Omega(x,y) &\leq p_{s+t}^V(x,y) + \left[1 - P_t^U \mathbf{1}_U(x)\right]\sup_{s<t'\leq t+s}\left\|p_{t'}^\Omega(\cdot,y)\right\|_{L^\infty(U_\rho\setminus K_1)} \\ &+ \left[1 - P_s^V \mathbf{1}_V(y)\right]\sup_{t<t'\leq t+s}\left\|p_{t'}^\Omega(\cdot,x)\right\|_{L^\infty(V_\rho\setminus K_2)}. \tag{5.12}\end{aligned}$$

On the other hand, if $U \cap V = \emptyset$, then using the fact that $p_s^V(z, y) = 0$ for μ-almost all $z \in U$, we obtain that

$$\int_M p_t^U(x, z) p_s^V(z, y) d\mu(z) = \int_U p_t^U(x, z) p_s^V(z, y) d\mu(z) = 0,$$

so that (5.4) becomes

$$p_{t+s}^\Omega(x, y) \leq \left[1 - P_t^U \mathbf{1}_U(x)\right] \sup_{s < t' \leq t+s} \left\|p_{t'}^\Omega(\cdot, y)\right\|_{L^\infty(U_\rho \setminus K_1)}$$

$$+ \left[1 - P_s^V \mathbf{1}_V(y)\right] \sup_{t < t' \leq t+s} \left\|p_{t'}^\Omega(\cdot, x)\right\|_{L^\infty(V_\rho \setminus K_2)}. \qquad (5.13)$$

6. Pointwise off-diagonal estimates of heat kernels

In this section we introduce a techniques for self-improvement of pointwise upper estimates of the heat kernel of a local, conservative, regular Dirichlet form. This issue was addressed in [10, 11, 5, 6] on abstract metric measure spaces, and in [1, 2, 9] on some fractals sets. Motivated by the application of symmetric comparison inequalities for the heat kernels in [8], we here present an alternative approach to such results, which is based on Theorem 5.1.

Let $\{P_t\}_{t \geq 0}$, $\{P_t^\Omega\}_{t \geq 0}$ be the semigroups of the Dirichlet forms $(\mathcal{E}, \mathcal{F})$, $(\mathcal{E}, \mathcal{F}(\Omega))$ respectively as before. For any $x \in M$ and $r > 0$, define the metric ball

$$B(x, r) = \{y \in M : d(x, y) < r\}.$$

For any ball $B = B(x, r)$ and any positive constant λ, denote by λB the ball $B(x, \lambda r)$.

Recall that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$ is called *conservative* if the heat semigroup $\{P_t\}_{t \geq 0}$ of $(\mathcal{E}, \mathcal{F})$ satisfies the following property:

$$P_t 1 = 1 \text{ in } M \text{ for any } t > 0.$$

Lemma 6.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is a conservative, regular Dirichlet form in $L^2(M, \mu)$, and let $\{P_t\}_{t \geq 0}$ be the heat semigroup of $(\mathcal{E}, \mathcal{F})$. Assume that $\phi(r, t)$ is a non-negative function on $(0, \infty) \times (0, \infty)$ such that $\phi(r, \cdot)$ is increasing in $(0, \infty)$ for every $r > 0$. If, for any $t > 0$ and any ball B in M of radius r,*

$$P_t \mathbf{1}_{B^c} \leq \phi(r, t) \text{ in } \frac{1}{4}B, \qquad (6.1)$$

then

$$1 - P_t^B \mathbf{1}_B \leq 2\phi\left(\frac{r}{4}, t\right) \text{ in } \frac{1}{4}B. \qquad (6.2)$$

Remark 6.2. A version of this statement appeared in [1, proof of Lemma 3.9] where a probabilistic proof was given. We follow the argument of [5, Theorem 3.1], [6, Theorem 5.13] where this statement was proved with some additional restrictions.

Proof. Applying the estimate (4.2) with $\Omega = M$, $U = B$, $K = \frac{3}{4}\overline{B}$ and $f = \mathbf{1}_{\frac{1}{2}B}$, we obtain that, for any $t > 0$ and almost everywhere in M,

$$P_t^B \mathbf{1}_{\frac{1}{2}B} \geq P_t \mathbf{1}_{\frac{1}{2}B} - \sup_{0 < s \leq t} \|P_s \mathbf{1}_{\frac{1}{2}B}\|_{L^\infty((\frac{3}{4}\overline{B})^c)}. \qquad (6.3)$$

For any $x \in \frac{1}{4}B$, we have that $B(x, r/4) \subset \frac{1}{2}B$ (see Fig. 6). Using the identity $P_t 1 = 1$, we have that, for any $x \in \frac{1}{4}B$,

$$P_t \mathbf{1}_{\frac{1}{2}B} = 1 - P_t \mathbf{1}_{\left(\frac{1}{2}B\right)^c} \geq 1 - P_t \mathbf{1}_{B(x,r/4)^c}.$$

Applying (6.1) for the ball $B(x, r/4)$, we see that

$$P_t \mathbf{1}_{B(x,r/4)^c} \leq \phi(r/4, t) \text{ in } B(x, r/16).$$

It follows that, for any $x \in \frac{1}{4}B$,

$$P_t \mathbf{1}_{\frac{1}{2}B} \geq 1 - \phi(r/4, t) \text{ in } B(x, r/16).$$

Covering $\frac{1}{4}B$ by a countable family of balls $B(x_k, r/16)$ where $x_k \in \frac{1}{4}B$, we obtain that

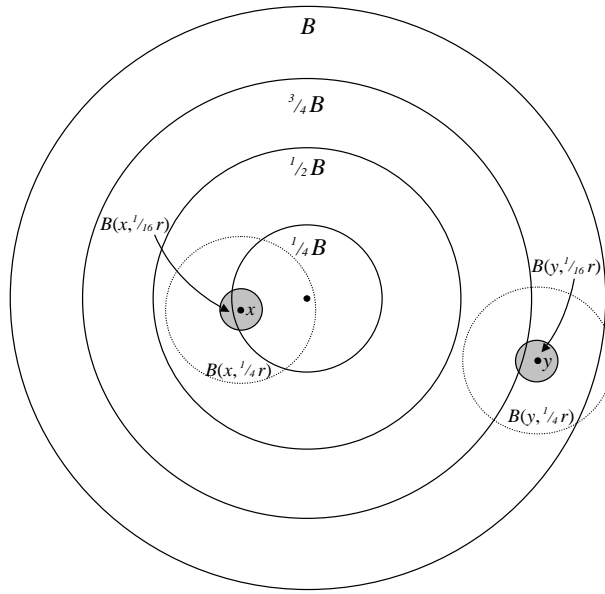$$P_t \mathbf{1}_{\frac{1}{2}B} \geq 1 - \phi(r/4, t) \text{ in } \frac{1}{4}B. \tag{6.4}$$

FIGURE 6. Illustration to the proof of Lemma 6.1

On the other hand, for any $y \in \left(\frac{3}{4}\overline{B}\right)^c$, we have that $\frac{1}{2}B \subset B(y, r/4)^c$, and so

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq P_s \mathbf{1}_{B(y,r/4)^c}.$$

Applying (6.1) for the ball $B(y, r/4)$ at time s and using the monotonicity of $\phi(r, s)$ in s, we obtain that, for any $0 < s \leq t$,

$$P_s \mathbf{1}_{B(y,r/4)^c} \leq \phi(r/4, s) \leq \phi(r/4, t) \text{ in } B(y, r/16).$$

It follows that, for any $y \in \left(\frac{3}{4}\overline{B}\right)^c$ and any $0 < s \leq t$,

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \phi(r/4, t) \text{ in } B(y, r/16),$$

which implies that

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \phi(r/4, t) \text{ in } \left(\frac{3}{4}\overline{B}\right)^c. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5), we obtain that, for any $t > 0$,

$$P_t^B \mathbf{1}_B \geq P_t^B \mathbf{1}_{\frac{1}{2}B} \geq 1 - 2\phi(r/4, t) \quad \text{in } \frac{1}{4}B, \tag{6.6}$$

which was to be proved. ∎

In the next statement, we use a function $F : M \times M \times (0, \infty) \to (0, \infty)$ with the following properties:

(F1): $F(x, y, s) = F(y, x, s)$ for all $x, y \in M$ and $s > 0$;
(F2): $F(x, y, s)$ is decreasing in s for any $x, y \in M$;
(F3): there exist $\alpha, C > 0$ such that

$$\frac{F(z, y, s)}{F(x, y, s)} \leq C \left(1 + \frac{d(x, z)}{s}\right)^{\alpha} \tag{6.7}$$

for all $x, y, z \in M$ and $s > 0$.

Theorem 6.3. *Let $(\mathcal{E}, \mathcal{F})$ be a conservative, local, regular Dirichlet form in $L^2(M, \mu)$. Let h be a positive increasing function on $(0, +\infty)$. Assume in addition that the following two conditions hold:*

(1) *The heat kernel p_t of $(\mathcal{E}, \mathcal{F})$ exists and satisfies the inequality*

$$p_t(x, y) \leq F(x, y, h(t)), \tag{6.8}$$

for all $t > 0$, μ-almost all $x, y \in M$, where F is a function that satisfies the conditions (F1)-(F3) above.

(2) *There exist $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\delta > 0$ such that, for any ball B of radius $r > 0$ and for any $t > 0$, we have*

$$P_t \mathbf{1}_{B^c} \leq \varepsilon \quad \text{in } \frac{1}{4}B \tag{6.9}$$

whenever $h(t) \leq \delta r$.

Then, for all $\lambda, t > 0$ and μ-almost all $x, y \in M$,

$$p_t(x, y) \leq CF\left(x, y, h\left(\frac{t}{2}\right)\right) \exp\left(-c't\Psi\left(\frac{cr}{t}\right)\right) \tag{6.10}$$

where $r = d(x, y)$, the constant C_0 is the same as in (6.7), and Ψ is defined by

$$\Psi(s) = \sup_{\lambda > 0} \left\{\frac{s}{h(1/\lambda)} - \lambda\right\}. \tag{6.11}$$

Proof. Fix $t > 0$, two distinct points $x_0, y_0 \in M$ and set $r = \frac{1}{2}d(x_0, y_0)$. Applying (5.13) with $U = B(x_0, r)$, $V = B(y_0, r)$, $\Omega = M$ and $\rho = 0$, we obtain that, for μ-almost all $x \in B(x_0, r)$ and $y \in B(y_0, r)$,

$$p_t(x, y) \leq [1 - P_{t/2}^U \mathbf{1}_U(x)] \sup_{t/2 < s \leq t} \operatorname*{esup}_{z \in B(x_0, r)} p_s(z, y) \tag{6.12}$$

$$+ [1 - P_{t/2}^V \mathbf{1}_V(y)] \sup_{t/2 < s \leq t} \operatorname*{esup}_{z \in B(y_0, r)} p_s(z, x). \tag{6.13}$$

In what follows, we estimate the term on the right-hand side of (6.12), while the term in (6.13) can be treated similarly. We claim that, for all $\lambda > 0$,

$$1 - P_{t/2}^U \mathbf{1}_U \leq C \exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right) \quad \text{in } \frac{1}{4}U. \tag{6.14}$$

Indeed, we see from (6.9) that the hypothesis (6.1) of Lemma 6.1 is satisfied with

$$\phi(r,t) = \begin{cases} \varepsilon, & \text{if } h(t) \le \delta r, \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, by Lemma 6.1, we obtain that, for all balls B of radius r,

$$1 - P_t^B \mathbf{1}_B \le 2\phi\left(\frac{r}{4}, t\right) \le 2\varepsilon \text{ in } \frac{1}{4}B,$$

provided that $h(t) \le \delta r/4$. It follows from [5, Theorem 3.4] (see also [6, Theorem 5.7]) that, for any ball B of radius r and for any $\lambda > 0$,

$$P_t \mathbf{1}_{B^c} \le C \exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right) \text{ in } \frac{1}{2}B.$$

Using Lemma 6.1 again, this time with the function

$$\phi(r,t) = C \exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right),$$

we obtain

$$1 - P_t^B \mathbf{1}_B \le 2C \exp\left(c'\lambda t - \frac{cr/4}{h(1/\lambda)}\right) \text{ in } \frac{1}{4}B,$$

which proves (6.14).

On the other hand, for all $z \in B(x_0, r)$ and $x \in B(x_0, r)$, we have that $z \in B(x, 2r)$, whence by condition $(F3)$

$$\frac{F(z, y, h(t/2))}{F(x, y, h(t/2))} \le C_0 \left(1 + \frac{2r}{h(t/2)}\right)^\alpha \le 2^\alpha C_0 \left(1 + \frac{r}{h(t/2)}\right)^\alpha.$$

Noting that h is increasing and $F(x, y, \cdot)$ is decreasing, we have from (6.8) that, for all $\frac{t}{2} \le s \le t$ and for μ-almost all $z \in B(x_0, r)$ and $y \in B(y_0, r)$,

$$p_s(z, y) \le F(z, y, h(s)) \le F(z, y, h(t/2))$$

$$= F(x, y, h(t/2)) \frac{F(z, y, h(t/2))}{F(x, y, h(t/2))} \le 2^\alpha C_0 F(x, y, h(t/2)) \left(1 + \frac{r}{h(t/2)}\right)^\alpha.$$

Therefore, we have, for almost all $y \in B(y_0, r)$,

$$\sup_{t/2 < s \le t} \operatorname*{esup}_{z \in B(x_0, r)} p_s(z, y) \le CF(x, y, h(t/2)) \left(1 + \frac{r}{h(t/2)}\right)^\alpha. \tag{6.15}$$

Combining (6.14) and (6.15) and estimating similarly the term in (6.13), we obtain from (6.12)-(6.13) that, for μ-almost all $x \in B\left(x_0, \frac{1}{4}r\right)$, $y \in B\left(y_0, \frac{1}{4}r\right)$,

$$p_t(x, y) \le CF(x, y, h(t/2)) \left(1 + \frac{r}{h(t/2)}\right)^\alpha \exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right). \tag{6.16}$$

In order to absorb the middle term to the exponential on the right-hand side in (6.16), fix r, t and consider the function

$$G(\lambda) := \frac{c'r}{h(1/\lambda)} - c\lambda t.$$

Using this with $\lambda = 2/t$ and the elementary inequality

$$\alpha \log(1 + s) \le \frac{c'}{2}s + c'', \quad s \ge 0,$$

where c' is as above and $c'' = c''(c', \alpha)$ is large enough, we obtain that

$$
\begin{aligned}
\alpha \log \left(1 + \frac{r}{h(t/2)}\right) &\leq \frac{1}{2} \frac{c'r}{h(t/2)} + c'' \\
&= \frac{1}{2} G(2/t) + c + c'' \leq \frac{1}{2} \sup_{\lambda > 0} G(\lambda) + c + c''.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left(1 + \frac{r}{h(t/2)}\right)^\alpha \exp \left(-\sup_{\lambda > 0} G(\lambda)\right) &\leq \exp \left(-\frac{1}{2} \sup_{\lambda > 0} G(\lambda) + c + c''\right) \\
&\leq C \exp \left(-\frac{1}{2} \sup_{\lambda > 0} G(\lambda)\right) \\
&\leq C \exp \left(-\frac{1}{2} G(\lambda)\right).
\end{aligned}
$$

Therefore, we obtain from (6.16) that, for any $\lambda > 0$ and μ-almost all $x \in B\left(x_0, \frac{1}{4}r\right)$, $y \in B\left(y_0, \frac{1}{4}r\right)$,

$$
p_t(x, y) \leq CF(x, y, h(t/2)) \exp \left(-\frac{1}{2} G(\lambda)\right). \tag{6.17}
$$

Since $M \times M \backslash \text{diag}$ can be covered by a countable family of sets $B\left(x_0, \frac{1}{4}r\right) \times B\left(y_0, \frac{1}{4}r\right)$ as above, it follows that (6.17) holds for μ-almost all $x, y \in M$. Taking sup in $\lambda > 0$, we obtain (6.10). ∎

Let us give an example to illustrate Theorem 6.3. Set

$$
V(x, r) := \mu(B(x, r))
$$

and assume in the sequel that the following volume doubling condition (VD) is satisfied: there is a constant $C_D \geq 1$ such that

$$
V(x, 2r) \leq C_D V(x, r). \tag{6.18}
$$

for all $x \in M$ and $r > 0$. It is known that (VD) implies the existence of a constant $\alpha > 0$ such that

$$
\frac{V(x, R)}{V(y, r)} \leq C_D \left(\frac{d(x, y) + R}{r}\right)^\alpha \tag{6.19}
$$

for all $x, y \in M$ and $0 < r \leq R$ (see, for example, [6]).

Define functions h and F as follows:

$$
h(t) = t^{1/\beta}
$$

and

$$
F(x, y, s) = \frac{C}{\sqrt{V(x, h(s)) V(y, h(s))}},
$$

for all $t, s > 0$ and $x, y \in M$, where $\beta > 1$ is some constant. It follows from (6.19) that F satisfies conditions $(F1)$-$(F3)$. It is easy to see that the supremum in (6.11) is attained at $\lambda = cs^{\frac{\beta}{\beta - 1}}$ so that

$$
\Psi(s) = cs^{\frac{\beta}{\beta - 1}}.
$$

The estimate (6.10) becomes

$$p_t(x,y) \le \frac{C}{\sqrt{V(x,t^{1/\beta})V(y,t^{1/\beta})}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right),$$

for all $t > 0$ and almost all $x, y \in M$. Using (6.19) again and applying the same argument as in the proof of Theorem 6.3, we obtain that

$$p_t(x,y) \le \frac{C}{V(x,t^{1/\beta})} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right). \tag{6.20}$$

In particular, if $V(x,r) \simeq r^\alpha$ for some $\alpha > 0$, then (6.20) becomes

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right). \tag{6.21}$$

Remark 6.4. The estimate of type (6.21) was obtained in [3] for the Sierpinski gasket, and in [2] for the Sierpinski carpet, and in [9] for a certain class of post-critically finite self-similar sets. The estimate (6.20) with $\beta = 2$ was obtained by Li and Yau [13] for Riemannian manifolds of non-negative curvature, and with any $\beta > 1$ by Kigami [12] for some general class of self-similar sets.

7. Appendix

Proposition 7.1. *Let $F(x, y)$ be a non-negative μ-measurable function of $x, y \in M$. Then the function*

$$f(x) = \operatorname*{esup}_{y} F(x,y)$$

is measurable.

Proof. Fix a pointwise realization of F. Assume first that F is bounded. For any $x \in M$, consider the mapping

$$L^1 \ni \varphi \longmapsto T\varphi(x) := \int_M F(x,y)\varphi(y)d\mu(y)$$

which is a bounded linear functional on L^1. We have

$$f(x) = \sup_{\|\varphi\|_1 \le 1} T\varphi(x).$$

Since T is continuous in φ, the supremum can be replaced by the one over a dense subset $S \subset L^1$, that is,

$$f(x) = \sup_{\|\varphi\|_1 \le 1, \varphi \in S} T\varphi(x).$$

Since $T\varphi$ is a measurable function, the supremum over a countable family is also measurable, and hence, the function f is measurable.

For an arbitrary F, consider $F_k = F \wedge k$, we have from above that $f_k(x) := \operatorname{esup}_y F_k(x,y)$ is measurable. Note that the sequence $\{f_k\}_{k=1}^\infty$ increases and converges to f pointwise as $k \to \infty$. Hence, the function f is measurable. ■

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