Heat Kernels and Besov Spaces on Metric Measure Spaces

Jun Cao and Alexander Grigor'yan

Abstract. Let (M, ρ, μ) be a metric measure space satisfying the volume doubling condition. Assume also that (M, ρ, μ) supports a heat kernel satisfying the upper and lower Gaussian bounds. We study the problem of identity of two families of Besov spaces $B^s_{p,q}$ and $B^{s,\mathcal{L}}_{p,q}$, where the former one is defined using purely the metric measure structure of M, while the latter one is defined by means of the heat semigroup associated with the heat kernel. We prove that the identity $B^s_{p,q} = B^{s,\mathcal{L}}_{p,q}$ holds for a range of parameters p, q, s given by some Hardy-Littlewood-Sobolev-Kato diagram.

Contents

1 Introduction

1.1 Motivation and background

This work is devoted to the notion of Besov spaces in the setting of metric measure spaces. It is customary to use various scales of function spaces, in particular, Besov spaces $B^s_{p,q}$, in order to measure the degree of smoothness of functions. Introduction of the Besov spaces in $\mathbb{R}^d$ was motivated

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by the imbedding and extension problems [8]. Besides, the interpolation of Sobolev spaces leads also to Besov spaces. For the theory of Besov spaces in $\mathbb{R}^d$ there is an extensive literature, see, for example, [4, 15, 42, 44, 45, 46, 47, 50].

In a general metric measure space there are various natural ways to define the family of Besov spaces. One possibility is to use directly the metric and measure of the underlying space in order to define the Besov seminorm (see (1.8) and (1.9a) below). The function spaces obtained in this way are called *Lipschitz type Besov spaces* and are denoted by $B^s_{p,q}$. Another possibility to define the Besov seminorm arises in the presence of a heat semigroup $e^{-t\mathcal{L}}$ with the generator $\mathcal{L}$ acting in L^2 (see (1.10) and (1.11a) below). We refer to such spaces as $B^{s,\mathcal{L}}_{p,q}$.

The Lipschitz type Besov spaces were considered in [18, 20, 21, 34, 38, 40, 43, 49]) while the spaces $B^{s,\mathcal{L}}_{p,q}$ were dealt with in [9, 10, 17, 28, 29, 33, 36]. For other definitions of Besov spaces, we refer the reader to [1, 2, 3, 25, 27, 35, 48].

Jonsson [32] introduced the spaces $B^s_{p,q}$ on the Sierpiński gasket and proved that the domain of the associated Dirichlet form coincides with $B^{\beta/2}_{2,\infty}$, where β is the walk dimension (see also [20, 21] for an extension of this result to general metric measure spaces). Hu and Zähle [29] proved that, in the setting of metric measure spaces, $B^{s,\mathcal{L}}_{2,2}$ coincides with some Bessel potential space H^s_2.

In 2010 Pietruska-Pałuba raised in [41] the following question:

Under what conditions the two spaces $B^s_{p,q}$ and $B^{s,\mathcal{L}}_{p,q}$ are identical?

This question has attracted a lot of attention. In $\mathbb{R}^d$ with $\mathcal{L} = -\Delta$, the identity

$$B^s_{p,q} = B^{s,\mathcal{L}}_{p,q} \tag{1.1}$$

has been known for long time for all $p, q \in (1, \infty)$ and $s \in (0, 1)$ (see [45, 46]). However, in the case when $\mathcal{L} = -\mathrm{div}(A\nabla)$ is a uniformly elliptic operator in $\mathbb{R}^d$ with real symmetric measurable coefficients, the identity (1.1) can be only guaranteed when $(\frac{1}{p}, s)$ lies in certain convex polygon (shaded area on Fig.1), while $q \in (1, \infty)$ is any (see [11] for the details).

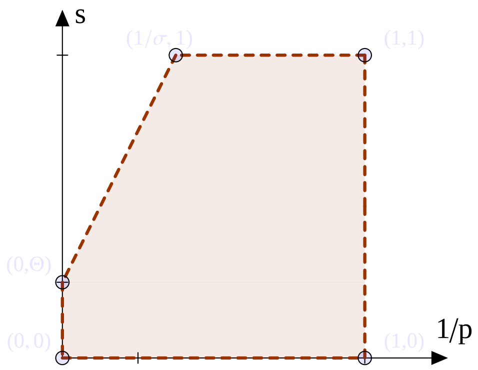

Figure 1: the range for p and s for $\mathcal{L} = -\mathrm{div}(A\nabla)$

Here $\Theta \in (0, 1)$ is the Hölder exponent of the heat kernel as is described in (1.7) below, and $\sigma > 2$ is determined by the range of $p \in (1, \infty)$ such that the Riesz transform associated with $\mathcal{L}$ is L^p bounded.

Hu and Zähle [29] proved the identity (1.1) on metric measure spaces for $p = q = 2$ and for all $s \in (0, 1)$ assuming that the heat semigroup $e^{-t\mathcal{L}}$ has the heat kernel satisfying certain upper and lower bounds (see also [41] for a similar result).

It was shown later in [22] that, under similar hypothesis, (1.1) holds for all $p, q \in (1, \infty)$ and $s \in (0, \Theta)$ (see also [49] for some similar results in the setting of RD-space). On Fig. 2, the range of the parameters p and s is shown for which (1.1) is true according to [22, 29, 41, 49].

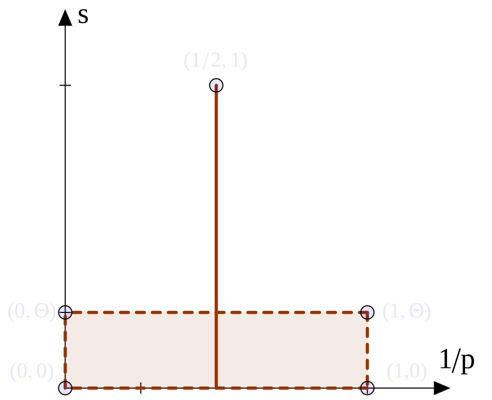

Figure 2: previously known range of p and s

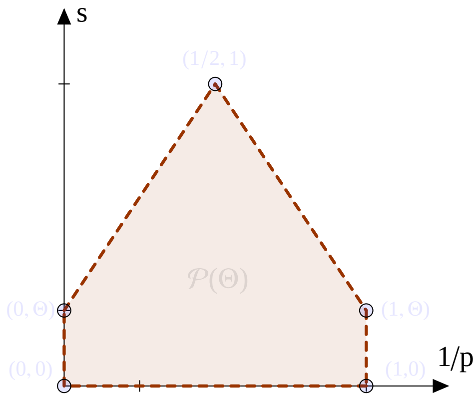

Figure 3: the new range of p and s

In the present paper we study the aforementioned problem in the setting of metric measure spaces, under the hypothesis that the heat kernel satisfies Gaussian upper and lower bounds. Our main result, Theorem 1.2, says that (1.1) holds for any $q \in (1, \infty]$ and any $(\frac{1}{p}, s)$ lying in a Hardy-Littlewood-Sobolev-Kato diagram $\mathcal{P}(\Theta)$ as shown on Fig. 3, which clearly significantly enlarges the domain of p, s from Fig. 2.

Besides, we prove in Theorem 1.3 that the identity (1.1) is true for the full range $(\frac{1}{p}, s) \in (0, 1)^2$, provided a further assumption on the domain of the square root of $\mathcal{L}$ in the L^p scale is satisfied.

Our proofs use completely new techniques based on wavelets with almost Lipschitz continuity. Such wavelets were constructed in [6, 31] merely from the metric structure of the underlying space. We use the wavelets to determine the interpolation spaces of certain Lipschitz type function spaces, which together with the hypothesis about the heat kernel estimates enables us to prove some Hardy-Littlewood-Sobolev-Kato estimates associated with $\mathcal{L}$. These estimates finally give us the range of the parameters p, s ensuring the validity of (1.1).

This paper is organized as follows. In Section 1.2, we state the main results of this paper: Theorems 1.2 and 1.3 as well as introduce some necessary notions and notation. In Section 2, we give the wavelet characterizations of the Lipschitz-type function spaces. In Section 3, we establish the real and complex interpolations of those spaces. Finally, in Section 4, we prove Theorems 1.2 and 1.3.

1.2 Setup and main results

Let (M, ρ, μ) be a locally compact complete separable metric measure space, where ρ is a metric and μ is a nonnegative Radon measure with full support on M. We say that (M, ρ, μ) satisfies *volume doubling* **(VD)** if for any $x \in M$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \le C_0 \mu(B(x, r)), \tag{1.2}$$

where $B(x, r) := \{y \in M : \ \rho(y, x) < r\}$ denotes the open ball centered at x of radius r and $C_0 > 1$ is a positive constant independent of x and r. It is easy to see that the condition **(VD)** implies that, for all

$x \in M$, $r \in (0, \infty)$ and $\lambda \in (1, \infty)$,

$$\mu(B(x, \lambda r)) \leq C_0 \lambda^d \mu(B(x, r)), \tag{1.3}$$

where $d := \log_2 C_0 > 0$. For any $p \in [1, \infty]$, consider the Lebesgue space $L^p(M) := L^p(M, \mu)$.

To conduct a smoothness analysis on (M, ρ, μ), we use the notion of a heat kernel.

Definition 1.1 ([20]). A family $\{p_t\}_{t>0}$ of $\mu \otimes \mu$-measurable functions on $M \times M$ is called a *heat kernel* if the following conditions are satisfied for μ-almost all $x, y \in M$ and all $s, t > 0$:

(i) Positivity: $p_t(x, y) \geq 0$.

(ii) Stochastic completeness: $\int_M p_t(x, y)\, d\mu(y) \equiv 1$.

(iii) Symmetry: $p_t(x, y) = p_t(y, x)$.

(iv) Semigroup property: $p_{t+s}(x, y) = \int_M p_s(x, z) p_t(z, y)\, d\mu(y)$.

(v) Approximation of identity: for any $f \in L^2(M)$,

$$\lim_{t \to 0_+} \int_M p_t(\cdot, y) f(y)\, d\mu(y) = f$$

in $L^2(M)$.

In many occasions, a heat kernel appears as the integral kernel of a heat semigroup $\{P_t\}_{t \geq 0}$ that is associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(M)$ (see [16]). Conversely, given a heat kernel $\{p_t\}_{t>0}$ as in Definition 1.1, one constructs an associated heat semigroup $\{P_t\}_{t \geq 0}$ acting on $L^2(M)$ by

$$P_t f(x) := \int_M p_t(x, y) f(y)\, d\mu(y) \tag{1.4}$$

for any $f \in L^2(M)$, $t \in (0, \infty)$ and μ-almost all $x \in M$, and $P_0 f = f$. Denote by $\mathcal{L}$ be the generator of $\{P_t\}_{t \geq 0}$ so that $P_t = e^{-t\mathcal{L}}$.

The metric measure space (M, ρ, μ) is said to satisfy the *Gaussian bounds* **(GB)** if there exists a heat kernel $\{p_t\}_{t>0}$ on $M \times M$ such that

$$\frac{1}{C_1 t^{d/2}} \exp\left\{-\frac{c_0 \rho(x, y)^2}{t}\right\} \leq p_t(x, y) \leq \frac{C_1}{t^{d/2}} \exp\left\{-\frac{c_1 \rho(x, y)^2}{t}\right\} \tag{1.5}$$

for μ-almost all x, $y \in M$ and any $t \in (0, \infty)$, where C_1, d, c_0 and c_2 are positive constants that are independent of x, y and t.

For example, the classical Gaussian-Weierstrass heat kernel in $\mathbb{R}^d$

$$p_t(x, y) = \frac{1}{(4\pi t)^{d/2}} \exp\left\{-\frac{|x - y|^2}{4t}\right\} \tag{1.6}$$

is associated with the Dirichlet form

$$\mathcal{E}(f, f) := \int_{\mathbb{R}^d} |\nabla f|^2\, dx$$

with domain $\mathcal{F} = W^{1,p}(\mathbb{R}^d)$, and its generator is $-\Delta$, where Δ is the Laplace operator. Clearly, (1.6) satisfies **(GB)**.

The condition **(GB)** implies the following two conditions:

(i) the volume regularity: for any $x \in M$ and $r \in (0, \infty)$,

$$\mu(B(x,r)) \simeq r^d,$$

where the implicit constants are independent of x and r (see [20, Theorem 3.1]). In this case, it is easy to see **(VD)** is true and $\mu(M) = \infty$;

(ii) the Hölder regularity: the heat kernel $\{p_t\}_{t>0}$ satisfies the estimate

$$\left|p_t(x, y) - p_t(x, y')\right| \le C_2 \left(\frac{\rho(y, y')}{\sqrt{t}}\right)^{\Theta} \frac{1}{t^{d/2}} \exp\left\{-\frac{c_2[\rho(y, y')]^2}{t}\right\} \tag{1.7}$$

for any $t \in (0, \infty)$ and all $x, y, y' \in M$ such that $\rho(y, y') \le \sqrt{t}$, where the constants Θ, C_2, c_2 are positive and depend only on M; besides $\Theta \in (0, 1)$ (see [23]).

Now let (M, ρ, μ) be a metric measure space satisfying **(VD)**. We introduce the *Lipschitz Besov space* $B^s_{p,q}$ for any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, \infty)$ by

$$B^s_{p,q} := \left\{ f \in L^p(M) : \ \|f\|_{B^s_{p,q}} := \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}, \tag{1.8}$$

where

$$\|f\|_{\dot{B}^s_{p,q}} := \left\{ \int_0^\infty r^{-sq} \left[\int_M \fint_{B(x,r)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \right]^{q/p} \frac{dr}{r} \right\}^{1/q} \tag{1.9a}$$

with the usual modification when $q = \infty$. Here and hearafter,

$$\fint_B := \frac{1}{\mu(B)} \int_B$$

denotes the integral mean over the set B.

Note that this definition of $B^s_{p,q}$ does not depend on the operator $\mathcal{L}$ or the heat kernel.

On the other hand, let (M, ρ, μ) be a metric measure space satisfying **(GB)**. We introduce the *heat Besov space* $B^{s,\mathcal{L}}_{p,q}$ for any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in [0, \infty)$ by

$$B^{s,\mathcal{L}}_{p,q} := \left\{ f \in L^p(M) : \ \|f\|_{B^{s,\mathcal{L}}_{p,q}} := \|f\|_{L^p} + \|f\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} < \infty \right\}, \tag{1.10}$$

where

$$\|f\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} := \left\{ \int_0^\infty \left[t^{-s/2} \left\| (t\mathcal{L})^k e^{-t\mathcal{L}} f \right\|_{L^p} \right]^q \frac{dt}{t} \right\}^{1/q} \tag{1.11a}$$

with some $k \in \mathbb{Z}_+ \cap (s/2, \infty)$ and we make the usual modification when $q = \infty$. As it was pointed out in [22, Proposition 2.9] (see also [33, Theorem 6.1]), the norms $\|\cdot\|_{B^{s,\mathcal{L}}_{p,q}}$ in (1.10) are equivalent for different choices of $k \in \mathbb{Z}_+ \cap (s/2, \infty)$, so that the space $B^{s,\mathcal{L}}_{p,q}$ does not depend on k. Both $B^s_{p,q}$ and $B^{s,\mathcal{L}}_{p,q}$ are Banach spaces. We refer to [22, 33, 49] for further properties of these spaces.

We use the Hölder exponent $\Theta \in (0, 1)$ from (1.7) in order to define the following domain

$$\mathcal{P}(\boldsymbol{\Theta}) := \left\{ \left(\frac{1}{p}, s\right) \in (0, 1) \times (0, 1) : \ \frac{1}{p} \in \begin{cases} (0, 1), & s \in (0, \Theta), \\ \left(\frac{s-\Theta}{2(1-\Theta)}, \frac{2-s-\Theta}{2(1-\Theta)}\right), & s \in [\Theta, 1) \end{cases} \right\} \tag{1.12}$$

that is a convex polygon in the $(\frac{1}{p}, s)$-plane as illustrated on Fig. 3. Following the terminology in [5], we refer to $\mathcal{P}(\Theta)$ as a *Hardy-Littlewood-Sobolev-Kato diagram*.

Our main result is stated in the next theorem.

Theorem 1.2. *Let (M, ρ, μ) be a metric measure space satisfying* **(GB)**, *and let $\mathcal{P}(\Theta)$ be as on Fig. 3 (see also (1.12)). Then, for any $(1/p, s) \in \mathcal{P}(\Theta)$ and $q \in (1, \infty]$, we have the identity*

$$B^s_{p,q} = B^{s,\mathcal{L}}_{p,q}. \tag{1.13}$$

For the proof of Theorem 1.2 we use the same strategy that we employed in [11] in the setting of elliptic operators in Euclidean spaces. For that, we first consider the corresponding question for Triebel-Lizorkin spaces at the endpoint values $s = 1$ and $s < \Theta$, and use interpolation to produce the desired range of parameters. However, unlike the Euclidean setting, in the present abstract setting there is no interpolation theory for Lipschitz-type function spaces. To overcome this difficulty, we apply the technique of wavelets that enables us to establish the desired interpolation. The wavelets on metric measure spaces were constructed by Hytönen and Tapiola [31]. The almost Lipschitz regularity of these wavelets is essential in Theorem 1.2 – this allows to extend the range of the parameter s from $s \in (0, \Theta)$ as in [22] to $s \in (0, 1)$.

Another difficulty in the metric measure setting occurs at the endpoint $s = 1$, which is related to the domain of the square root $\mathcal{L}^{1/2}$ of the generator $\mathcal{L}$. Recall that, for any $p \in (1, \infty)$ and $s \in (0, 1]$, the domain of the fractional power $\mathcal{L}^{s/2}$ of $\mathcal{L}$ in the space L^p is defined to be the space

$$\mathrm{dom}_p(\mathcal{L}^{s/2}) := \left\{ f \in L^p : \ \mathcal{L}^{s/2} f \in L^p \right\} \tag{1.14}$$

endowed with the norm

$$\|f\|_{\mathrm{dom}_p(\mathcal{L}^{s/2})} := \|f\|_{L^p} + \left\|\mathcal{L}^{s/2} f\right\|_{L^p}. \tag{1.15}$$

Recall that the fractional power $\mathcal{L}^{s/2}$ is defined via the functional calculus (see, for example, [24]).

In the Euclidean case with $\mathcal{L} = -\mathrm{div}(A\nabla)$, we usually have

$$\mathrm{dom}_p(\mathcal{L}^{1/2}) = W^{1,p}$$

for any $p \in (1, \sigma)$ with $\sigma > 2$ depending on $\mathcal{L}$ (see Figure 1), where $W^{1,p}$ is the classical Sobolev space (see [5, Theorem 4.15]). However, in general metric measure spaces, we only have

$$\mathrm{dom}_2(\mathcal{L}^{1/2}) = B^1_{2,\infty}$$

see [20, Theorem 5.1]. Let us emphasize that the aforementioned difference in the characterizations of the domain of $\mathcal{L}^{1/2}$ leads to the difference in the ranges of the parameters on Fig. 1 and Fig. 2.

To obtain the full range of parameters for the identity (1.13), we introduce the following condition for the characterization of $\mathrm{dom}_p(\mathcal{L}^{1/2})$.

(DF). $\mathrm{dom}_p(\mathcal{L}^{1/2}) = B^1_{p,\infty}$ for any $p \in (1, \infty)$.

The condition (**DF**) holds true if $M = \mathbb{R}^d$ and $\mathcal{L} = \Delta$ is the Laplace operator. In the general setting of metric measure space (M, ρ, μ) satisfying **(GB)**, let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form with the generator $\mathcal{L}$. It was proved in [2, Corollary 4.10] that the condition **(DF)** holds provided $(\mathcal{E}, \mathcal{F})$ is strongly local, regular, and satisfies some strong Bakry-Émery curvature condition. On the other hand, there exist examples of manifolds and graphs where **(DF)** is not satisfied for any $p \neq 2$ (see [12, Theorem 5.1]).

Under Assumption **(DF)**, the next theorem establishes the identity (1.1) for the full range of parameters.

Theorem 1.3. *Let (M,ρ,μ) be the metric measure space satisfying* **(GB)** *and* **(DF)**. *Then, for all $p \in (1,\infty)$, $q \in (1,\infty]$ and $s \in (0,1)$, we have*

$$B^s_{p,q} = B^{s,\mathcal{L}}_{p,q}.$$

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Notation. Let $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any subset $E \subset M$, $\mathbf{1}_E$ denotes its *characteristic function*. We use C to denote a *positive constant* that is independent of the main parameters involved, whose value may differ on each occurrence. On the contrary, the constants with subscripts, such as C_1, keep the same value during the argument. For any function f on M, let $\mathcal{M}(f)$ be its *Hardy-Littlewood maximal function* defined by for any $x \in M$ by

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, dy, \tag{1.16}$$

where the supremum is taken over all the balls in M containing x. For nonnegative functions f, g, we write $f \lesssim g$ if $f \le Cg$ in a specified range, for some constant C. We write $f \simeq g$ if $f \lesssim g \lesssim f$.

2 Wavelet characterizations

The wavelets on a metric measure space are certain functions with "good" properties that serve as basic bricks to build objects with more complicated structures. Usually, the wavelets form an orthonormal basis in $L^2(M)$ provided $\mu(M) = \infty$. *The latter condition will be always assumed throughout the paper.* Note that if (M,ρ,μ) satisfies **(GB)**, then $\mu(M) = \infty$.

The aim of this section is to establish the wavelet characterizations of the homogeneous Lipschitz-type function spaces. In Section 2.1 we review some basic properties of wavelets on metric measure spaces. In Section 2.2, we define the homogeneous Lipschitz-type function spaces and state their wavelet characterizations. Finally, in Section 2.3, we prove these wavelet characterizations.

2.1 Wavelets on metric measure space

Let (M,ρ,μ) be a metric measure space satisfying **(VD)**. The following definition of dyadic points is taken from [31]; this is a collection of reference points in M endowed with a partial order (see also [6, 13, 30]).

Definition 2.1. Let $\delta \in (0,1)$ and $\{\mathcal{A}_k := \{x_{k,\alpha}\}_{\alpha \in \mathcal{I}_k}\}_{k \in \mathbb{Z}}$ be a sequence of points in M where $\mathcal{I}_k$ is a countable index set. The family $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ is called *a sequence of dyadic points* if $\mathcal{A}_k \subset \mathcal{A}_{k+1}$ and if it satisfies the following two properties:

(I) $\mathcal{A}_k$ is a maximal set of δ^k-separated points for any $k \in \mathbb{Z}$, namely, for any $\alpha, \beta \in \mathcal{I}_k$,

(I-1) $\rho(x_{k,\alpha}, x_{k,\beta}) \ge \delta^k$ for any $\alpha \ne \beta$,

(I-2) $\min_{\alpha \in \mathcal{I}_k} \rho(x, x_{k,\alpha}) < \delta^k$ for any $x \in M$;

(II) let $\mathcal{K} := \{(k,\alpha) : k \in \mathbb{Z}, \alpha \in \mathcal{I}_k\}$ be the parameter set associated with the dyadic points $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$. There exists a *partial order* $\le$ in $\mathcal{K}$ such that for any $k \in \mathbb{Z}$ and $r_k \in [\frac{1}{4}\delta^k, \frac{1}{2}\delta^k]$, (II-1) if $x_{k+1,\beta} \in B(x_{k,\alpha}, r_k)$, then $(k+1,\beta) \le (k,\alpha)$;

(II-2) if $(k+1,\beta) \le (k,\alpha)$, then $x_{k+1,\alpha} \in B(x_{k,\alpha}, 4r_k)$;

(II-3) for every $(k+1,\beta)$, there exists exactly one (k,α), called its *parent*, such that $(k+1,\beta) \le (k,\alpha)$;

(II-4) for every (k,α), there are between one and N_0 pairs $(k+1,\beta)$, called its *children*, such that $(k+1,\beta) \le (k,\alpha)$. Here, $N_0 \in \mathbb{N}$ depends only on the doubling constant C_0 in (1.2);

(II-5) $(l,\beta) \le (k,\alpha)$ if and only if $l \ge k$ and there exists a chain of ordered pairs $(j+1,\gamma_{j+1}) \le (j,\gamma_j)$ for $j = k, k+1, \dots, l-1$ with $\gamma_k = \alpha$ and $\gamma_l = \beta$. In this case, we called (l,β) and (k,α) are one another's *descendant* and *ancestor*, respectively.

The dyadic points lead to the following definition of dyadic cubes in M that are the analogues of the dyadic cubes in the Euclidean space.

Definition 2.2. Let $\delta \in (0,1)$ and $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ be a sequence of dyadic points in (M,ρ,μ) as in Definition 2.1. A collection of open sets $\{Q_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{I}_k} \subset M$ is called *a system of (open) dyadic cubes associated with* $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ if for any $k, l \in \mathbb{Z}$ and $\alpha, \beta \in \mathcal{I}_k$,

(i) $M = \bigcup_\alpha \overline{Q}_{k,\alpha}$, where $\overline{Q}_{k,\alpha}$ denotes the closure of $Q_{k,\alpha}$;

(ii) $\overline{Q}_{k,\alpha} \cap Q_{k,\beta} = \emptyset$ when $\alpha \ne \beta$;

(iii) $B(x_{k,\alpha}, \frac{1}{5}\delta^k) \subset Q_{k,\alpha} \subset \overline{Q}_{k,\alpha} \subset B(x_{k,\alpha}, 3\delta^k)$;

(iv) $\overline{Q}_{k,\alpha} = \bigcup_{\beta:(l,\beta)\le(k,\alpha)} \overline{Q}_{l,\beta}$ for any $l \ge k$.

Based on the notion of dyadic cubes, the following definition of wavelets was introduced in [31, Definition 6.9].

Definition 2.3. Let $\delta \in (0,1)$, $\eta \in [0,1]$ and $\{Q_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{I}_k}$ be a system of dyadic cubes associated with the dyadic points $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ as in Definition 2.2. For any $k \in \mathbb{Z}$, let $\mathcal{J}_k := \mathcal{I}_{k+1} \setminus \mathcal{I}_k$. A set of real-valued functions $\{\psi_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ on M is called *a basis of wavelets with exp-localization and Hölder-continuous of order* η, if the following properties are satisfied for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{J}_k$:

(i) (vanishing mean) $\int_M \psi_{k,\alpha}(x)\, d\mu(x) = 0$;

(ii) (localization) for any $x \in M$,

$$\left|\psi_{k,\alpha}(x)\right| \le \frac{C_3}{\sqrt{\mu(B(x_{k,\alpha}, \delta^k))}} \exp\left\{-\frac{\rho(x, x_{k,\alpha})}{\delta^k}\right\}, \tag{2.1}$$

where $\{x_{k,\alpha}\}_{k,\alpha}$ are the dyadic points as in Definition 2.1 and the constant $C_3 > 0$ is independent of x, k and α;

(iii) (Hölder continuity) for any $x, y \in M$,

$$\left|\psi_{k,\alpha}(x) - \psi_{k,\alpha}(y)\right| \le \frac{C_4}{\sqrt{\mu(B(x_{k,\alpha}, \delta^k))}} \exp\left\{-\frac{\rho(x, x_{k,\alpha})}{\delta^k}\right\} \left(\frac{\rho(x,y)}{\delta^k}\right)^\eta, \tag{2.2}$$

where the constant $C_4 > 0$ is independent of x, y, k and α;

(iv) (orthonormal basis) the functions $\{\psi_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ form an orthonormal basis of $L^2(M,\mu)$.

The existence of the wavelets satisfying Definition 2.3 is proved in [31, Corollary 6.13].

Proposition 2.4 ([31]). *Let* (M,ρ,μ) *satisfy* **(VD)**. *For any* $\eta\in(0,1)$, *there exist* $\delta\in(0,1)$ *small enough and a basis of wavelets* $\{\psi_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ *associated with a system of dyadic cubes* $\{Q_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{I}_k}$ *such that* $\{\psi_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ *is exp-localization and Hölder-continuous of order* η.

Remark 2.5. Let $p\in(1,\infty)$. The wavelets $\{\psi_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ also form an unconditional basis of the space $L^p(M)=L^p(M,\mu)$ (see [6, Corollary 10.2]). This implies that any $f\in L^p(M)$ has the following wavelet expansion in $L^p(M)$:

$$f=\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{J}_k}\langle f,\psi_{k,\alpha}\rangle\psi_{k,\alpha}\,. \tag{2.3}$$

The wavelet expansion (2.3) can be extended from $L^p(M)$ to a larger class of distributions on M. For that let us recall the definition of test functions and distributions from [26, 27].

Definition 2.6. Let $\eta\in(0,1)$ be as in Definition 2.3. A function $\varphi:M\to\mathbb{R}$ is said to be in the *test function class* $\mathcal{G}(x_0,r,\beta,\gamma)$ for some $x_0\in M$, $r\in(0,\infty)$, $\beta\in(0,\eta]$ and $\gamma\in(0,\infty)$, if the following three assertions hold:

(i) for any $x\in M$,

$$|\varphi(x)|\le\frac{C}{V_r(x_0)+V(x_0,x)}\left(\frac{r}{r+\rho(x_0,x)}\right)^{\gamma}, \tag{2.4}$$

where $V_r(x_0):=\mu(B(x_0,r))$, $V(x_0,x):=\mu(B(x_0,\rho(x_0,x)))$ and the positive C is independent of x;

(ii) for any $x,y\in M$ satisfying $\rho(x,y)\le\frac{1}{2}(r+\rho(x_0,x))$,

$$|\varphi(x)-\varphi(y)|\le\frac{C}{V_r(x_0)+V(x_0,x)}\left(\frac{r}{r+\rho(x_0,x)}\right)^{\gamma}\left(\frac{\rho(x,y)}{r+\rho(x_0,x)}\right)^{\beta}, \tag{2.5}$$

where the positive C is independent of x and y;

(iii) $\int_M\varphi(x)\,d\mu(x)=0$.

For any $\varphi\in\mathcal{G}(x_0,r,\beta,\gamma)$, endow φ with a norm by setting

$$\|\varphi\|_{\mathcal{G}(x_0,r,\beta,\gamma)}:=\inf\{C>0:\ \text{(i) and (ii) hold}\}\,. \tag{2.6}$$

Further properties of the test function class can be fund in [26, 27]. It is known that the space $(\mathcal{G}(x,r,\beta,\gamma),\|\cdot\|_{\mathcal{G}(x_0,r,\beta,\gamma)})$ is a Banach space that is invariant under the changes of x and r. Thus, we can fix a reference point $x_0\in M$ and denote $\mathcal{G}(\beta,\gamma):=\mathcal{G}(x_0,1,\beta,\gamma)$. It is easy to see the embedding $\mathcal{G}(\beta',\gamma)\subset\mathcal{G}(\beta,\gamma)$ holds for any $\beta\le\beta'$.

Now for any $\beta\in(0,\eta]$, let $\mathring{\mathcal{G}}(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\eta,\gamma)$ in the norm of $\mathcal{G}(\beta,\gamma)$. Then $(\mathring{\mathcal{G}}(\beta,\gamma))'$ is defined to be the set of all *continuous linear functionals* $\mathscr{L}$ on $\mathring{\mathcal{G}}(\beta,\gamma)$ with the property that, for all $\varphi\in\mathring{\mathcal{G}}(\beta,\gamma)$,

$$|\mathscr{L}(\varphi)|\lesssim\|\varphi\|_{\mathring{\mathcal{G}}(\beta,\gamma)}.$$

The following proposition extends the wavelet expansion to the space of distributions.

Proposition 2.7. *[26, Corollary 3.5] Let* $\beta,\gamma\in(0,\eta)$. *Then the wavelet expansion* (2.3) *also holds for any* $f\in(\mathring{\mathcal{G}}(\beta,\gamma))'$.

2.2 Homogeneous function spaces and their wavelet characterizations

The vanishing mean condition in Definition 2.3(i) indicates that the wavelets used in this paper are mother wavelets. As the mother wavelets characterize homogeneous function spaces (see [37, 42, 50]), we need the following definition of the homogenous version of the Lipschitz Besov space (cf. (1.8) and (1.9a)).

Definition 2.8. (i) For any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, \infty)$, the *homogeneous Lipschitz Besov space* $\dot{B}^s_{p,q}$ is defined to be

$$\dot{B}^s_{p,q} := \left\{ f \in L^p_{\mathrm{loc}}(M) : \ \|f\|_{\dot{B}^s_{p,q}} < \infty \right\},$$

where $\|f\|_{\dot{B}^s_{p,q}}$ is defined as in (1.9a).

(ii) For any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, \infty)$, the *homogeneous Lipschitz Triebel-Lizorkin space* $\dot{F}^s_{p,q}$ is defined to be

$$\dot{F}^s_{p,q} := \left\{ f \in L^p_{\mathrm{loc}}(M) : \ \|f\|_{\dot{F}^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\dot{F}^s_{p,q}} := \left\| \left[\int_0^\infty r^{-sq} \left(\fint_{B(\cdot,r)} |f(\cdot) - f(y)| \, d\mu(y) \right)^q \frac{dr}{r} \right]^{1/q} \right\|_{L^p} \tag{2.7}$$

with the usual modification when $q = \infty$.

As the spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ share many common properties, we will use the notation $\dot{A}^s_{p,q}$ to denote either space $\dot{B}^s_{p,q}$ or $\dot{F}^s_{p,q}$ when there is no confusion. In particular, for any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, \infty)$, it can be proved that $(\dot{A}^s_{p,q}/C, \|\cdot\|_{\dot{A}^s_{p,q}})$ is a Banach space, where $\dot{A}^s_{p,q}/C$ denotes the quotient space and C is the space of all constant functions on M (see [38, Propositions 3.1 and 3.2] and [49, Proposition 2.2]). Furthermore, it is easy to see that for all $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, \infty)$,

$$B^s_{p,q} = L^p \cap \dot{B}^s_{p,q}.$$

For any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, \infty)$, define the *inhomogeneous Triebel-Lizorkin space* $F^s_{p,q} := L^p \cap \dot{F}^s_{p,q}$ endow with the norm

$$\|f\|_{F^s_{p,q}} := \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}.$$

For functions in the above homogeneous function spaces, its wavelet coefficients are usually belong to the following sequence spaces.

Definition 2.9. Let $\delta \in (0, 1)$ and $\{Q_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{I}_k}$ be a system of dyadic cubes as in Definition 2.2. For any $k \in \mathbb{Z}$, denote by $\mathcal{J}_k = \mathcal{I}_{k+1} \setminus \mathcal{I}_k$.

(i) For any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in [0, \infty)$, the *homogeneous Besov sequence space* $\dot{b}^s_{p,q}$ is defined to be the space of all sequences $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \subset \mathbb{R}$ satisfying

$$\left\| \{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \right\|_{\dot{b}^s_{p,q}} := \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} \left|\lambda_{k,\alpha}\right| \right)^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} < \infty \tag{2.8}$$

with the usual modification when $q = \infty$.

(ii) For any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in [0, \infty)$, the *homogeneous Triebel-Lizorkin sequence space* $\dot{f}^s_{p,q}$ is defined to be the space of all sequences $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \subset \mathbb{R}$ satisfying

$$\left\|\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}\right\|_{\dot{f}^s_{p,q}} := \left\|\left[\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left(\sum_{\alpha\in\mathcal{J}_k} \mu(Q_{k,\alpha})^{-\frac{1}{2}} \mathbf{1}_{Q_{k,\alpha}}(\cdot) \left|\lambda_{k,\alpha}\right|\right)^q\right]^{\frac{1}{q}}\right\|_{L^p} < \infty \tag{2.9}$$

with the usual modification when $q = \infty$.

The next lemma collects some of the basic properties of the aforementioned spaces.

Lemma 2.10. *Let $p \in (1, \infty)$, $q \in (1, \infty]$ and $\beta, \gamma \in (0, \eta)$ with $\eta \in (0, 1)$ being as in (2.2). Then*

(i) *for any $s \in (0, 1)$ and $q_1, q_2 \in (1, \infty]$ with $q_1 \le q_2$, then*

$$\dot{B}^s_{p,q_1} \subset \dot{B}^s_{p,q_2}; \tag{2.10}$$

(ii) *for any $s \in (0, \gamma)$,*

$$\dot{B}^s_{p,q} \subset (\mathring{\mathcal{G}}(\beta, \gamma))'; \tag{2.11}$$

(iii) *for any $s \in [0, \gamma)$ and $\{\lambda_{k,\alpha}\}_{k,\alpha} \in \dot{b}^s_{p,q}$, the series*

$$\sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha}\psi_{k,\alpha}$$

converges in $(\mathring{\mathcal{G}}(\beta, \gamma))'$.

Proof. The assertion (iii) was proved in [25, Proposition 1.1], so that we need to prove (i) and (ii). We first show (i). By (1.9a), we know that

$$\|f\|_{\dot{B}^s_{p,q}} \simeq \left\|\left\{\delta^{-ks}\left[\int_M \fint_{B(x,c\delta^k)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x)\right]^{1/p}\right\}_{k\in\mathbb{Z}}\right\|_{l^q}$$

for any fixed $c > 0$. (i) then follows immediately from the increase property of the l^q-norm.

We now prove (ii). By applying (i), it suffices to show

$$\dot{B}^s_{p,\infty} \subset (\mathring{\mathcal{G}}(\beta, \gamma))'. \tag{2.12}$$

Indeed, let $f \in \dot{B}^s_{p,\infty}$. Then, for any $g \in \mathring{\mathcal{G}}(\beta, \gamma)$, by Definition 2.6, **(VD)** and $s < \gamma$, we have

$$\begin{aligned}
&\left|\int_M f(y)\varphi(y) \, d\mu(y)\right| \\
&\quad = \left|\int_M \left(f(y) - \fint_{B(x_0,1)} f(x) \, d\mu(x)\right) \varphi(y) \, d\mu(y)\right| \\
&\quad \le \frac{1}{\mu(B(x_0, 1))} \int_{B(x_0,1)} \left(\int_M |f(y) - f(x)| \, |\varphi(y)| d\mu(y)\right) d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\{\frac{1}{\mu(B(x_0,1))}\int_{B(x_0,1)}\left[\int_M \frac{|f(y)-f(x)|}{V_1(x_0)+V(x_0,y)}\left(\frac{1}{1+\rho(x_0,y)}\right)^{\gamma} d\mu(y)\right]^p d\mu(x)\right\}^{1/p}\\
&\lesssim \left\{\frac{1}{\mu(B(x_0,1))}\int_{B(x_0,1)}\left[\sum_{j=0}^{\infty}\delta^{j\gamma}\fint_{\rho(y,x)<\delta^{-(j+1)}}|f(y)-f(x)|\,d\mu(y)\right]^p d\mu(x)\right\}^{1/p}\\
&\lesssim \sum_{j=0}^{\infty}\delta^{j(\gamma-s)}\delta^{js}\left\{\int_M\left[\fint_{\rho(y,x)<\delta^{-(j+1)}}|f(y)-f(x)|\,d\mu(y)\right]^p d\mu(x)\right\}^{1/p}\\
&\lesssim \sup_{j\in\mathbb{Z}}\delta^{js}\left\{\int_M\left[\fint_{\rho(y,x)<\delta^{-(j+1)}}|f(y)-f(x)|\,d\mu(y)\right]^p d\mu(x)\right\}^{1/p}\left(\sum_{j=0}^{\infty}\delta^{j(\gamma-s)}\right)\lesssim \|f\|_{\dot B^s_{p,\infty}},
\end{aligned}
$$

which implies that (2.12) holds true. This finishes the proof of (ii) and hence Lemma 2.10. □

We now state the first main result of this section that establishes the wavelet characterizations of the homogeneous Besov and Triebel-Lizorkin spaces $\dot B^s_{p,q}$.

Theorem 2.11. *Let $p\in(1,\infty)$, $q\in(1,\infty]$ and $s\in(0,1)$. Assume (M,ρ,μ) satisfies the condition* **(VD)** *and that $\{\psi_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ is a basis of wavelets as in Definition 2.3 with $\eta\in(s,1)$. Then the following assertions hold:*

(i) *for any $f\in\dot B^s_{p,q}$, let*

$$E(f):=\{\langle f,\psi_{k,\alpha}\rangle\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}. \tag{2.13}$$

Then $E(f)\in\dot b^s_{p,q}$ with

$$\|E(f)\|_{\dot b^s_{p,q}}\le C\,\|f\|_{\dot B^s_{p,q}},$$

where the positive constant C is independent of f;

(ii) *for any $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}\in\dot b^s_{p,q}$, let*

$$R(\{\lambda_{k,\alpha}\}_{k,\alpha}):=\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{J}_k}\lambda_{k,\alpha}\psi_{k,\alpha}. \tag{2.14}$$

Then $R(\{\lambda_{k,\alpha}\}_{k,\alpha})\in\dot B^s_{p,q}$ with

$$\left\|R(\{\lambda_{k,\alpha}\}_{k,\alpha})\right\|_{\dot B^s_{p,q}}\le C\left\|\{\lambda_{k,\alpha}\}_{k,\alpha}\right\|_{\dot b^s_{p,q}},$$

where the positive constant C is independent of $\{\lambda_{k,\alpha}\}_{k,\alpha}$.

Theorem 2.12. *Let $p\in(1,\infty)$, $q\in(1,\infty]$ and $s\in(0,1)$. Assume (M,ρ,μ) satisfies the condition* **(VD)** *and that $\{\psi_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ is a basis of wavelets as in Definition 2.3 with $\eta\in(s,1)$. Then the following assertions hold:*

(i) *for any $f\in\dot F^s_{p,q}$, let $E(f)$ be as in (2.13). Then $E(f)\in\dot f^s_{p,q}$ with*

$$\|E(f)\|_{\dot f^s_{p,q}}\le C\,\|f\|_{\dot F^s_{p,q}},$$

where the positive constant C is independent of f;

(ii) *for any* $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \in \dot{f}^s_{p,q}$*, let* $R(\{\lambda_{k,\alpha}\}_{k,\alpha})$ *be as in* (2.14)*. Then* $R(\{\lambda_{k,\alpha}\}_{k,\alpha}) \in \dot{F}^s_{p,q}$ *with*

$$\left\|R(\{\lambda_{k,\alpha}\}_{k,\alpha})\right\|_{\dot{F}^s_{p,q}} \le C \left\|\{\lambda_{k,\alpha}\}_{k,\alpha}\right\|_{\dot{f}^s_{p,q}},$$

where the positive constant C *is independent of* $\{\lambda_{k,\alpha}\}_{k,\alpha}$*.*

Theorems 2.11 and 2.12 will be proved in Section 2.3. In the remainder of this subsection, we assume that the two theorems are true and consider their consequences.

Remark 2.13. (i) For any $p \in (1, \infty)$, $q \in (1, \infty]$, $s \in (0, 1)$ and $f \in L^p_{\rm loc}(M)$, let

$$\|f\|_{\widetilde{\dot{B}^s_{p,q}}} := \left\{\int_0^\infty r^{-sq}\left[\int_M \left(\fint_{B(x,r)} |f(x) - f(y)|\, d\mu(y)\right)^p d\mu(x)\right]^{q/p} \frac{dr}{r}\right\}^{1/q} \tag{2.15}$$

with the usual modification when $q = \infty$. By the Hölder inequality, it is easy to see that

$$\|f\|_{\widetilde{\dot{B}^s_{p,q}}} \le \|f\|_{\dot{B}^s_{p,q}}. \tag{2.16}$$

On the other hand, in the proof of Theorem 2.11 (see (2.23) below), we will prove that

$$\|E(f)\|_{\dot{b}^s_{p,q}} \lesssim \|f\|_{\widetilde{\dot{B}^s_{p,q}}}. \tag{2.17}$$

By Proposition 2.7 and Lemma 2.10(ii), we know that

$$R \circ E = I \tag{2.18}$$

on $\dot{B}^s_{p,q}$. This combined with Theorem 2.11(ii) implies that for any $f \in \dot{B}^s_{p,q}$,

$$\|f\|_{\dot{B}^s_{p,q}} \lesssim \|E(f)\|_{\dot{b}^s_{p,q}},$$

which together with (2.16) and (2.17) implies the following equivalence of norms:

$$\|f\|_{\widetilde{\dot{B}^s_{p,q}}} \simeq \|f\|_{\dot{B}^s_{p,q}}. \tag{2.19}$$

(ii) In view of Theorem 2.11, we can introduce the *homogeneous Besov space* $\dot{B}^0_{p,q}$ *with zero order smoothness*. To be precise, for any $p \in (1, \infty)$ and $q \in (1, \infty]$, let

$$\dot{B}^0_{p,q} := \left\{f \in L^p_{\rm loc}(M) :\ E(f) \in \dot{b}^0_{p,q}\right\}, \tag{2.20}$$

where $E(f)$ is defined as in (2.13). By the increase property of the l^q-norm in (2.8), it is easy to see that for any $p \in (1, \infty)$ and $q_1, q_2 \in (1, \infty]$ with $q_1 \le q_2$,

$$\dot{B}^0_{p,q_1} \subset \dot{B}^0_{p,q_2}, \tag{2.21}$$

which is a limiting case of Lemma 2.10(i).

Similarly, for any $p \in (1, \infty)$ and $q \in (1, \infty]$, the *homogeneous Triebel-Lizorkin space* $\dot{F}^0_{p,q}$ *with zero smoothness* is defined by

$$\dot{F}^0_{p,q} := \left\{f \in L^p_{\rm loc}(M) :\ E(f) \in \dot{f}^0_{p,q}\right\}. \tag{2.22}$$

It is easy too see that $\dot{F}^0_{p,2} = L^p$ for any $p \in (1, \infty)$ due to the Littlewood-Paley square function characterization of L^p (see [26, Theorem 4.3]). These two kinds of spaces will be useful in the endpoint interpolation of Besov spaces (see the proof of Theorem 3.9 below).

Corollary 2.14. *Let $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in [0, \infty)$. Then*

(i) $\dot{B}^s_{p,\min\{p,q\}} \subset \dot{F}^s_{p,q} \subset \dot{B}^s_{p,\max\{p,q\}}$. *In particular,* $\dot{B}^s_{p,p} = \dot{F}^s_{p,p}$*;*

(ii) $\dot{b}^s_{p,\min\{p,q\}} \subset \dot{f}^s_{p,q} \subset \dot{b}^s_{p,\max\{p,q\}}$. *In particular,* $\dot{b}^s_{p,p} = \dot{f}^s_{p,p}$.

Proof. The proof of Corollary 2.14 is similar to the corresponding result in the classical Euclidean space (see, for example, [47, Section 11.4 and Proposition 13.6]), the details being omitted. □

2.3 Proofs of Theorems 2.11 and 2.12

Proof of Theorem 2.11(i). For any $f \in \dot{B}^s_{p,q}$, let $E(f)$ be as in (2.13). As claimed in Remark 2.13, we only need to prove

$$\|E(f)\|_{\dot{b}^s_{p,q}} \lesssim \|f\|_{\widetilde{\dot{B}^s_{p,q}}} \tag{2.23}$$

with $\|f\|_{\widetilde{\dot{B}^s_{p,q}}}$ as in (2.15). By (2.13) and (2.8), we know

$$\|E(f)\|_{\dot{b}^s_{p,q}} = \left\{\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} |\langle f, \psi_{k,\alpha}\rangle|\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}. \tag{2.24}$$

We first estimate the term $|\langle f, \psi_{k,\alpha}\rangle|$. By Definition 2.3, we know for any $k \in \mathbb{Z}$, $\alpha \in \mathcal{J}_k$ and $x \in B(x_{k,\alpha}, 3\delta^k)$,

$$\begin{aligned}
\left|\langle f, \psi_{k,\alpha}\rangle\right| &= \left|\int_M \psi_{k,\alpha}(y)\left(f(y) - \fint_{Q_{k,\alpha}} f\, d\mu\right) d\mu(y)\right| \\
&\lesssim \left[\mu(B(x_{k,\alpha}, \delta^k))\right]^{-\frac{1}{2}} \int_M \exp\left\{-\frac{\rho(y, x_{k,\alpha})}{\delta^k}\right\} \left|f(y) - \fint_{Q_{k,\alpha}} f\, d\mu\right| d\mu(y) \\
&\lesssim \left[\mu(B(x_{k,\alpha}, \delta^k))\right]^{-\frac{1}{2}} \sum_{j=0}^{\infty} \int_{S_{k-j}(B_{k,\alpha})} \exp\left\{-\frac{\rho(y, x_{k,\alpha})}{\delta^k}\right\} \left|f(y) - \fint_{Q_{k,\alpha}} f\, d\mu\right| d\mu(y),
\end{aligned} \tag{2.25}$$

where $S_{k-j}(B_{k,\alpha}) := B(x_{k,\alpha}, \delta^{k-j}) \setminus B(x_{k,\alpha}, \delta^{k-j+1})$ for any $j \in \mathbb{N}$ and $S_k(B_{k,\alpha}) := B(x_{k,\alpha}, \delta^k)$. Thus by (2.25), **(VD)** and Definition 2.2(iii), we conclude that

$$\begin{aligned}
\left|\langle f, \psi_{k,\alpha}\rangle\right| &\lesssim \left[\mu(B(x_{k,\alpha}, \delta^k))\right]^{-\frac{1}{2}} \sum_{j=0}^{\infty} \exp\left\{-\delta^{1-j}\right\} \frac{\mu(B(x_{k,\alpha}, \delta^{k-j}))}{\mu(Q_{k,\alpha})} \\
&\quad \times \int_{Q_{k,\alpha}} \left(\fint_{B(x_{k,\alpha},\delta^{k-j})} |f(y) - f(x)|\, d\mu(y)\right) d\mu(x) \\
&\lesssim [\mu(Q_{k,\alpha})]^{-\frac{1}{2}} \sum_{j=0}^{\infty} \exp\left\{-\delta^{1-j}\right\} \delta^{-jd} \int_{Q_{k,\alpha}} \left(\fint_{B(x_{k,\alpha},\delta^{k-j})} |f(y) - f(x)|\, d\mu(y)\right) d\mu(x) \\
&=: [\mu(Q_{k,\alpha})]^{-\frac{1}{2}} \sum_{j=0}^{\infty} \exp\left\{-\delta^{1-j}\right\} \delta^{-jd} \mathrm{I}_{k,j}(f, Q_{k,\alpha}).
\end{aligned} \tag{2.26}$$

Note that for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{J}_k$, we have

$$[\mu(Q_{k,\alpha})]^{\frac{1}{p}-1}\, \mathrm{I}_{k,j}(f, Q_{k,\alpha}) = [\mu(Q_{k,\alpha})]^{\frac{1}{p}} \fint_{Q_{k,\alpha}} \left(\fint_{B(x_{k,\alpha},\delta^{k-j})} |f(y) - f(x)|\, d\mu(y) \right) d\mu(x)$$
$$\leq \left[\int_{Q_{k,\alpha}} \left(\fint_{B(x_{k,\alpha},\delta^{k-j})} |f(y) - f(x)|\, d\mu(y) \right)^p d\mu(x) \right]^{\frac{1}{p}},$$

which combining (2.24), (2.26), (2.15) and Definition 2.2 implies that

$$\|E(f)\|_{\dot{b}^s_{p,q}} \tag{2.27}$$
$$\lesssim \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} |\langle f, \psi_{k,\alpha}\rangle| \right)^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$
$$\lesssim \sum_{j=0}^{\infty} \exp\left\{-\delta^{1-j}\right\} \delta^{-jd} \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{J}_k} \int_{Q_{k,\alpha}} \left(\fint_{B(x_{k,\alpha},\delta^{k-j})} |f(y) - f(x)|\, d\mu(y) \right)^p d\mu(x) \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$
$$\lesssim \sum_{j=0}^{\infty} \exp\left\{-\delta^{1-j}\right\} \delta^{-jd} \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{J}_k} \int_{Q_{k,\alpha}} \left(\fint_{B(x,4\delta^{k-j})} |f(y) - f(x)|\, d\mu(y) \right)^p d\mu(x) \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$
$$\lesssim \sum_{j=0}^{\infty} \exp\left\{-\delta^{1-j}\right\} \delta^{-j(d+s)} \left\{ \sum_{k\in\mathbb{Z}} \delta^{-(k-j)sq} \left[\int_{M} \left(\fint_{B(x,4\delta^{k-j})} |f(y) - f(x)|\, d\mu(y) \right)^p d\mu(x) \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$
$$\lesssim \|f\|_{\widetilde{\dot{B}^s_{p,q}}},$$

where in the third inequality, we have used Definition 2.2(iii) and the fact that $j \geq 0$. This shows (2.23) and hence finishes the proof of Theorem 2.11(i).

□

To prove Theorem 2.11(ii), we need the following lemma.

Lemma 2.15. *For any $p \in (1, \infty)$, $q \in (1, \infty]$, $s \in [0, \eta)$ and $\beta, \gamma \in (0, \eta)$ with η as in (2.2). Let $\{\lambda_{k,\alpha}\}_{k,\alpha} \in \dot{b}^s_{p,q}$ and $f := \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha}\psi_{k,\alpha}$ converges in $(\mathring{\mathcal{G}}(\beta, \gamma))'$. Then $f \in L^p_{\mathrm{loc}}$.*

Proof. As $f \in (\mathring{\mathcal{G}}(\beta, \gamma))'$, by Definition 2.6(iii), we know that for any $x_0 \in M$ and $l_0 \in \mathbb{Z}$,

$$f = \sum_{k=-\infty}^{l_0-1} \sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha} \left(\psi_{k,\alpha} - \psi_{k,\alpha}(x_0)\right) + \sum_{k=l_0}^{\infty} \sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha}\psi_{k,\alpha} =: f_1 + f_2 \tag{2.28}$$

in $(\mathring{\mathcal{G}}(\beta, \gamma))'$. Thus, to finish the proof of this lemma, we only need to show that for, any $x_0 \in M$ and $l_0 \in \mathbb{Z}$,

$$\mathrm{I}_1 := \left(\int_{B(x_0,\delta^{l_0})} |f_1(x)|^p\, d\mu(x) \right)^{\frac{1}{p}} < \infty \tag{2.29}$$

and

$$\mathrm{I}_2 := \left(\int_{B(x_0,\delta^{l_0})} |f_2(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty. \tag{2.30}$$

We first prove (2.29). By (2.28), Definition 2.3 and **(VD)**, we have

$$\begin{aligned} \mathrm{I}_1 &\lesssim \left\{ \int_{B(x_0,\delta^{l_0})} \left[\sum_{k=-\infty}^{l_0-1} \sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(B(x_{k,\alpha},\delta^k))^{-\frac{1}{2}} \exp\left\{ -\frac{\rho(x,x_{k,\alpha})}{\delta^k} \right\} \left(\frac{\rho(x,x_0)}{\delta^k} \right)^{\eta} \right]^p d\mu(x) \right\}^{\frac{1}{p}} \\ &\lesssim \sum_{k=-\infty}^{l_0-1} \left\{ \int_M \left[\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-\frac{1}{2}} \delta^{(l_0-k)\eta} \exp\left\{ -\frac{\rho(x,x_{k,\alpha})}{\delta^k} \right\} \right]^p d\mu(x) \right\}^{\frac{1}{p}}. \end{aligned} \tag{2.31}$$

We now need the following pointwise estimate on the Hardy-Littlewood maximal function from [14, 15] (see also [49, Lemma 3.10]): for any $x \in M$,

$$\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-\frac{1}{2}} \delta^{(l_0-k)\eta} \exp\left\{ -\frac{\rho(x,x_{k,\alpha})}{\delta^k} \right\} \lesssim \mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-\frac{1}{2}} \delta^{(l_0-k)\eta} \mathbf{1}_{Q_{k,\alpha}} \right)(x), \tag{2.32}$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function as in (1.16). Together with (2.31), the L^p boundedness of $\mathcal{M}$ and the fact $s < \eta$, this implies that

$$\begin{aligned} \mathrm{I}_1 &\lesssim \sum_{k=-\infty}^{l_0-1} \left\{ \int_M \left[\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-\frac{1}{2}} \delta^{(l_0-k)\eta} \mathbf{1}_{Q_{k,\alpha}}(x) \right]^p d\mu(x) \right\}^{\frac{1}{p}} \\ &\lesssim \sum_{k=-\infty}^{l_0-1} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} \delta^{(l_0-k)\eta} \right)^p \right]^{\frac{1}{p}} \\ &\lesssim \sum_{k=-\infty}^{l_0-1} \delta^{(l_0-k)(\eta-s)} \delta^{(l_0-k)s} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} \right)^p \right]^{\frac{1}{p}} \\ &\lesssim \left\{ \sum_{k=-\infty}^{l_0-1} \delta^{(l_0-k)sq} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} \right)^p \right]^{\frac{q}{p}} \right\}^{1/q} \lesssim \|\{\lambda_{k,\alpha}\}_{k,\alpha}\|_{\dot{b}^s_{p,q}} < \infty, \end{aligned} \tag{2.33}$$

which proves (2.29).

To prove (2.30), by an argument similar to that of (2.33), we see

$$\begin{aligned} \mathrm{I}_2 &\lesssim \sum_{k=l_0}^{\infty} \left\{ \int_M \left[\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-\frac{1}{2}} \mathbf{1}_{Q_{k,\alpha}}(x) \right]^p d\mu(x) \right\}^{\frac{1}{p}} \\ &\lesssim \sum_{k=l_0}^{\infty} \delta^{(k-l_0)s} \delta^{-(k-l_0)s} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} \right)^p \right]^{\frac{1}{p}} \\ &\lesssim \left\{ \sum_{k=l_0}^{\infty} \delta^{(l_0-k)sq} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} \right)^p \right]^{\frac{q}{p}} \right\}^{1/q} \lesssim \|\{\lambda_{k,\alpha}\}_{k,\alpha}\|_{\dot{b}^s_{p,q}} < \infty, \end{aligned}$$

which proves (2.30) and hence completes the proof of Lemma 2.15.

□

Now we prove Theorem 2.11(ii) using Lemma 2.15.

Proof of Theorem 2.11(ii). Let $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \in \dot{b}^s_{p,q}$ and $f := R(\{\lambda_{k,\alpha}\}_{k,\alpha})$ be as in (2.14). By Lemmas 2.10(iii) and 2.15, we know that $f \in (\mathring{\mathcal{G}}(\beta,\gamma))' \cap L^p_{\mathrm{loc}}(M)$. Thus, to finish the proof of Theorem 2.11(ii), we only need to show that

$$\|f\|_{\dot{B}^s_{p,q}} \lesssim \left\|\{\lambda_{k,\alpha}\}_{k,\alpha}\right\|_{\dot{b}^s_{p,q}}. \tag{2.34}$$

For any $j\in\mathbb{Z}$ and $x,y\in M$ satisfying $\rho(x,y)<\delta^j$, write

$$f = \sum_{k<j}\sum_{\alpha\in\mathcal{J}_k}\lambda_{k,\alpha}\psi_{k,\alpha} + \sum_{k\geq j}\sum_{\alpha\in\mathcal{J}_k}\lambda_{k,\alpha}\psi_{k,\alpha} =: f_1 + f_2. \tag{2.35}$$

Then, by Definition 2.3, the assumption $\rho(x,y)<\delta^j$ and by (2.32), we have

$$\begin{aligned}|f_1(x)-f_1(y)| &\leq \sum_{k<j}\sum_{\alpha\in\mathcal{J}_k}\left|\lambda_{k,\alpha}\right|\left|\psi_{k,\alpha}(x)-\psi_{k,\alpha}(y)\right| \\ &\lesssim \sum_{k<j}\sum_{\alpha\in\mathcal{J}_k}\left|\lambda_{k,\alpha}\right|\mu(Q_{k,\alpha})^{-1/2}\exp\left\{-\frac{\rho(x,x_{k,\alpha})}{\delta^k}\right\}\left(\frac{\rho(x,y)}{\delta^k}\right)^{\eta} \\ &\lesssim \sum_{k<j}\mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k}\left|\lambda_{k,\alpha}\right|\mu(Q_{k,\alpha})^{-1/2}\delta^{(j-k)\eta}\mathbf{1}_{Q_{k,\alpha}}\right)(x).\end{aligned}$$

This implies that, for any $x\in M$,

$$\begin{aligned}d_{j,p}(f_1)(x) &:= \left\{\fint_{B(x,\delta^j)}|f_1(x)-f_1(y)|^p\,d\mu(y)\right\}^{\frac{1}{p}} \\ &\lesssim \sum_{k<j}\delta^{-(k-j)\eta}\mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k}\left|\lambda_{k,\alpha}\right|\mu(Q_{k,\alpha})^{-1/2}\mathbf{1}_{Q_{k,\alpha}}\right)(x).\end{aligned} \tag{2.36}$$

By (1.9a), (2.36) and Young's convolution inequality (note that $\{\delta^{j(\eta-s)}\}_{j>0}\in l^1$ as $\eta>s$), we see

$$\begin{aligned}\|f_1\|_{\dot{B}^s_{p,q}} &\simeq \left\{\sum_{j\in\mathbb{Z}}\left\|\delta^{-js}d_{j,p}(f_1)\right\|^q_{L^p}\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\sum_{j\in\mathbb{Z}}\left[\sum_{k<j}\delta^{-(k-j)(\eta-s)}\delta^{-ks}\left\|\mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k}\left|\lambda_{k,\alpha}\right|\mu(Q_{k,\alpha})^{-1/2}\mathbf{1}_{Q_{k,\alpha}}\right)\right\|_{L^p}\right]^q\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\sum_{j\in\mathbb{Z}}\left[\sum_{k<j}\delta^{-(k-j)(\eta-s)}\delta^{-ks}\left(\sum_{\alpha\in\mathcal{J}_k}\left(|\lambda_{k,\alpha}|\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}}\right)^p\right)^{\frac{1}{p}}\right]^q\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\sum_{k\in\mathbb{Z}}\delta^{-ksq}\left[\sum_{\alpha\in\mathcal{J}_k}\left(|\lambda_{k,\alpha}|\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}}\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \simeq \left\|\{\lambda_{k,\alpha}\}_{k,\alpha}\right\|_{\dot{b}^s_{p,q}}.\end{aligned} \tag{2.37}$$

For f_2, we have

$$|f_2(x) - f_2(y)| \le |f_2(x)| + |f_2(y)|. \tag{2.38}$$

We first estimate $|f_2(y)|$. By (2.35), Definition 2.3, **(VD)** and (2.32), we obtain

$$|f_2(y)| \lesssim \sum_{k\ge j}\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-1/2} \exp\left\{-\frac{\rho(y,x_{k,\alpha})}{\delta^k}\right\} \lesssim \sum_{k\ge j} \mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-1/2}\mathbf{1}_{Q_{k,\alpha}}\right)(y).$$

Thus, similar to (2.37), we have

$$\begin{aligned} \mathrm{I} &:= \left\{\sum_{j\in\mathbb{Z}} \left\|\delta^{-js}\left(\fint_{B(\cdot,\delta^j)} |f_2(y)|^p\,d\mu(y)\right)^{\frac{1}{p}}\right\|_{L^p}^q\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\sum_{j\in\mathbb{Z}}\left[\sum_{k\ge j}\delta^{(k-j)s}\delta^{-ks}\left\|\mathcal{M}\circ\mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k}\left|\lambda_{k,\alpha}\right|\mu(Q_{k,\alpha})^{-1/2}\mathbf{1}_{Q_{k,\alpha}}\right)\right\|_{L^p}\right]^q\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\sum_{j\in\mathbb{Z}}\left[\sum_{k\ge j}\delta^{(k-j)s}\delta^{-ks}\left(\sum_{\alpha\in\mathcal{J}_k}\left(|\lambda_{k,\alpha}|\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}}\right)^p\right)^{\frac{1}{p}}\right]^q\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\sum_{k\in\mathbb{Z}}\delta^{-ksq}\left[\sum_{\alpha\in\mathcal{J}_k}\left(|\lambda_{k,\alpha}|\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}}\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \simeq \left\|\{\lambda_{k,\alpha}\}_{k,\alpha}\right\|_{\dot{b}^s_{p,q}}. \end{aligned} \tag{2.39}$$

Similarly, we obtain

$$\mathrm{J} := \left\{\sum_{j\in\mathbb{Z}} \left\|\delta^{-js}\left(\fint_{B(\cdot,\delta^j)} |f_2(x)|^p\,d\mu(y)\right)^{\frac{1}{p}}\right\|_{L^p}^q\right\}^{\frac{1}{q}} \lesssim \left\|\{\lambda_{k,\alpha}\}_{k,\alpha}\right\|_{\dot{b}^s_{p,q}}. \tag{2.40}$$

Combining (2.35) and (2.37) through (2.40), we conclude that (2.34) holds true. This finishes the proof of Theorem 2.11(ii).

□

We now prove Theorem 2.12.

Proof of Theorem 2.12. We first prove Theorem 2.12(i). Let $f \in \dot{F}^s_{p,q}$ and $E(f)$ be as in (2.13). By (2.9), we need to show that

$$\mathrm{I} := \left\|\left[\sum_{k\in\mathbb{Z}}\delta^{-ksq}\left(\sum_{\alpha\in\mathcal{J}_k}\mu(Q_{k,\alpha})^{-\frac{1}{2}}\mathbf{1}_{Q_{k,\alpha}}(\cdot)\left|\langle f,\psi_{k,\alpha}\rangle\right|\right)^q\right]^{\frac{1}{q}}\right\|_{L^p} \lesssim \|f\|_{\dot{F}^s_{p,q}}. \tag{2.41}$$

Similar to (2.25), we know that for any $k\in\mathbb{Z}$, $\alpha\in\mathcal{J}_k$ and $x\in Q_{k,\alpha}$,

$$\left|\langle f,\psi_{k,\alpha}\rangle\right| = \left|\int_M \psi_{k,\alpha}(y)\,(f(y)-f(x))\,d\mu(y)\right| \tag{2.42}$$

$$\begin{aligned}
&\lesssim [\mu(Q_{k,\alpha})]^{-\frac{1}{2}} \int_M \exp\left\{-\frac{\rho(y, x_{k,\alpha})}{\delta^k}\right\} |f(y) - f(x)|\, d\mu(y)\\
&\lesssim [\mu(Q_{k,\alpha})]^{-\frac{1}{2}} \sum_{j=0}^{\infty} \int_{S_{k-j}(B_{k,\alpha})} \exp\left\{-\frac{\rho(y, x_{k,\alpha})}{\delta^k}\right\} |f(y) - f(x)|\, d\mu(y)\\
&\lesssim [\mu(Q_{k,\alpha})]^{-\frac{1}{2}} \sum_{j=0}^{\infty} \exp\left\{-\delta^{-j}\right\} \mu(B(x_{k,\alpha}, \delta^{k-j})) \fint_{B(x_{k,\alpha},\delta^{k-j})} |f(y) - f(x)|\, d\mu(y)\\
&\lesssim [\mu(Q_{k,\alpha})]^{\frac{1}{2}} \sum_{j=0}^{\infty} \exp\left\{-\delta^{-j}\right\} \delta^{-jd} \fint_{B(x,4\delta^{k-j})} |f(y) - f(x)|\, d\mu(y)\\
&=: [\mu(Q_{k,\alpha})]^{\frac{1}{2}} \sum_{j=0}^{\infty} \exp\left\{-\delta^{-j}\right\} \delta^{-jd} D_{j,k,\alpha}(f)(x).
\end{aligned}$$

By (2.41) together with (2.42) and (2.7), we conclude

$$\begin{aligned}
\mathrm{I} &\lesssim \left\|\left[\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left(\sum_{\alpha\in\mathcal{J}_k} \mu(Q_{k,\alpha})^{-\frac{1}{2}} \mathbf{1}_{Q_{k,\alpha}}(\cdot)\, [\mu(Q_{k,\alpha})]^{\frac{1}{2}} \sum_{j=0}^{\infty} \exp\left\{-\delta^{-j}\right\} \delta^{-jd} D_{j,k,\alpha}(f)(\cdot)\right)^q\right]^{\frac{1}{q}}\right\|_{L^p}\\
&\lesssim \sum_{j=0}^{\infty} \exp\left\{-\delta^{-j}\right\} \delta^{-j(d+s)} \left\|\left[\sum_{k\in\mathbb{Z}} \delta^{-(k-j)sq} \left(D_{j,k,\alpha}(f)(\cdot)\right)^q\right]^{\frac{1}{q}}\right\|_{L^p}\\
&\lesssim \sum_{j=0}^{\infty} \exp\left\{-\delta^{-j}\right\} \delta^{-j(d+s)} \left\|\left[\sum_{k\in\mathbb{Z}} \delta^{-(k-j)sq} \left(\fint_{B(\cdot,4\delta^{k-j})} |f(y) - f(\cdot)|\, d\mu(y)\right)^q\right]^{\frac{1}{q}}\right\|_{L^p}\\
&\lesssim \|f\|_{\dot F^s_{p,q}},
\end{aligned}$$

which implies that (2.41) holds true and hence completes the proof of Theorem 2.12(i).

We now prove Theorem 2.12(ii). For any $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \in \dot f^s_{p,q}$, let $f := R(\{\lambda_{k,\alpha}\}_{k,\alpha})$ be as in (2.14). By Corollary 2.14(ii) and Lemma 2.15, we know that $f \in (\mathring{\mathcal{G}}(\beta,\gamma))' \cap L^p_{\mathrm{loc}}$. Thus, to finish the proof of Theorem 2.12(ii), it suffices to prove that $\|f\|_{\dot F^s_{p,q}} \lesssim \|\{\lambda_{k,\alpha}\}_{k,\alpha}\|_{\dot f^s_{p,q}}$, namely,

$$\mathrm{J} := \left\|\left[\sum_{j\in\mathbb{Z}} \delta^{-jsq} \left(\fint_{B(x,\delta^j)} |f(\cdot) - f(y)|\, d\mu(y)\right)^q\right]^{1/q}\right\|_{L^p} \lesssim \|\{\lambda_{k,\alpha}\}_{k,\alpha}\|_{\dot f^s_{p,q}}. \tag{2.43}$$

By (2.14), write

$$f = \sum_{k<j} \sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha}\psi_{k,\alpha} + \sum_{k<j} \sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha}\psi_{k,\alpha} =: f_1 + f_2. \tag{2.44}$$

For f_1, by (2.36), we know that $x \in M$,

$$\begin{aligned}
d_{j,p}(f_1)(x) &:= \left\{\fint_{B(x,\delta^j)} |f_1(x) - f_1(y)|^p\, d\mu(y)\right\}^{\frac{1}{p}}\\
&\lesssim \sum_{k<j} \delta^{-(k-j)\eta} \mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-1/2} \mathbf{1}_{Q_{k,\alpha}}\right)(x).
\end{aligned}$$

Combined with the Fefferman-Stein vector valued maximal inequality (see [19, Theorem 1.2]), this yields

$$\begin{aligned}
&\left\| \left[\sum_{j\in\mathbb{Z}} \delta^{-jsq} \left(d_{j,p}(f_1) \right)^q \right]^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left[\sum_{j\in\mathbb{Z}} \delta^{-jsq} \left(\sum_{k<j} \delta^{-(k-j)\eta} \mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-1/2} \mathbf{1}_{Q_{k,\alpha}} \right) \right)^q \right]^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left[\sum_{j\in\mathbb{Z}} \left(\sum_{k<j} \delta^{-(k-j)(\eta-s)} \mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k} \delta^{-ksq} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-1/2} \mathbf{1}_{Q_{k,\alpha}} \right) \right)^q \right]^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left[\sum_{k\in\mathbb{Z}} \left(\mathcal{M}\left(\sum_{\alpha\in\mathcal{J}_k} \delta^{-ks} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-1/2} \mathbf{1}_{Q_{k,\alpha}} \right) \right)^q \right]^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left[\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left(\sum_{\alpha\in\mathcal{J}_k} \left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{-1/2} \mathbf{1}_{Q_{k,\alpha}} \right)^q \right]^{1/q} \right\|_{L^p} \lesssim \|\{\lambda_{k,\alpha}\}_{k,\alpha}\|_{\dot{f}^s_{p,q}}
\end{aligned}$$

The estimates for f_2 is similar.

□

3 Real and complex interpolations

In this section, we establish the real and complex interpolations of the homogeneous Lipschitz-type function spaces and some of their inhomogeneous versions. Throughout this section, we assume that the underlying metric measure space (M, ρ, μ) is unbounded and satisfy the condition **(VD)**.

We first in Section 3.1 review some basic facts and properties of interpolation; then in Section 3.2, we consider the interpolations of the homogeneous Lipschitz-type function spaces with smoothness parameter $s \in (0, 1)$. Finally, in Section 3.3, we extend the interpolations of Section 3.2 to the endpoint case $s = 0$ and also to some of their inhomogeneous versions.

3.1 Preliminaries on interpolation

Let $(\mathbb{X}_0, \mathbb{X}_1)$ be a compatible couple of Banach spaces, namely, there exists a Hausdorff topological vector space $\mathbb{Y}$ such that for any $i \in \{1, 2\}$, $\mathbb{X}_i \subset \mathbb{Y}$. For any compatible Banach couple $(\mathbb{X}_0, \mathbb{X}_1)$, the sum $\mathbb{X}_0 + \mathbb{X}_1$ is defined to the Banach space under the norm

$$\|a\|_{\mathbb{X}_0+\mathbb{X}_1} := \inf\left\{\|a_0\|_{\mathbb{X}_0} + \|a_1\|_{\mathbb{X}_1} :\ a = a_0 + a_1 \text{ with } a_0 \in \mathbb{X}_0, a_1 \in \mathbb{X}_1\right\}.$$

For any $a \in \mathbb{X}_0 + \mathbb{X}_1$ and $t \in (0, \infty)$, the *K-functional* of f is defined by

$$K(a, t; \mathbb{X}_0, \mathbb{X}_1) := \inf\left\{\|a_0\|_{\mathbb{X}_0} + t\|a_1\|_{\mathbb{X}_1} :\ a = a_0 + a_1 \text{ with } a_0 \in \mathbb{X}_0, a_1 \in \mathbb{X}_1\right\}. \tag{3.1}$$

Notice that $K(a, t; \mathbb{X}_0, \mathbb{X}_1)$ is increase in t.

Definition 3.1. Let $(\mathbb{X}_0, \mathbb{X}_1)$ be a compatible Banach couple and $\theta \in (0,1)$, $q \in (1, \infty]$. The *real interpolation space* $(\mathbb{X}_0, \mathbb{X}_1)_{\theta,q}$ is defined to be the space of all $a \in \mathbb{X}_0 + \mathbb{X}_1$ such that

$$\|f\|_{(\mathbb{X}_0,\mathbb{X}_1)_{\theta,q}} := \left[\int_0^\infty \left(t^{-\theta} K(a, t; \mathbb{X}_0, \mathbb{X}_1)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} \tag{3.2}$$

with the usual modification when $q = \infty$.

Let

$$S_0 := \{z \in \mathbb{C} : \ 0 < \operatorname{Re} z < 1\} \tag{3.3}$$

be an open strip in the complex plane $\mathbb{C}$ and

$$S := \{z \in \mathbb{C} : \ 0 \le \operatorname{Re} z \le 1\} \tag{3.4}$$

be its closure. Let $\mathcal{A}_{(\mathbb{X}_0,\mathbb{X}_1)}$ be the set of all bounded analytic functions $F : \ S_0 \to \mathbb{X}_0 + \mathbb{X}_1$, which can be extended to continuous functions on S and satisfy that for any $j \in \{0, 1\}$, the function $t \mapsto F(j + it) : \mathbb{R} \to \mathbb{X}_j$ is bounded and continuous. For any $F \in \mathcal{A}_{(\mathbb{X}_0,\mathbb{X}_1)}$, endow with the norm

$$\|F\|_{\mathcal{A}_{(\mathbb{X}_0,\mathbb{X}_1)}} := \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{\mathbb{X}_0}, \sup_{t\in\mathbb{R}} \|F(1 + it)\|_{\mathbb{X}_1}\right\}. \tag{3.5}$$

Definition 3.2. Let $(\mathbb{X}_0, \mathbb{X}_1)$ be a compatible Banach couple and $\theta \in (0, 1)$. The *complex interpolation space* $[\mathbb{X}_0, \mathbb{X}_1]_\theta$ is defined to be the space of all

$$a \in \mathcal{A}_{(\mathbb{X}_0,\mathbb{X}_1)}(\theta) := \{F(\theta) : \ F \in \mathcal{A}_{(\mathbb{X}_0,\mathbb{X}_1)}\}$$

endow with the norm

$$\|a\|_{[\mathbb{X}_0,\mathbb{X}_1]_\theta} := \inf\left\{\|F\|_{\mathcal{A}_{(\mathbb{X}_0,\mathbb{X}_1)}} : \ F(\theta) = a\right\}. \tag{3.6}$$

The real and complex interpolations are the two most important interpolation methods in the literature (see [7, 44]). In particular, they satisfy the following interpolation property (see [7, Theorems 3.1.2 and 4.1.2]).

Lemma 3.3. *Let $(\mathbb{X}_0, \mathbb{X}_1)$ and $(\mathbb{Y}_0, \mathbb{Y}_1)$ be two compatible couples of Banach spaces. Consider a bounded linear operator $T : \ \mathbb{X}_j \to \mathbb{Y}_j$ for $j \in \{0, 1\}$. Then for any $\theta \in (0, 1)$ and $q \in (1, \infty]$, T induces a bounded linear operator T_θ satisfying*

$$T_\theta : \ (\mathbb{X}_0, \mathbb{X}_1)_{\theta,q} \to (\mathbb{Y}_0, \mathbb{Y}_1)_{\theta,q}$$

and

$$T_\theta : \ [\mathbb{X}_0, \mathbb{X}_1]_\theta \to [\mathbb{Y}_0, \mathbb{Y}_1]_\theta$$

with the operator norm $\|T_\theta\| \le \|T\|^{1-\theta}_{\mathbb{X}_0\to\mathbb{Y}_0} \|T\|^{\theta}_{\mathbb{X}_1\to\mathbb{Y}_1}$.

Let $(\mathbb{X}_0, \mathbb{X}_1)$ and $(\mathbb{Y}_0, \mathbb{Y}_1)$ be two compatible couples of Banach spaces. We call that $(\mathbb{Y}_0, \mathbb{Y}_1)$ is a *retract* of $(\mathbb{X}_0, \mathbb{X}_1)$ if there exist two bounded linear operators such that

(i) $E : \ \mathbb{Y}_j \to \mathbb{X}_j$ for $j \in \{0, 1\}$;

(ii) $R: \mathbb{X}_j \to \mathbb{Y}_j$ for $j \in \{0,1\}$;

(iii) $R \circ E = I$ on $\mathbb{Y}_j$ for $j \in \{0,1\}$.

The following result on the retract of interpolation can be found in [44, p. 22].

Lemma 3.4. *Let* $(\mathbb{X}_0, \mathbb{X}_1)$ *and* $(\mathbb{Y}_0, \mathbb{Y}_1)$ *be two compatible couples of Banach spaces. Assume that* $(\mathbb{Y}_0, \mathbb{Y}_1)$ *is a retract of* $(\mathbb{X}_0, \mathbb{X}_1)$*. Then for any* $\theta \in (0,1)$ *and* $q \in (1, \infty]$*,*

$$(\mathbb{Y}_0, \mathbb{Y}_1)_{\theta,q} = R\left((\mathbb{X}_0, \mathbb{X}_1)_{\theta,q}\right)$$

and

$$[\mathbb{Y}_0, \mathbb{Y}_1]_\theta = R\left([\mathbb{X}_0, \mathbb{X}_1]_\theta\right).$$

The advantage of Lemma 3.4 is that it provides an approach to reduce the interpolation of the spaces $(\mathbb{Y}_0, \mathbb{Y}_1)$ to that of $(\mathbb{X}_0, \mathbb{X}_1)$, whose interpolation is usually easier to establish. One typical example of such $\mathbb{X}$-space is the following mixed norm Lebesgue space. To be precise, for any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in [0, \infty)$, let

$$l_q^s\left(L_p\right) := \left\{\{f_k(\cdot)\}_{k\in\mathbb{Z}} : \ f_k(\cdot) \in L^p(M) \text{ and } \|\{f_k(\cdot)\}_{k\in\mathbb{Z}}\|_{l_q^s(L_p)} < \infty\right\},$$

where

$$\|\{f_k(\cdot)\}_{k\in\mathbb{Z}}\|_{l_q^s(L_p)} := \left(\sum_{k\in\mathbb{Z}} \delta^{-ksq} \|f_k(\cdot)\|_{L^p}^q\right)^{\frac{1}{q}} \tag{3.7}$$

with $\delta \in (0,1)$.

Let

$$L_p\left(l_q^s\right) := \left\{\{f_k(\cdot)\}_{k\in\mathbb{Z}} : \ \|\{f_k(\cdot)\}_{k\in\mathbb{Z}}\|_{L_p(l_q^s)} < \infty\right\},$$

where

$$\|\{f_k(\cdot)\}_{k\in\mathbb{Z}}\|_{L_p(l_q^s)} := \left\|\left\{\sum_{k\in\mathbb{Z}} \delta^{-ksq} |f_k(\cdot)|^q\right\}^{\frac{1}{q}}\right\|_{L^p}. \tag{3.8}$$

The following interpolation of mixed norm Lebesgue spaces can been found in [7, Chapter 5] (see also [44, Section 1.18]).

Lemma 3.5. *Let* $p_0, p_1 \in (1, \infty)$*,* $q_0, q_1 \in (1, \infty]$*,* $s_0, s_1 \in [0, \infty)$ *and* $\theta \in (0,1)$*.*

(i) *For* $p \in (1, \infty)$*,* $q \in (1, \infty]$ *and* $s \in [0, \infty)$ *satisfying* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*,* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *and* $s = (1-\theta)s_0 + \theta s_1$*, it holds*

$$\left[l_{q_0}^{s_0}\left(L_{p_0}\right), l_{q_1}^{s_1}(L_{p_1})\right]_\theta = l_q^s(L_p) \tag{3.9}$$

and

$$\left[L_{p_0}\left(l_{q_0}^{s_0}\right), L_{p_1}\left(l_{q_1}^{s_1}\right)\right]_\theta = L_p\left(l_q^s\right). \tag{3.10}$$

(ii) *If $s_0 \neq s_1$, then for any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in [0, \infty)$ satisfying $s = (1-\theta)s_0 + \theta s_1$, it holds*

$$\left(\dot{l}^{s_0}_{q_0}\left(L_p\right), \dot{l}^{s_1}_{q_1}(L_p)\right)_{\theta,q} = \dot{l}^{s}_{q}(L_p) \tag{3.11}$$

and

$$\left(L_p\left(\dot{l}^{s_0}_{q_0}\right), L_p\left(\dot{l}^{s_1}_{q_1}\right)\right)_{\theta,q} = \dot{l}^{s}_{q}\left(L_p\right). \tag{3.12}$$

3.2 Interpolations of Besov and Triebel-Lizorkin spaces

Let (M, ρ, μ) satisfy **(VD)**. The following two theorems give the real and complex interpolations of the homogeneous Besov and Triebel-Lizorkin spaces with $s \in (0, 1)$.

Theorem 3.6. *Let p_0, $p_1 \in (1, \infty)$, q_0, $q_1 \in (1, \infty]$, s_0, $s_1 \in (0, 1)$ and $\theta \in (0, 1)$. Then for any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, 1)$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $s = (1-\theta)s_0 + \theta s_1$,*

$$\left[\dot{B}^{s_0}_{p_0,q_0}, \dot{B}^{s_1}_{p_1,q_1}\right]_{\theta} = \dot{B}^{s}_{p,q} \tag{3.13}$$

and

$$\left[\dot{F}^{s_0}_{p_0,q_0}, \dot{F}^{s_1}_{p_1,q_1}\right]_{\theta} = \dot{F}^{s}_{p,q}. \tag{3.14}$$

Theorem 3.7. *Let q_0, $q_1 \in (1, \infty]$, s_0, $s_1 \in (0, 1)$ with $s_0 \neq s_1$ and $\theta \in (0, 1)$. Then for any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, 1)$ satisfying $s = (1-\theta)s_0 + \theta s_1$,*

$$\left(\dot{B}^{s_0}_{p,q_0}, \dot{B}^{s_1}_{p,q_1}\right)_{\theta,q} = \dot{B}^{s}_{p,q} \tag{3.15}$$

and

$$\left(\dot{F}^{s_0}_{p,q_0}, \dot{F}^{s_1}_{p,q_1}\right)_{\theta,q} = \dot{B}^{s}_{p,q}. \tag{3.16}$$

We prove Theorems 3.6 and 3.7 by using Lemma 3.4. To this end, we need the following retract operators. For any sequence $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \subset \mathbb{R}$ with $\mathcal{J}_k$ as in (2.8), let

$$\widetilde{E}\left(\{\lambda_{k,\alpha}\}_{k,\alpha}\right) := \{f_k\}_{k\in\mathbb{Z}} \tag{3.17}$$

be a sequence of functions on M with

$$f_k := \sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha} \mathbf{1}_{Q_{k,\alpha}} \mu(Q_{k,\alpha})^{-1/2}, \tag{3.18}$$

where $\{Q_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}$ denotes the dyadic cubes as in Definition 2.2.

On the other hand, for any sequence of functions $\{f_k\}_{k\in\mathbb{Z}}$ in $L^1_{\mathrm{loc}}(M)$, let

$$\widetilde{R}(\{f_k\}_k) := \{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \tag{3.19}$$

be a sequence of numbers in $\mathbb{R}$ with

$$\lambda_{k,\alpha} := \mu(Q_{k,\alpha})^{-1/2} \int_{Q_{k,\alpha}} f_k(x)\, d\mu(x). \tag{3.20}$$

Lemma 3.8. *Let* $p \in (1, \infty)$, $q \in (1, \infty]$, $s \in (0, 1)$ *and* $\widetilde{E}$, $\widetilde{R}$ *be respectively as in* (3.17) *and* (3.19). *Then,*

(i) $\widetilde{E} : \dot{b}^s_{p,q} \to \dot{l}^s_q(L_p)$, $\dot{f}^s_{p,q} \to L_p(\dot{l}^s_q)$ *are bounded;*

(ii) $\widetilde{R} : \dot{l}^s_q(L_p) \to \dot{b}^s_{p,q}$, $L_p(\dot{l}^s_q) \to \dot{f}^s_{p,q}$ *are bounded;*

(iii) $\widetilde{R} \circ \widetilde{E} = I$ *on* $\dot{b}^s_{p,q}$ *and* $\dot{f}^s_{p,q}$.

Proof. We first prove (iii). For any sequence of numbers $\{\lambda_{k,\alpha}\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k} \subset \mathbb{R}$, by (3.18), (3.20) and Definition 2.2(ii), we have

$$\widetilde{R} \circ \widetilde{E}\,(\{\lambda_{k,\alpha}\}_{k,\alpha}) = \widetilde{R}\left(\left\{\sum_{\widetilde{\alpha}\in\mathcal{J}_k} \lambda_{k,\widetilde{\alpha}} \mathbf{1}_{Q_{k,\widetilde{\alpha}}} \mu(Q_{k,\widetilde{\alpha}})^{-1/2}\right\}_k\right)$$
$$= \left\{\sum_{\widetilde{\alpha}\in\mathcal{J}_k} \lambda_{k,\widetilde{\alpha}} \int_{Q_{k,\alpha}} \mathbf{1}_{Q_{k,\widetilde{\alpha}}}(x)\, d\mu(x) \mu(Q_{k,\widetilde{\alpha}})^{-1/2} \mu(Q_{k,\alpha})^{-1/2}\right\}_{k,\alpha} = \{\lambda_{k,\alpha}\}_{k,\alpha},$$

which immediately implies that (iii) holds true.

Let us now prove (i). We first show that $\widetilde{E}: \dot{b}^s_{p,q} \to \dot{l}^s_q(L_p)$ is bounded. Indeed, for any $\{\lambda_{k,\alpha}\}_{k,\alpha} \in \dot{b}^s_{p,q}$, by (3.7), (3.17), (3.18) and Definition 2.2, we know

$$\left\|\widetilde{E}\,(\{\lambda_{k,\alpha}\}_{k,\alpha})\right\|_{\dot{l}^s_q(L_p)} = \left(\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left\|\sum_{\alpha\in\mathcal{J}_k} \lambda_{k,\alpha} \mathbf{1}_{Q_{k,\alpha}} \mu(Q_{k,\alpha})^{-1/2}\right\|^q_{L_p}\right)^{\frac{1}{q}}$$
$$\lesssim \left\{\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{J}_k} \left(\left|\lambda_{k,\alpha}\right| \mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}}\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \simeq \left\|\{\lambda_{k,\alpha}\}_{k,\alpha}\right\|_{\dot{b}^s_{p,q}},$$

which implies that $\widetilde{E}: \dot{b}^s_{p,q} \to \dot{l}^s_q(L_p)$ is bounded. The proof of the boundedness of $\widetilde{E}: \dot{f}^s_{p,q} \to L_p(\dot{l}^s_q)$ is similar, the details being omitted.

We now prove (ii). As in the proof of (i), we only prove one of the claimed boundedness. In particular, we will show that $\widetilde{R}: L_p(\dot{l}^s_q) \to \dot{f}^s_{p,q}$ is bounded. Indeed, for any $\{f_k\}_{k\in\mathbb{Z}} \in L_p(l^s_q)$, by (2.9), (3.19), (3.20) and (3.7), we see

$$\left\|\widetilde{R}\,(\{f_k\}_k)\right\|_{\dot{f}^s_{p,q}} \le \left\|\left[\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left(\sum_{\alpha\in\mathcal{J}_k} \left\|f_k \mathbf{1}_{Q_{k,\alpha}}\right\|_{L^1} \mu(Q_{k,\alpha})^{-1/2} \mathbf{1}_{Q_{k,\alpha}} \mu(Q_{k,\alpha})^{-1/2}\right)^q\right]^{\frac{1}{q}}\right\|_{L^p}$$
$$\lesssim \left\|\left[\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left(\sum_{\alpha\in\mathcal{J}_k} \left[\fint_{Q_{k,\alpha}} |f_k|\, d\mu(x)\right] \mathbf{1}_{Q_{k,\alpha}}\right)^q\right]^{\frac{1}{q}}\right\|_{L^p}$$
$$\lesssim \left\|\left[\sum_{k\in\mathbb{Z}} \delta^{-ksq} \left(\mathcal{M}(f_k)\right)^q\right]^{\frac{1}{q}}\right\|_{L^p}$$

$$\lesssim \left\| \left[\sum_{k\in\mathbb{Z}} \delta^{-ksq} |f_k|^q \right]^{\frac{1}{q}} \right\|_{L^p} \simeq \left\| \{f_k\}_k \right\|_{L_p(l_q^s)}.$$

This implies that $\widetilde{R}: L_p(\dot{l}_q^s) \to \dot{f}_{p,q}^s$ is bounded. The proof of the boundedness $\widetilde{R}: \dot{l}_q^s(L_p) \to \dot{b}_{p,q}^s$ is similar. This finishes the proof of (ii).

Altogether, we finish the proof of Lemma 3.8.

□

With the help of Lemma 3.8, we now turn to the proof of Theorems 3.6 and 3.7.

Proofs of Theorems 3.6 and 3.7. Let $p_0,\ p_1 \in (1,\ \infty)$, $q_0,\ q_1 \in (1, \infty]$ and $s_0,\ s_1 \in (0,1)$. Let E and R be respectively as in (2.13) and (2.14). Then by Theorem 2.11, Proposition 2.7 and Lemma 2.10(ii), we know that for $j \in \{0,1\}$,

(i) $E: \dot{B}_{p_j,q_j}^{s_j} \to \dot{b}_{p_j,q_j}^{s_j}$ is bounded;

(ii) $R: \dot{b}_{p_j,q_j}^{s_j} \to \dot{B}_{p_j,q_j}^{s_j}$ is bounded;

(iii) $R \circ E = I$ on $\dot{B}_{p_j,q_j}^{s_j}$.

Thus, $(\dot{B}_{p_0,q_0}^{s_0}, \dot{B}_{p_1,q_1}^{s_1})$ is a retract of $(\dot{b}_{p_0,q_0}^{s_0}, \dot{b}_{p_1,q_1}^{s_1})$ as described in Section 3.1. By Lemma 3.4, we know that for any $\theta \in (0,1)$ and $q \in (1,\infty]$,

$$\left(\dot{B}_{p_0,q_0}^{s_0}, \dot{B}_{p_1,q_1}^{s_1}\right)_{\theta,q} = R\left(\left(\dot{b}_{p_0,q_0}^{s_0}, \dot{b}_{p_1,q_1}^{s_1}\right)_{\theta,q}\right) \tag{3.21}$$

and

$$\left[\dot{B}_{p_0,q_0}^{s_0}, \dot{B}_{p_1,q_1}^{s_1}\right]_{\theta} = R\left(\left[\dot{b}_{p_0,q_0}^{s_0}, \dot{b}_{p_1,q_1}^{s_1}\right]_{\theta}\right). \tag{3.22}$$

On the other hand, let $\widetilde{E}$ and $\widetilde{R}$ be respectively as in (3.17) and (3.19). By Lemmas 3.8 and 3.4, we know that for any $\theta \in (0,1)$ and $q \in (1,\infty]$,

$$\left(\dot{b}_{p_0,q_0}^{s_0}, \dot{b}_{p_1,q_1}^{s_1}\right)_{\theta,q} = \widetilde{R}\left(\left(\dot{l}_{q_0}^{s_0}(L_{p_0}), \dot{l}_{q_1}^{s_1}(L_{p_1})\right)_{\theta,q}\right) \tag{3.23}$$

and

$$\left[\dot{b}_{p_0,q_0}^{s_0}, \dot{b}_{p_1,q_1}^{s_1}\right]_{\theta} = \widetilde{R}\left(\left[\dot{l}_{q_0}^{s_0}(L^{p_0}), \dot{l}_{q_1}^{s_1}(L^{p_1})\right]_{\theta}\right). \tag{3.24}$$

Moreover, by Lemma 3.5, we find for any $p_0,\ p_1 \in (1,\ \infty)$, $q_0,\ q_1 \in (1,\infty]$ and $s_0,\ s_1 \in (0,1)$,

(a) if $s_0 \neq s_1$, then for any $p \in (1,\ \infty)$, $q \in (1,\ \infty]$ and $s \in (0,1)$ satisfying $s = (1-\theta)s_0 + \theta s_1$,

$$\left(\dot{l}_{q_0}^{s_0}\left(L_p\right), \dot{l}_{q_1}^{s_1}(L_p)\right)_{\theta,q} = \dot{l}_q^s(L_p); \tag{3.25}$$

(b) for $p \in (1,\ \infty)$, $q \in (1,\ \infty]$ and $s \in (0,1)$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $s = (1-\theta)s_0 + \theta s_1$,

$$\left[\dot{l}_{q_0}^{s_0}\left(L_{p_0}\right), \dot{l}_{q_1}^{s_1}(L_{p_1})\right]_{\theta} = \dot{l}_q^s(L_p). \tag{3.26}$$

Note that $R \circ \widetilde{R}(\dot{l}_q^s(L_p)) = R(\dot{b}_{p,q}^s) = \dot{B}_{p,q}^s$. This combined with (3.21) through (3.26) implies that

$$\left(\dot{B}^{s_0}_{p_0,q_0}, \dot{B}^{s_1}_{p_1,q_1}\right)_{\theta,q} = \dot{B}^s_{p,q},$$

$$\left[\dot{B}^{s_0}_{p_0,q_0}, \dot{B}^{s_1}_{p_1,q_1}\right]_{\theta} = \dot{B}^s_{p,q}.$$

and hence proves Theorems 3.6 and 3.7 for the Besov spaces.

The proofs of Theorem 3.6 and 3.7 for the Triebel-Lizorkin spaces are similar, we only need to replace the sequence spaces $\dot{b}^s_{p,q}$ and $\dot{l}^s_q(L_p)$ respectively by $\dot{f}^s_{p,q}$ and $L_p(\dot{l}^s_q)$, the details being omitted. This finishes the proofs of Theorems 3.6 and 3.7.

□

3.3 Interpolations at the endpoint case

The next theorem extends some of the interpolations of Section 3.2 to the case $s = 1$.

Theorem 3.9. *Let $p_0, p_1 \in (1, \infty)$, $q_1 \in (1, \infty]$ and $s, \theta \in (0, 1)$. Then,*

(i) $\left(L^{p_0}, \dot{B}^s_{p_0,q_1}\right)_{\theta,q} = \dot{B}^{\theta s}_{p_0,q}$ *for any $q \in (1, \infty]$;*

(ii) $\left[L^{p_0}, \dot{F}^s_{p_1,q_1}\right]_{\theta} = \dot{F}^{\theta s}_{p,q}$ *with $p \in (1, \infty)$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $q \in (1, \infty]$ satisfying $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{q_1}$.*

Before proving Theorem 3.9, we need the following wavelet characterization of the Lebesgue space $L^p(M)$ from [26, Theorem 4.3].

Lemma 3.10 ([26])**.** *Suppose $\beta, \gamma \in (0, \eta)$ and $p \in (1, \infty)$. Then for any $f \in L^p(M)$,*

$$\|f\|_{L^p} \simeq \left\|\{\langle f, \psi_{k,\alpha}\rangle\}_{k\in\mathbb{Z},\alpha\in\mathcal{J}_k}\right\|_{\dot{f}^0_{p,2}},$$

where the implicit constants are independent of f.

We now turn to the proof of Theorem 3.9.

Proof of Theorem 3.9. Observe that (ii) follows immediately from Lemma 3.10 and an argument similar to the proof of (3.14) in Theorem 3.6. Thus, it suffices to prove (i). To simply the notation we set $p_0 = p$ in the remainder of the proof. We divide the proof into three steps.

Step I: we first show that for any $\widetilde{q} \in (\max\{2, q_1\}, \infty]$ (here we take $\widetilde{q} = \infty$ if $q_1 = \infty$),

$$\left(L^p, \dot{B}^s_{p,\widetilde{q}}\right)_{\theta,q} \subset \dot{B}^{s\theta}_{p,q}. \tag{3.27}$$

Indeed, for any $f \in (L^p, \dot{B}^s_{p,\widetilde{q}})_{\theta,q} \subset L^p + \dot{B}^s_{p,\widetilde{q}} \subset L^p_{\mathrm{loc}}$. Let $f = f_0 + f_1$ be an arbitrary decomposition with $f_0 \in L^p$ and $f_1 \in \dot{B}^s_{p,\widetilde{q}}$. Assume first $q < \infty$. By Theorem 2.11, we have

$$\begin{aligned} \|f\|^q_{\dot{B}^{s\theta}_{p,q}} &\lesssim \int_0^\infty t^{-\theta sq}\left(E_p(f_0, t)\right)^q \frac{dt}{t} + \|f_1\|^q_{B^{s\theta}_{p,q}} \\ &\lesssim \sum_{k\in\mathbb{Z}} \delta^{-k\theta sq}\left(E_p(f_0, \delta^k)\right)^q + \sum_{k\in\mathbb{Z}} \delta^{-k\theta sq}\left[\sum_{\alpha\in\mathcal{J}_k}\left(\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}}\left|\langle f_1, \psi_{k,\alpha}\rangle\right|\right)^p\right]^{\frac{q}{p}}, \end{aligned} \tag{3.28}$$

where, for any $t \in (0, \infty)$ and $g \in L^p_{\mathrm{loc}}$,

$$E_p(g, t) := \left(\int_M \fint_{B(x,t)} |g(x) - g(y)|^p \, d\mu(y) d\mu(x) \right)^{\frac{1}{p}}.$$

As $p \in (1, \infty)$, it is easy to see

$$E_p(g, t) \lesssim \|g\|_{L^p}.$$

This implies that

$$\sum_{k \in \mathbb{Z}} \delta^{-k\theta sq} \left(E_p(f_0, \delta^k) \right)^q \lesssim \sum_{k \in \mathbb{Z}} \delta^{-k\theta sq} \|f_0\|^q_{L^p}. \tag{3.29}$$

On the other hand, by Lemma 2.10 and Theorem 2.11 again, we find

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} \delta^{-k\theta sq} \left[\sum_{\alpha \in \mathcal{J}_k} \left(\mu(Q_{k,\alpha})^{\frac{1}{p} - \frac{1}{2}} \left| \langle f_1, \psi_{k,\alpha} \rangle \right| \right)^p \right]^{\frac{q}{p}} \\ &\quad \lesssim \sum_{k \in \mathbb{Z}} \delta^{-k(\theta - 1)sq} \left\{ \delta^{-ks} \left[\sum_{\alpha \in \mathcal{J}_k} \left(\mu(Q_{k,\alpha})^{\frac{1}{p} - \frac{1}{2}} \left| \langle f_1, \psi_{k,\alpha} \rangle \right| \right)^p \right]^{\frac{1}{p}} \right\}^q \lesssim \sum_{k \in \mathbb{Z}} \delta^{-k(\theta - 1)sq} \|f_1\|^q_{\dot{B}^s_{p,\widetilde{q}}}, \end{aligned}$$

which combined with (3.28) and (3.29) implies that

$$\|f\|^q_{\dot{B}^{s\theta}_{p,q}} \lesssim \sum_{k \in \mathbb{Z}} \delta^{-k\theta sq} \left[\|f_0\|_{L^p} + \delta^{ks} \|f_1\|_{\dot{B}^s_{p,\widetilde{q}}} \right]^q.$$

By (3.1), (3.2) and the arbitrariness of the decomposition $f = f_0 + f_1$, we conclude that

$$\|f\|^q_{\dot{B}^{s\theta}_{p,q}} \lesssim \sum_{k \in \mathbb{Z}} \delta^{-k\theta sq} K^q(f, \delta^{ks}; L^p, \dot{B}^s_{p,\widetilde{q}}) \simeq \int_0^\infty t^{-\theta q} K^q(f, t; L^p, \dot{B}^s_{p,\widetilde{q}}) \frac{dt}{t} \simeq \|f\|^q_{(L^p, \dot{B}^s_{p,\widetilde{q}})_{\theta,q}},$$

which proves (3.27) for $q < \infty$. The case $q = \infty$ follows from a similar argument with a minor modification on the norm $\|f\|_{\dot{B}^{s\theta}_{p,\infty}}$.

Step II: we show that for any $1 < r < \min\{2, p, q\}$,

$$\dot{B}^{\theta s}_{p,q} \subset \left(\dot{B}^0_{p,r}, \dot{B}^s_{p,r} \right)_{\theta,q}, \tag{3.30}$$

where $\dot{B}^0_{p,r}$ is as in (2.20).

Indeed, for any $f \in \dot{B}^{\theta s}_{p,q}$, write

$$f = \sum_{k > j} \left(\sum_{\alpha \in \mathcal{J}_k} \langle f, \psi_{k,\alpha} \rangle \right) \psi_{k,\alpha} + \sum_{k \le j} \left(\sum_{\alpha \in \mathcal{J}_k} \langle f, \psi_{k,\alpha} \rangle \right) \psi_{k,\alpha} =: f_0 + f_1, \tag{3.31}$$

where $j \in \mathbb{Z}$ will be determined later.

By Theorem 2.11, we have

$$\|f_0\|^r_{\dot{B}^0_{p,r}} \lesssim \sum_{k=j+1}^\infty \left[\sum_{\alpha \in \mathcal{J}_k} \left(\mu(Q_{k,\alpha})^{\frac{1}{p} - \frac{1}{2}} \left| \langle f, \psi_{k,\alpha} \rangle \right| \right)^p \right]^{\frac{r}{p}} =: \sum_{k=j+1}^\infty \xi^r_k. \tag{3.32}$$

By the Hölder inequality and the assumption that $f \in \dot{B}^{\theta s}_{p,q}$, we know that $\|f_0\|^r_{\dot{B}^0_{p,r}} < \infty$ and hence $f_0 \in \dot{B}^0_{p,r}$.

Similarly, we have

$$\|f_1\|^r_{\dot{B}^s_{p,r}} \lesssim \sum_{k=-\infty}^{j} \delta^{-ksr} \left[\sum_{\alpha \in \mathcal{J}_k} \left(\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}} \left| \langle f, \psi_{k,\alpha} \rangle \right| \right)^p \right]^{\frac{r}{p}} =: \sum_{k=-\infty}^{j} \delta^{-ksr} \xi_k^r \tag{3.33}$$

and $f_1 \in \dot{B}^s_{p,r}$. Thus the decomposition in (3.31) is a decomposition of f in $\dot{B}^0_{p,r} + \dot{B}^s_{p,r}$. Combining (3.1), (3.32) with (3.33), we find

$$K(f, \delta^{js}; \dot{B}^0_{p,r}, \dot{B}^s_{p,r}) \lesssim \left(\|f_0\|^r_{\dot{B}^0_{p,r}} + \delta^{jsr} \|f_1\|^r_{\dot{B}^s_{p,r}} \right)^{\frac{1}{r}} \lesssim \left(\sum_{k=j+1}^{\infty} \xi_k^r + \delta^{jsr} \sum_{k=-\infty}^{j} \delta^{-ksr} \xi_k^r \right)^{\frac{1}{r}}.$$

This implies that

$$\begin{aligned} \mathrm{I} &:= \int_0^\infty t^{-\theta q} K^q \left(f, t; \dot{B}^0_{p,r}, \dot{B}^s_{p,r} \right) \frac{dt}{t} \\ &\lesssim \sum_{j \in \mathbb{Z}} \delta^{-j\theta sq} \left(\sum_{k=j+1}^{\infty} \xi_k^r + \delta^{jsr} \sum_{k=-\infty}^{j} \delta^{-ksr} \xi_k^r \right)^{\frac{q}{r}} \\ &\lesssim \sum_{j \in \mathbb{Z}} \delta^{-j\theta sq} \left(\sum_{k=j+1}^{\infty} \xi_k^r \right)^{\frac{q}{r}} + \sum_{j \in \mathbb{Z}} \delta^{j(1-\theta)sq} \left(\sum_{k=-\infty}^{j} \delta^{-ksr} \xi_k^r \right)^{\frac{q}{r}} =: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{3.34}$$

For I_1, let $0 < \alpha_2 < \theta s < \alpha_1 < s$ and $\sigma > r$ satisfying $\frac{r}{q} + \frac{r}{\sigma} = 1$. We find

$$\begin{aligned} \mathrm{I}_1 &\lesssim \sum_{j \in \mathbb{Z}} \delta^{-j\theta sq} \left(\sum_{k=j+1}^{\infty} \delta^{k\alpha_2 r} \delta^{-k\alpha_2 r} \xi_k^r \right)^{\frac{q}{r}} \lesssim \sum_{j \in \mathbb{Z}} \delta^{-j\theta sq} \left(\sum_{k=j+1}^{\infty} \delta^{-k\alpha_2 q} \xi_k^q \right) \left(\sum_{k=j+1}^{\infty} \delta^{k\alpha_2 \sigma} \right)^{\frac{q}{\sigma}} \\ &\lesssim \sum_{j \in \mathbb{Z}} \delta^{-jq(\theta s - \alpha_2)} \left(\sum_{k=j+1}^{\infty} \delta^{-k\alpha_2 q} \xi_k^q \right) \simeq \sum_{j \in \mathbb{Z}} \sum_{k=j+1}^{\infty} \delta^{-(j-k)q(\theta s - \alpha_2)} \delta^{-k\theta sq} \xi_k^q \\ &\simeq \sum_{k \in \mathbb{Z}} \sum_{j=-\infty}^{k-1} \left(\delta^{-(j-k)q(\theta s - \alpha_2)} \right) \delta^{-k\theta sq} \xi_k^q \simeq \sum_{k \in \mathbb{Z}} \delta^{-k\theta sq} \xi_k^q. \end{aligned} \tag{3.35}$$

For I_2, we have

$$\begin{aligned} \mathrm{I}_2 &\lesssim \sum_{j \in \mathbb{Z}} \delta^{jsq(1-\theta)} \left[\sum_{k=-\infty}^{j} \delta^{-k(s-\alpha_1)r} \delta^{-k\alpha_1 r} \xi_k^r \right]^{\frac{q}{r}} \\ &\lesssim \sum_{j \in \mathbb{Z}} \delta^{jsq(1-\theta)} \left[\sum_{k=-\infty}^{j} \delta^{-k\alpha_1 q} \xi_k^q \right] \left[\sum_{k=-\infty}^{j} \delta^{-k(s-\alpha_1)\sigma} \right]^{\frac{q}{\sigma}} \\ &\lesssim \sum_{j \in \mathbb{Z}} \delta^{jq[s(1-\theta)-s+\alpha_1]} \left[\sum_{k=-\infty}^{j} \delta^{-k\alpha_1 q} \xi_k^q \right] \end{aligned} \tag{3.36}$$

$$\lesssim \sum_{k\in\mathbb{Z}}\left[\sum_{j=k}^{\infty}\delta^{(j-k)q[s(1-\theta)-s+\alpha_1]}\right]\delta^{-k\theta sq}\xi_k^q \simeq \sum_{k\in\mathbb{Z}}\delta^{-k\theta sq}\xi_k^q.$$

Combining (3.33) through (3.36) and Theorem 2.11, we conclude that

$$\|f\|_{(\dot B^0_{p,r},\dot B^s_{p,r})_{\theta,q}} = \left\{\int_0^\infty t^{-\theta q}K^q\left(f,t;\dot B^0_{p,r},\dot B^s_{p,r}\right)\frac{dt}{t}\right\}^{\frac{1}{q}} = \mathrm{I}^{1/q}$$
$$\lesssim \left\{\sum_{k\in\mathbb{Z}}\delta^{-k\theta sq}\xi_k^q\right\}^{\frac{1}{q}} \simeq \left\{\sum_{k\in\mathbb{Z}}\delta^{-k\theta sq}\left[\sum_{\alpha\in\mathcal{J}_k}\left(\mu(Q_{k,\alpha})^{\frac{1}{p}-\frac{1}{2}}\left|\langle f,\psi_{k,\alpha}\rangle\right|\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \simeq \|f\|_{\dot B^{\theta s}_{p,r}}.$$

This proves (3.30).

Step III: We finally prove (i). Let $1<r<\min\{2,p,q\}$ be as in Step II. By (2.21) and Corollary 2.14, we have

$$\dot B^0_{p,r}\subset \dot B^0_{p,\min\{p,2\}}\subset \dot F^0_{p,2}=L^p.$$

This combined with Steps I and II implies that

$$\dot B^{\theta s}_{p,q}\subset\left(\dot B^0_{p,r},\dot B^s_{p,r}\right)_{\theta,q}\subset\left(L^p,\dot B^s_{p,r}\right)_{\theta,q}\subset\left(L^p,\dot B^s_{p,\widetilde q}\right)_{\theta,q}\subset \dot B^{\theta s}_{p,q},$$

which completes the proof of Theorem 3.9.

□

Based on Theorem 3.9 and Corollary 2.14, we immediately obtain the following endpoint real interpolation of the homogeneous Triebel-Lizorkin spaces.

Corollary 3.11. *Let $p\in(1,\infty)$, $q_1\in(1,\infty]$ and $s\in(0,1)$. Then, for any $\theta\in(0,1)$ and $q\in(1,\infty]$,*

$$\left(L^p,\dot F^s_{p,q_1}\right)_{\theta,q}=\dot F^{\theta s}_{p,q}.$$

The following theorem establishes the endpoint real interpolation of the inhomogeneous spaces.

Theorem 3.12. *Let $p\in(1,\infty)$, $q_1\in(1,\infty]$ and $s\in(0,1)$. Then for any $\theta\in(0,1)$ and $q\in(1,\infty]$,*

(i) $\left(L^p,B^s_{p,q_1}\right)_{\theta,q}=B^{\theta s}_{p,q}$;

(ii) $\left(L^p,F^s_{p,q_1}\right)_{\theta,q}=B^{\theta s}_{p,q}$.

For the proof of Theorem 3.12, we need the following lemma.

Lemma 3.13. *Let $p\in(1,\infty)$ and $\mathbb{X}\subset L^p_{\mathrm{loc}}(M)$ be a Banach space satisfying that $(L^p,\mathbb{X}\cap L^p)$ is a compatible Banach couple. Then for any $t\in(0,\infty)$ and $f\in L^p$,*

$$\min\{1,t\}\|f\|_{L^p}+K(f,t;L^p,\mathbb{X})\simeq K(f,t;L^p,\mathbb{X}\cap L^p).$$

Proof. We prove this lemma using the idea from the proof of [18, Theorem 4.2]. Let $f \in L^p(M)$. By (3.1), it is easy to see that $K(f,t;L^p,\mathbb{X}) \le K(f,t;L^p,\mathbb{X}\cap L^p)$. Moreover, as $L^p + (\mathbb{X}\cap L^p) \subset L^p$, we have

$$\min\{1,t\}\|f\|_{L^p} \lesssim \min\{1,t\}\|f\|_{L^p+(\mathbb{X}\cap L^p)} \lesssim K(f,t;L^p,\mathbb{X}\cap L^p),$$

which implies that

$$\min\{1,t\}\|f\|_{L^p} + K(f,t;L^p,\mathbb{X}) \lesssim K(f,t;L^p,\mathbb{X}\cap L^p). \tag{3.37}$$

We now prove the opposite inequality. By the definition of K-functional, it is easy to see that

$$K(f,t;L^p,\mathbb{X}\cap L^p) \le \|f\|_{L^p}.$$

Thus, to finish the proof, we only need to show that for any $t \in (0,1)$,

$$K(f,t;L^p,\mathbb{X}\cap L^p) \lesssim K(f,t;L^p,\mathbb{X}) + t\|f\|_{L^p}. \tag{3.38}$$

Indeed, for any $\epsilon \in (0,1)$ small enough, let $f = f_0 + f_1$ be a decomposition satisfying $f_0 \in L^p$, $f_1 \in \mathbb{X}$ and

$$\|f_0\|_{L^p} + t\|f_1\|_{\mathbb{X}} < K(f,t;L^p,\mathbb{X}) + \epsilon/2.$$

Since $f \in L^p$, we see that $f_1 \in \mathbb{X}\cap L^p$. Since $t \in (0,1)$, we obtain

$$\begin{aligned} K(f,t;L^p,\mathbb{X}\cap L^p) &\le \|f_0\|_{L^p} + t\left(\|f_1\|_{L^p} + \|f_1\|_{\mathbb{X}}\right) \\ &< K(f,t;L^p,\mathbb{X}) + t\|f_1\|_{L^p} + \epsilon/2 \\ &< K(f,t;L^p,\mathbb{X}) + \|f_0\|_{L^p} + t\|f\|_{L^p} + \epsilon/2 < 2K(f,t;L^p,\mathbb{X}) + t\|f\|_{L^p} + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we obtain (3.38) and finish the proof. □

With the help of Lemma 3.13, we now prove Theorem 3.12.

Proof of Theorem 3.12. Without loss of generality, we only prove (i). The inclusion $(L^p, B^s_{p,q_1})_{\theta,q} \subset B^{\theta s}_{p,q}$ is an easy consequence of Theorem 3.9(i) and the facts $B^s_{p,q_1} = \dot{B}^s_{p,q_1} \cap L^p$ and $B^{\theta s}_{p,q} = \dot{B}^{\theta s}_{p,q} \cap L^p$.

To prove the converse inclusion, let $f \in L^p(M)$, by (3.2), Lemma 3.13 and Theorem 3.9, we have

$$\begin{aligned} \|f\|_{(L^p,B^s_{p,q_1})_{\theta,q}} &= \left[\int_0^\infty \left(t^{-\theta}K(f,t;L^p,\dot{B}^s_{p,q_1}\cap L^p)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} \\ &\lesssim \left[\int_0^\infty \left(t^{-\theta}K(f,t;L^p,\dot{B}^s_{p,q_1})\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} + \left[\int_0^\infty \left(t^{-\theta}\min\{1,t\}\|f\|_{L^p}\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} \\ &\lesssim \|f\|_{(L^p,\dot{B}^s_{p,q_1})_{\theta,q}} + \|f\|_{L^p} \simeq \|f\|_{B^{\theta s}_{p,q_1}}, \end{aligned}$$

which implies the inclusion $B^{\theta s}_{p,q} \subset (L^p, B^s_{p,q_1})_{\theta,q}$ and hence (i). By (i) and Corollary 2.14(i), we conclude that (ii) is also satisfied, which finishes the proof of Theorem 3.12.

□

4 Proofs of main results

In this section, we prove the main results of this paper. To that end, we first prove in Section 4.1 a Hardy-Littlewood-Sobolev-Kato estimates for parameters in $\mathcal{P}(\Theta)$ as on Fig. 3; then in Section 4.2, we prove Theorems 1.2 and 1.3.

4.1 The Hardy-Littlewood-Sobolev-Kato estimates

Let $\Theta \in (0,1)$ be as in (1.7) and $\mathcal{P}(\Theta)$ be as in (1.12) (see also Figure 3). The following proposition gives a Hardy-Littlewood-Sobolev-Kato estimates for parameters p and s in $\mathcal{P}(\Theta)$.

Proposition 4.1. *Let (M,ρ,μ) be a metric measure space satisfying the condition* **(GB)**. *Let $U := (\frac{1}{p}, s)$ and $N := (\frac{1}{q}, r) \in \mathcal{P}(\Theta)$ as in (1.12). Assume that $\nu \in (0,\pi)$ and $\varphi \in \mathscr{E}(\Sigma_\nu)$ is in the extended Dunford-Riesz class and satisfies the following estimate*

$$\left\| z^{\alpha(U,N)}\varphi \right\|_{L^\infty(\Sigma_\nu)} < \infty$$

with

$$\alpha(U,N) := \frac{r-s}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \tag{4.1}$$

where d denotes the Hausdorff dimension of M as in (1.5). Then for any $f \in \dot{F}^s_{p,2}$,

$$\|\varphi(\mathcal{L})f\|_{\dot{F}^r_{q,2}} \lesssim \left\| z^{\alpha(U,N)}\varphi \right\|_{L^\infty} \|f\|_{\dot{F}^s_{p,2}}. \tag{4.2}$$

To prove Proposition 4.1, we need the following result on the characterization on the domain of the fractional power of the generator $\mathcal{L}$.

Lemma 4.2. *Let (M,ρ,μ) be a metric measure space satisfying* **(GB)**. *Then the following is true.*

(i) *For any $s \in (0,1)$, we have $\mathrm{dom}_2(\mathcal{L}^{s/2}) = F^s_{2,2}$. Moreover, for all $f \in \mathrm{dom}_2(\mathcal{L}^{s/2})$,*

$$\left\| \mathcal{L}^{s/2} f \right\|_{L^2} \simeq \|f\|_{\dot{F}^s_{2,2}}.$$

(ii) *For any $s \in (0,\Theta)$ and $p \in (1,\infty)$, we have $\mathrm{dom}_p(\mathcal{L}^{s/2}) = F^s_{p,2}$. Moreover, for all $f \in \mathrm{dom}_p(\mathcal{L}^{s/2})$,*

$$\left\| \mathcal{L}^{s/2} f \right\|_{L^p} \simeq \|f\|_{\dot{F}^s_{p,2}}.$$

Proof. The assertion (i) was proved in [20, Corollary 5.5]. Thus, it suffices to prove (ii). As (M,ρ,μ) satisfies the conditions **(VD)** and **(GB)**, we have

$$\mathrm{dom}_p(\mathcal{L}^{s/2}) = F^{s,\mathcal{L}}_{p,2}$$

and

$$\left\| \mathcal{L}^{s/2} f \right\|_{L^p} \simeq \|f\|_{\dot{F}^{s,\mathcal{L}}_{p,2}}$$

where $\dot{F}^{s,\mathcal{L}}_{p,2}$ and $F^{s,\mathcal{L}}_{p,2} = L^p \cap \dot{F}^{s,\mathcal{L}}_{p,2}$ denote respectively the homogeneous and inhomogeneous heat Triebel-Lizorkin spaces (see [33, Theorem 7.8] and [17, Theorem 6.5]). Moreover, by using an argument similar to the proof of [11, Theorem 3.1], we obtain that, for any $s \in (0, \Theta)$ and $p, q \in (1, \infty)$,

$$\dot{F}^{s,\mathcal{L}}_{p,q} = \dot{F}^{s}_{p,q}, \tag{4.3}$$

which implies (ii). Note that although in [11, Theorem 3.1], (4.3) is proved only in the setting of the Euclidean space, the proof can be extended easily to the present setting by using **(VD)** and **(GB)**. □

We also need the following result of the boundedness of the Riesz potential $\mathcal{L}^{-\alpha/2}$ from L^p to L^q.

Lemma 4.3. *Let (M, ρ, μ) be a metric measure space satisfying the condition* **(GB)**. *Then for any $1 < p < q < \infty$ and $\alpha \in (0, d)$ satisfying $\frac{\alpha}{d} = \frac{1}{p} - \frac{1}{q}$, the Riesz potential $\mathcal{L}^{-\alpha/2}$ is bounded from L^p to L^q.*

Proof. By the functional calculus for the Riesz potential (see [24, Corollary 3.3.6]), we know that

$$\mathcal{L}^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2} e^{-t\mathcal{L}} \frac{dt}{t},$$

which combined with the condition **(GB)** implies that $\mathcal{L}^{-\alpha/2}$ has an integral kernel $K(\cdot, \cdot)$ satisfying that for any $x, y \in M$,

$$|K(x, y)| \lesssim [\rho(x, y)]^{\alpha - d}.$$

This implies that $\mathcal{L}^{-\alpha/2}$ is a generalized fractional integral on M defined as in [39]. By [39, Corollary 2.5], we know that $\mathcal{L}^{-\alpha/2}$ is bounded from L^p to L^q, which completes the proof of Lemma 4.3. □

We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. Let $U := (\frac{1}{p}, s)$, $N := (\frac{1}{q}, r) \in \mathcal{P}(\Theta)$ be as as in (1.12) and $m(U, N)$ the slop of the vector $\overrightarrow{UN}$. We consider three cases based on the size of $|m(U, N)|$.

Case I: $|m(U, N)| = \infty$. In this case, we always have $p = q$ and hence $\alpha(U, N) = \frac{1}{2}(r - s)$. If further $p = q = 2$, then for any $f \in \dot{F}^s_{2,2}$, by Lemma 4.2(i), we know

$$\|\varphi(\mathcal{L})f\|_{\dot{F}^r_{2,2}} \simeq \left\|\mathcal{L}^{r/2}\varphi(\mathcal{L})f\right\|_{L^2} \lesssim \left\|z^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty} \left\|\mathcal{L}^{s/2}f\right\|_{L^2} \simeq \left\|z^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty} \|f\|_{\dot{F}^s_{2,2}}, \tag{4.4}$$

which verifies (4.2) in this subcase.

On the other hand, if $\max\{r, s\} < \Theta$, then by Lemma 4.2(ii) and the bounded H_∞ functional calculus, we have for any $f \in \dot{F}^s_{p,2}$,

$$\|\varphi(\mathcal{L})f\|_{\dot{F}^r_{p,2}} \simeq \left\|\mathcal{L}^{r/2}\varphi(\mathcal{L})f\right\|_{L^p} \lesssim \left\|z^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty} \|f\|_{\dot{F}^s_{p,2}}, \tag{4.5}$$

which shows that (4.2) also holds in this subcase.

If $p = q \neq 2$ and $\max\{r, s\} \geq \Theta$, without loss of generality, we assume that $|r - s| < \Theta$. Otherwise, we may decompose the vector $\overrightarrow{UN}$ into a finite number of vectors with equally small length $\leq \Theta$ and then use the above estimates by composition (see the proof of [11, Theorem 4.3] in the Euclidean case). As $|r-s|<\Theta$, we know that there exist $U_0 := (\frac{1}{p_0}, s_0)$, $N_0 := (\frac{1}{p_0}, r_0)$, $U_1 := (\frac{1}{2}, s_1)$, $N_1 := (\frac{1}{2}, r_1) \in \mathcal{P}(\Theta)$ and $\theta \in (0,1)$ satisfying

$$\begin{cases} \max\{s_0, r_0\} < \Theta, \\ r_0 - s_0 = r - s = r_1 - s_1, \\ \theta = \frac{r-r_0}{r_1-r_0} = \frac{s-s_0}{s_1-s_0}. \end{cases} \tag{4.6}$$

Note that by the definition of $\mathcal{P}(\Theta)$ as illustrated on Figure 3, such points always exist. By Theorem 3.6 and (4.6), we find

$$\begin{cases} \dot{F}^r_{p,2} = \left[\dot{F}^{r_0}_{p_0,2}, \dot{F}^{r_1}_{2,2}\right]_\theta, \\ \\ \dot{F}^s_{p,2} = \left[\dot{F}^{s_0}_{p_0,2}, \dot{F}^{s_1}_{2,2}\right]_\theta. \end{cases} \tag{4.7}$$

Moreover, (4.6) implies that $\overrightarrow{U_0N_0}$ and $\overrightarrow{U_1N_1}$ belong to the above sub-cases which have already been dealt with. This yields that

$$\begin{cases} \|\varphi(\mathcal{L})f\|_{\dot{F}^{r_0}_{p_0,2}} \lesssim \left\|z^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty} \|f\|_{\dot{F}^{s_0}_{p_0,2}}, \\ \|\varphi(\mathcal{L})f\|_{\dot{F}^{r_1}_{2,2}} \lesssim \left\|z^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty} \|f\|_{\dot{F}^{s_1}_{2,2}}, \end{cases}$$

which together with (4.6), (4.7) and Lemma 3.3 shows that for any $f \in \dot{F}^s_{p,2}$,

$$\|\varphi(\mathcal{L})f\|_{\dot{F}^r_{p_0,2}} \lesssim \left\|z^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty} \|f\|_{\dot{F}^s_{p_0,2}}.$$

Thus (4.2) holds under Case I.

Case II: $|m(U,N)| = 0$. In this case, we always have $r = s$ and hence $\alpha(U,N) = \frac{d}{2}(\frac{1}{p} - \frac{1}{q})$. Now consider two subcases: a) $r = s \in (0,\Theta)$; b) $r = s \in [\Theta, 1)$.

For the case II-a), by Lemmas 4.2(ii) and 4.3, we have that for any $f \in \dot{F}^r_{p,2}$,

$$\begin{aligned} \|\varphi(\mathcal{L})f\|_{\dot{F}^r_{q,2}} &\lesssim \left\|\mathcal{L}^{r/2}\varphi(\mathcal{L})f\right\|_{L^q} \\ &\lesssim \left\|z^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty} \left\|\mathcal{L}^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\mathcal{L}^{r/2}\varphi(\mathcal{L})f\right\|_{L^q} \lesssim \left\|z^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty} \|f\|_{\dot{F}^r_{p,s}}, \end{aligned}$$

which verifies (4.2) under Case II-a).

For the case II-b), let $U_0 := (\frac{1}{p}, r_0)$ and $N_0 := (\frac{1}{q}, r_0)$ with $r_0 \in (0,\Theta)$. It is easy to see that $\overrightarrow{MM_0}$ and $\overrightarrow{N_0N}$ belong to Case I, while $\overrightarrow{M_0N_0}$ belong to Case II-a). This implies that for any $f \in \dot{F}^r_{p,2}$

$$\begin{aligned} \|\varphi(\mathcal{L})f\|_{\dot{F}^r_{q,2}} &\simeq \left\|\mathcal{L}^{\frac{1}{2}(r_0-r)}\varphi(\mathcal{L})\mathcal{L}^{-\frac{1}{2}(r_0-r)}f\right\|_{\dot{F}^r_{q,2}} \lesssim \left\|\varphi(\mathcal{L})\mathcal{L}^{-\frac{1}{2}(r_0-r)}f\right\|_{\dot{F}^{r_0}_{q,2}} \\ &\lesssim \left\|z^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty} \left\|\mathcal{L}^{-\frac{1}{2}(r_0-r)}f\right\|_{\dot{F}^{r_0}_{p,2}} \lesssim \left\|z^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty} \|f\|_{\dot{F}^r_{p,2}}, \end{aligned}$$

which implies (4.2) under Case II-b) and hence Case II.

The **Case III:** $|m(U,N)| \in (0,\infty)$. In this case, let $U_0 := (\frac{1}{q}, s)$. By the fact $\overrightarrow{UN} = \overrightarrow{UU_0} + \overrightarrow{U_0N}$, we know that (4.2) follows from a composition argument similar to that used in Case II-b), the details being omitted. This finishes the proof of Proposition 4.1.

□

Corollary 4.4. *Let* $(\frac{1}{p}, s) \in \mathcal{P}(\Theta)$. *Then* $\mathrm{dom}_p(\mathcal{L}^{s/2}) = F^s_{p,2}$.

Proof. For any $(\frac{1}{p}, s) \in \mathcal{P}(\Theta)$, let $U := (\frac{1}{p}, 0)$, $N := (\frac{1}{p}, s)$, $\varphi(z) := z^{-s/2}$ and $f \in \mathrm{dom}_p(\mathcal{L}^{s/2})$. By the fact $\mathcal{L}^{s/2} f \in L^p = \dot{F}^0_{p,2}$ and Proposition 4.1, we know that $\|f\|_{\dot{F}^s_{p,2}} = \left\|\varphi(\mathcal{L})\mathcal{L}^{s/2} f\right\|_{\dot{F}^s_{p,2}} \lesssim \|\mathcal{L}^{s/2} f\|_{L^p}$. This combined with (1.15) and the fact $F^s_{p,2} = L^p \cap \dot{F}^s_{p,2}$ shows that

$$\|f\|_{F^s_{p,2}} \lesssim \|f\|_{\mathrm{dom}_p(\mathcal{L}^{s/2})},$$

which implies that inclusion $\mathrm{dom}_p(\mathcal{L}^{s/2}) \subset F^s_{p,2}$.

On the other hand, for any $f \in \dot{F}^s_{p,2}$, let $\widetilde{U} := (\frac{1}{p}, s)$, $\widetilde{N} := (\frac{1}{p}, 0)$ and $\widetilde{\varphi}(z) := z^{s/2}$. By Proposition 4.1 again, we find

$$\left\|\mathcal{L}^{s/2} f\right\|_{L^p} \lesssim \|f\|_{\dot{F}^s_{p,2}},$$

which implies the converse inclusion $F^s_{p,2} \subset \mathrm{dom}_p(\mathcal{L}^{s/2})$ and hence finishes the proof of Corollary 4.4.

□

4.2 Proofs of Theorems 1.2 and 1.3

We now prove Theorem 1.2.

Proof of Theorem 1.2. For any $(\frac{1}{p}, s) \in \mathcal{P}(\Theta)$ and $q \in (1, \infty]$, let $\epsilon \in (0, 1)$ small enough such that $(\frac{1}{p}, s + \epsilon) \in \mathcal{P}(\Theta)$. As $\mathcal{P}(\Theta)$ is open, we know that such ϵ exists (see Figure 3). By Corollary 4.4, we find

$$\mathrm{dom}_p(\mathcal{L}^{(s+\epsilon)/2}) = F^{s+\epsilon}_{p,2}. \tag{4.8}$$

Moreover, from [24, Chapter 6], it follows that there exists $\theta = \frac{s}{s+\epsilon} \in (0, 1)$ so that

$$\left(L^p, \mathrm{dom}_p(\mathcal{L}^{(s+\epsilon)/2})\right)_{\theta,q} = B^{s,\mathcal{L}}_{p,q}. \tag{4.9}$$

On the other hand, as $\theta = \frac{s}{s+\epsilon}$, by Theorem 3.12, we see

$$\left(L^p, F^{s+\epsilon}_{p,2}\right)_{\theta,q} = B^s_{p,q}. \tag{4.10}$$

Combining (4.8) through (4.10), we conclude that

$$B^{s,\mathcal{L}}_{p,q} = B^s_{p,q},$$

which completes the proof of Theorem 1.2. □

Finally we prove Theorem 1.3.

Proof of Theorem 1.3. For any $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, 1)$, by [22, Theorem 1.5(a)], we know that

$$B^s_{p,q} \subset B^{s,\mathcal{L}}_{p,q}. \tag{4.11}$$

We now turn to the proof of the converse inclusion. From [24, Chapter 6], it follows that

$$B^{s,\mathcal{L}}_{p,q} = \left(L^p, \mathrm{dom}_p(\mathcal{L}^{1/2})\right)_{s,q}.$$

This together with **(DF)** and (3.2) implies that for any $f \in B^{s,\mathcal{L}}_{p,q}$,

$$\|f\|_{B^{s,\mathcal{L}}_{p,q}} \simeq \left\{\int_0^\infty t^{-sq} K^q(f, t; L^p, B^1_{p,\infty}) \frac{dt}{t}\right\}^{\frac{1}{q}}. \tag{4.12}$$

On the other hand, let

$$\mathrm{E}_p(f, t) := \left\{\int_M \fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y)\, d\mu(x)\right\}^{\frac{1}{p}}. \tag{4.13}$$

For any decomposition $f = f_0 + f_1$ with $f_0 \in L^p$ and $f_1 \in B^1_{p,\infty}$, it follows from **(VD)**, that $\mathrm{E}_p(f_0, t) \lesssim \|f_0\|_{L^p}$ and

$$\mathrm{E}_p(f_1, t) \lesssim t \sup_{t>0} \left\{t^{-1} \left[\int_M \fint_{B(x,t)} |f_1(x) - f_1(y)|^p \, d\mu(y)\, d\mu(x)\right]^{\frac{1}{p}}\right\} \lesssim t\|f_1\|_{B^1_{p,\infty}}.$$

By this, (4.13) and the arbitrariness of the decomposition $f = f_0 + f_1$, we conclude that, for any $t \in (0, \infty)$,

$$\mathrm{E}_p(f, t) \lesssim K(f, t; L^p, B^1_{p,\infty}),$$

which together with (4.12) yields

$$\|f\|_{\dot{B}^s_{p,q}} \simeq \left[\int_0^\infty t^{-sq} E^q_p(f, t) \frac{dt}{t}\right]^{\frac{1}{q}} \lesssim \left[\int_0^\infty t^{-sq} K^q(f, t; L^p, B^1_{p,\infty}) \frac{dt}{t}\right]^{\frac{1}{q}} \simeq \|f\|_{B^{s,\mathcal{L}}_{p,q}}.$$

This implies the inclusion $B^{s,\mathcal{L}}_{p,q} \subset B^s_{p,q}$ and hence finishes the proof of Theorem 1.3. □

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