SET OF POSITIVE SOLUTIONS OF LAPLACE-BELTRAMI EQUATION ON SPECIAL TYPE OF RIEMANNIAN MANIFOLDS

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INTRODUCTION

This article considers Riemannian manifolds that can be graphically represented in the following manner: a finite number n of curved tubes running to infinity (the tubes can expand) is "drawn out" of a bounded domain in $\mathbf{R}^d$. We designate the resultant domain as R and examine the problem of finding all positive harmonic functions in R satisfying the homogeneous Neumann condition at the boundary of R. It turns out that, if the tubes are domains of the "cone type" (see Sec. 1), two situations are possible: either any positive harmonic function is a constant (i.e., the Liouville theorem holds) or it is a linear combination with nonnegative coefficients of n fixed harmonic functions. The first situation occurs if all the tubes do not expand sufficiently rapidly (e.g., if the volume of that portion of R included in a sphere of radius r with a fixed center does not exceed const r^2), while the second is realized if this is not the case (see Sec. 1 for the exact formulation and Sec. 2-3 for the proof).

Despite the similarity of their formulations, the Dirichlet and Neumann problems differ qualitatively. In the case of Dirichlet conditions, the set of positive solutions is apparently n-dimensional regardless of the degree of expansion of R.

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1. FORMULATION OF FUNDAMENTAL RESULTS

Let R be a d-dimensional smooth bounded noncompact Riemannian manifold with the edge ∂R (possibly empty).

Definition 1. If Ω is an open subset of R, we will use the term "harmonic function in Ω" to refer to any function u continuous together with its first derivatives up to the edge $\partial R \cap \bar{\Omega}$, that satisfies in $\Omega \setminus \partial R$ the Laplace-Beltrami equation $\Delta u = 0$ and the Neumann condition

$$\partial u / \partial \nu = 0 \tag{1}$$

at the edge $\partial R \cap \bar{\Omega}$, where ν is a normal to ∂R (definitions of the operators Δ, $\partial/\partial\nu$ and other concepts in Riemannian geometry are given by Kobayasi and Nomidzu [1]).

Satisfaction of boundary condition (1) is thus included in the definition of a harmonic function, i.e., functions not satisfying this condition will not be considered.

Definition 2. That portion of the boundary of the set $\Omega \subset R$ $\partial_0 \Omega = \overline{\partial\Omega \cap (R \setminus \partial R)}$ will be called its interior boundary.

Definition 3. The function $\rho \in C^\infty(R)$ will be called the stripping function if, for any $t \in (-\infty, +\infty)$, the set $\{\rho < t\}$ is a compactum.

If D is a subset of R, we will call sets $D(t) = D \cap \{\rho = t\}$ sections of D. It follows from Sard's theorem that sections R(t) are smooth submanifolds of R transverse to the edge ∂R for almost all t. We will henceforth assume that the stripping function ρ(x) is specified on the manifold R.

We will now describe how the manifold R should be constructed. We will assume that the set $\{\rho > -1\}$ consists of n connectivity components $D_1, \ldots, D_n$; each component D_i is a domain of the "cone type."

Definition 4. Let D be one of the components $D_1, \ldots, D_n$. We call the set D a domain of the cone type if, for some $N > 0$, $\lambda > 0$, any section $D(t)$, $t \geq 0$, can be covered by N charts, and: a) in each chart, all eigenvalues of quadratic form I lie in the interval (λ, λ^{-1}); b) in each chart, we can select a pair of concentric Euclidean spheres (or hemispheres, if the chart adjoins the edge ∂R) with a radius ratio of 1:2 such that the smaller spheres (and hemispheres) in aggregate cover the section D(t). For example, the surface of rotation in the R^{d+1} graph of a Lipschitz function is a domain of the cone type.

We will distinguish parabolic and hyperbolic domains among D_i. Let D be one component of D_i. We denote by v_k a harmonic function in the domain $D \cap \{0 < \rho < k\}$, that satisfies the conditions

$$v_k|_{D(0)} = 1, \quad v_k|_{D(k)} = 0.$$

As $k \to \infty$, the sequence v_k increases and converges to the function v, $0 < v \leq 1$, harmonic in D, which we call the capacity potential of set D.

Definition 5. A component D is called parabolic if its capacity potential is exactly equal to 1 and hyperbolic otherwise (see also [2]).

It is easily understandable that the definition does not depend on the choice of stripping function. These concepts can be defined in precisely the same manner for the entire manifold R. It is obvious that a manifold is parabolic when and only when all D_i are parabolic. See the commentary below for more on Definition 5.

Definition 6. We call the number

$$\lim_D f = \lim_{\substack{\rho(x)\to\infty \\ x \in D}} f(x)$$

the limit of the function $f(x)$ if the limit on the right exists.

Definition 7. We call the number

$$\operatorname{str}_D u = \int_{D(t)} \frac{\partial u}{\partial \nu} d\sigma$$

the flow of the harmonic function u over the region D, where ν is the unit normal to D(t) running in the direction of increasing ρ; $d\sigma$ is a (d - 1)-dimensional volume element on the submanifold D(t); t > 0 is any regular value of the functions ρ, $\rho|_{\partial R}$.

The definition of the flow is independent of t by virtue of Green's formula. Actually, if $t_1 < t_2$, $Q = D \cap \{t_1 < \rho < t_2\}$, then

$$0 = \int_Q \Delta u \, d\mu = \int_{D(t_2)} \frac{\partial u}{\partial \nu} d\sigma - \int_{D(t_1)} \frac{\partial u}{\partial \nu} d\sigma + \int_{\partial R \cap \bar{Q}} \frac{\partial u}{\partial \nu} d\sigma = \int_{D(t_2)} \frac{\partial u}{\partial \nu} d\sigma - \int_{D(t_1)} \frac{\partial u}{\partial \nu} d\sigma$$

($d\mu$ is a d-dimensional volume element on R).

Finally, we formulate the basic result.

Theorem. Let on manifold R s regions of the cone type be parabolic and l regions be hyperbolic, $s + l = n$. Let $l \geq 1$, i.e., the manifold R be hyperbolic. Then for each collection of nonnegative real numbers $(a_1, \ldots, a_s, b_1, \ldots, b_l)$, not simultaneously vanishing, there also exists a unique positive function on the manifold R such that

$$\operatorname{str}_{A_i} u = a_i, \quad i = 1, 2, \ldots, s; \quad \lim_{B_j} u = b_j, \quad j = 1, 2, \ldots, l$$

(here $\lim_{A_i} u = +\infty$, if $a_i > 0$, and $\lim_{A_i} u$ exists and is finite if $a_i = 0$) and there are no other positive harmonic functions.

Corollary 1. There exists a unique (up to transposition and multiplication by a constant) collection of n positive harmonic functions $u_1, \ldots, u_n$ such that any positive harmonic function u on R is represented (uniquely) in the form

$$u = \sum_{k=1}^{n} \alpha_k u_k, \text{ where } \alpha_k \geq 0.$$

Corollary 2. The minimum Martin boundary of manifold R consists of n points (see [3] for the appropriate definitions).

Commentaries. 1. The concepts of parabolic and hyperbolic manifolds arose as generalizations of the corresponding notions for two-dimensional surfaces (see [2,4]). Any positive superharmonic function equals a constant on parabolic manifolds and the Liouville theorem specifically holds. If, under the conditions of the theorem, all domains D_i are parabolic, i.e., $l = 0$, then any positive harmonic function on R therefore equals a constant. In the language of potential theory, this means that, when $l = 0$, the minimal Martin boundary consists of a single point and, when $l \geq 1$, it consists of n points regardless of the ratio of s and l.

The geometric conditions for parabolicity are of great interest (see [4-7]). For example, if ρ is a Lipschitz function and the size of the set $D \cap \{\rho < t\}$ does not exceed const t^2, then the domain D is parabolic.

2. We wish to clarify why the cone type condition is required. Let a component D be a domain of the cone type. For any section D(t), we then find a neighborhood U(t) such that, if u is a positive harmonic function in U(t), the Harnack inequality

$$\sup_{D(t)} u \leqslant P \inf_{D(t)} u \tag{2}$$

holds, where the constant P depends only on N, λ, d. Actually, we can take as U(t) the combination of all the charts mentioned in paragraph b of Definition 4 and apply Moser's theorem [8] in each sphere of lesser radius selected. It can be assumed that the neighborhoods U(t) "move out" to infinity as $t \to \infty$ in the following sense: $\inf_{U(t)} \rho \underset{t\to\infty}{\longrightarrow} \infty$ (this can be achieved by reducing the radii of the spheres under consideration, increasing their number and hence the constant P). Thus, we need the cone type condition only for satisfaction of the Harnack inequality (2) (here it can be assumed that the constant P is the same for all domains D). If this inequality is satisfied for any other reason, the theorem holds as before. The assertion of the theorem is invalid without the Harnack inequality or some condition that replaces it. For more information on the Harnack inequality see [9,10].

2. PROOF OF EXISTENCE

We will term any (d - 1)-dimensional submanifold in R transversal to the edge ∂R a hypersurface.

Lemma 1. Let Ω be a precompact open set in R whose interior boundary $\partial_0 \Omega$ consists of nonintersecting compact hypersurfaces F_1 and F_2. Let u be a harmonic

function in Ω continuous in $\bar{\Omega}$; $u|_{F_1} < 0$, $u|_{F_2} > 0$. Then

$$\int_{F_1} \frac{\partial u}{\partial \nu} d\sigma = -\int_{F_2} \frac{\partial u}{\partial \nu} d\sigma > 0, \tag{3}$$

where ν is the interior unit normal.

Proof. Let ε be a regular value of the function u; $0 < \varepsilon < \min_{F_2} u$. Consider the open set $\Omega_\varepsilon = \{u > \varepsilon\}$ and part of its boundary $S_\varepsilon = \{u = \varepsilon\}$. Since the function u in $\bar{\Omega}_\varepsilon$ reaches its minimum at S_ε, by the normal derivative lemma [11]

$$\int_{S_\varepsilon} \frac{\partial u}{\partial \nu} d\sigma > 0, \tag{4}$$

where ν is a normal interior with respect to Ω_ε. Applying Green's formula in the domains Ω_ε and Ω, we find that each of the integrals in (3) equals integral (4).

Proposition 1. Let u be a positive harmonic function in the cone type domain $D \subset R$. There then exists the finite or infinite limit $\lim_D u$. In this case, $\lim_D u = \infty$ when and only when $\operatorname{str}_D u > 0$ and D is a parabolic domain.

Proof. Assume $M(t) = \sup_{D(t)} u$,

$$m(t) = \inf_{D(t)} u, \quad M = \lim_{t\to\infty} M(t), \quad m = \lim_{t\to\infty} m(t).$$

If for some $t_0 > 0$ $M(t_0) < M$, then by the maximum principle $M(t) < M$ with $t > t_0$ and $M(t) \to M$ $(t \to \infty)$. Applying the Harnack inequality to the harmonic function M - u on the section D(t) with sufficiently large t, we obtain $M - m(t) \leqslant P \times (M - M(t))$, whence $m(t) \to M$ $(t \to \infty)$, $\lim_D u = M = m$. The case $M = \infty$ is even simpler to consider.

Let for all $t > 0$ $M(t) \geqslant M$ and, analogously, $m(t) \leq m$. The function M(t) is then diminishing, while m(t) is increasing. For some $t_0 \geq 0$ we apply the Harnack inequality to the function $u - m(t_0)$ on the section $D(t)$, $t \gg t_0$:

$$M(t) - m(t_0) \leqslant P(m(t) - m(t_0)),$$

whence

$$M - m(t_0) < P(m - m(t_0)).$$

As $t_0 \to \infty$ we obtain $m(t_0) \to M$, $\lim_D u = M = m$.

We will now prove the second part of Proposition 1. Consider the following two cases.

a) D is a hyperbolic domain. Let $(1 - v)$ be its capacity potential. Assume $w = av + b$, where

$$a = \operatorname{str}_D u / \operatorname{str}_D v, \quad b = \sup_{D(0)} u.$$

Then $u - w|_{D(0)} < 0$, $\operatorname{str}_D (u - w) = 0$. By Lemma 1, on each section D(t), t > 0,

$$\inf_{D(t)} (u - w) < 0, \quad \inf_{D(t)} u < \sup_{D(t)} w,$$

whence by Harnack's inequality

$$\sup_{D(t)} u < P \sup_{D(t)} w.$$

Since the function w is bounded, boundedness of the function u and specifically $\lim_D u < \infty$ follows.

b) D is a parabolic domain. If $\lim_D u = \infty$, then with large t $\inf_{D(t)} u > \sup_{D(0)} u$, and by Lemma 1 $\operatorname{str}_D u > 0$.

Let $\lim_D u < \infty$, then for some C $\sup_D u < C$. We will prove that $\operatorname{str}_D u < 0$. Actually, let $v_k(x)$ be a sequence of harmonic functions specified by the conditions

$$v_k|_{D(0)} = 0, \quad v_k|_{D(k)} = 1.$$

Then $\{v_k\}$ decreases and converges to zero, so that $\operatorname{str}_D v_k \xrightarrow[k\to\infty]{} 0$. Since the function $Cv_k - u$ is negative on D(0) and positive on D(k), by Lemma 1

$$\operatorname{str}_D (Cv_k - u) > 0, \quad \operatorname{str}_D u < C \operatorname{str}_D v_k.$$

When $k \to \infty$, we obtain $\operatorname{str}_D u < 0$.

Corollary 3. The flow over a parabolic domain is nonnegative for a positive harmonic function.

Corollary 4. If v is the capacity potential of the hyperbolic domain D, then $\lim_D v = 0$, $\operatorname{str}_D v < 0$.

The proofs of Corollaries 3 and 4 are obvious.

Proposition 2. Under the conditions of the theorem, for any collection of nonnegative numbers $(a_1, \ldots, a_s, b_1, \ldots, b_l)$, that do not vanish simultaneously there exists a positive harmonic function u on R for which

$$\operatorname{str}_{A_i} u = a_i, \; i = 1, 2, \ldots, s; \quad \lim_{B_j} u = b_j, \; j = 1, 2, \ldots, l.$$

Proof. It is sufficient to consider two cases.

a) $b_1 = 1$ and all other numbers a_i, b_j equal zero. We relabel B_1 as D_1. Let V(x) be a function harmonic in the domain bounded by the hypersurfaces $D_i(t_i)$, $i = 1, 2, \ldots, n$, here

$$V|_{D_1(t_1)} = 1, \quad V|_{D_i(t_i)} = 0, \quad i \geq 2.$$

Successively tending $t_1 \to \infty$, $t_2 \to \infty$, ..., we obtain at the limit the function u sought. The flow U over the parabolic domains equals zero by virtue of the boundedness of u (see Proposition 1), while the limit for the hyperbolic domains equals 0 or 1 (for D_1) as a consequence of the existence of barriers, i.e., capacity potentials tending to zero (Corollary 4).

b) $a_1 = 1$ and the other numbers equal 0. As in the preceding paragraph, we first construct the harmonic function $V^\tau(x)$ such that $V^\tau|_{A_1(\tau)} = 1$, and the limits V^τ equal zero over all the hyperbolic domains, as do the flows over all the parabolic domains except A_1. We will prove that the family of functions $\{u^\tau\}$, where $u^\tau = \dfrac{V^\tau}{\operatorname{str}_{A_1} V^\tau}$, are compact. Since $\operatorname{str}_{A_1} u^\tau = 1$, the sum of the flows u^τ over the hyperbolic domains equals -1, whence it follows that, for any hyperbolic domain B, $\operatorname{str}_B u^\tau \geq -1$. Let v be a function proportional to the capacity potential B, with $\operatorname{str}_B v = -1$. Since

$v(x)+\varepsilon>\varepsilon>u^{\tau}(x)$ and $\underset{B}{\mathrm{str}}\,(v+\varepsilon)<\underset{B}{\mathrm{str}}\,u^{\tau}$ at large ρ(x), it follows from Lemma 1 that at some point $x\in B(t)$ $v(x)+\varepsilon>u^{\tau}(x)$. By virtue of the arbitrary choice of ε > 0, we obtain $\sup\limits_{B(t)} v \geq \inf\limits_{B(t)} u^{\tau}$, and by the Harnack inequality (2)

$$\sup_{B(t)} u^{\tau} \leq P \sup_{B(t)} v, \tag{5}$$

whence compactness of $\{u^{\tau}\}$ follows. Let u be the limit function as $\tau \to \infty$. Then $\lim\limits_{B_j} u=0$, $j=1, 2, \ldots, l$, by (5) and $\underset{A_1}{\mathrm{str}}\,u=1$; $\underset{A_i}{\mathrm{str}}\,u=0$, $i\geq 2$, so that $\{u^{\tau}\}$ exhibits uniform local convergence with the first derivatives.

Propositions 1 and 2 contain two assertions of the theorem. We will now turn to proof of the third, fundamental part of the theorem, i.e., the uniqueness of the function u with specified $(\underset{A_i}{\mathrm{str}}\,u,\ \lim\limits_{B_j} u)$.

3. PROOF OF UNIQUENESS

Central to the entire proof is

Lemma 2. Let u and v be positive harmonci functions in a cone-type parabolic domain A and let $u=v|_{A(0)}$, $\underset{A}{\mathrm{str}}\,u=\underset{A}{\mathrm{str}}\,v>0$. Then $u\equiv v$.

Proof. Let $\{v_k\}$ be a sequence of numbers growing with sufficient rapidity (i.e., it is necessary that, for any section A(t) intersecting the hypersurface $E_k=\{v=v_k\}$, the neighborhood U(t) from Sec. 1 lies in the domain $\{v_{k-1}<v<v_{k+1}\}$). We will prove that, if w is a positive harmonic function in the domain $\{v<v_{k+1}\}$, the Harnack inequality on the hypersurface E_k holds for it:

$$\max_{E_k} w \leq P^4 \min_{E_k} w. \tag{6}$$

Assume $M=\max\limits_{E_k} w$, $m=\min\limits_{E_k} w$. Let t_1 and t_2 be such that the minimum point $w|_{E_k}$ lies on the section $A(t_1)$ and the maximum point on $A(t_2)$. Let $t_1 < t_2$ for the sake of definiteness. Consider the function

$$V(x)=P^2m(v-v_{k-1})/(v_k-v_{k-1}).$$

On the sections $A(t_1)$ and $A(t_2)$, the function V(x) assumes the value P^2m at certain points (since E_k intersects $A(t_{1,2})$) and, since V is positive in the domain

$\{v>v_{k-1}\}$, the Harnack inequality

$$P^3m\geq V|_{A(t_1)\cup A(t_2)}\geq Pm$$

is satisfied on the sections $A(t_1)$, $A(t_2)$. On the other hand, we again have $w|_{A(t_1)}\leq Pm$ by virtue of the Harnack inequality, so that

$$V\geq w|_{A(t_1)}. \tag{7}$$

Further, on the set E_{k-1} the function V = 0 and w > 0, i.e., $V<w|_{E_{k-1}}$. Therefore by Lemma 1

$$\underset{A}{\mathrm{str}}\,V \geq \underset{A}{\mathrm{str}}\,w. \tag{8}$$

We now compare V and w on the section $A(t_2)$. By (7) and (8), at some point $x\in A(t_2)$ $V(x)\geq w(x)$. Since

$$V|_{A(t_2)}\leq P^3m,\quad w|_{A(t_2)}\geq P^{-1}M,$$

hence $P^3m\geq P^{-1}M$, $M\leq P^4m$.

We now turn directly to proof that $u\equiv v$. Let $M_k=\max\limits_{E_k} u$, $m_k=\min\limits_{E_k} u$. Compare the functions u and v in the domain $\Omega_k=\{v<v_k\}$. It should be noted that $\partial_0\Omega_k=A(0)\cup E_k$. Since $v|_{A(0)}\geq u|_{A(0)}$, $\mathrm{str}\,u=\mathrm{str}\,v$, at some point $x\in E_k$ $v(x)\geq u(x)$, by Lemma 1, whence $v_k \geq m_k$. Similarly, $M_k \geq v_k$.

Now compare the functions u and $(m_k/v_k)v$ in Ω_k. Since $u|_{A(0)}=v|_{A(0)}\geq (m_k/v_k)\,v|_{A(0)}$,

$$u|_{E_k}\geq m_k=(m_k/v_k)\,v|_{E_k},$$

then by the maximum principle $u \geq (m_k/v_k)v$ in Ω_k. We apply to the function $w = u - (m_k/v_k)v$ the Harnack inequality on E_{k-1}:

$$m_{k-1}-m_kv_{k-1}/v_k\geq P^{-4}(M_{k-1}-m_kv_{k-1}/v_k),$$

or, by virtue of the fact that $M_{k-1} > v_{k-1}$,

$$m_{k-1} - m_k v_{k-1}/v_k > P^{-4}(v_{k-1} - m_k v_{k-1}/v_k). \tag{9}$$

Analogously,

$$M_k v_{k-1}/v_k - M_{k-1} > P^{-4}(M_k v_{k-1}/v_k - v_{k-1}). \tag{10}$$

Adding (9) and (10), we find

$$(M_k - m_k) v_{k-1}/v_k - (M_{k-1} - m_{k-1}) > P^{-4}(M_k - m_k) v_{k-1}/v_k,$$
$$(1 - P^{-4})(M_k - m_k)/v_k > (M_{k-1} - m_{k-1})/v_{k-1}.$$

Hence we obtain by successive iterations

$$(M_k - m_k)/v_k < (1 - P^{-4})(M_{k+1} - m_{k+1})/v_{k+1} < \dots < (1 - P^4)^q (M_{k+q} - m_{k+q})/v_{k+q}.$$

Since $(M_{k+q} - m_{k+q})/v_{k+q} < P^4$, by the Harnack inequality, as $q \to \infty$ we obtain $M_k - m_k = 0$, $M_k = m_k = v_k$, i.e., $u \equiv v$. The lemma has been proved.

Corollary 5. Let u and v be positive harmonic functions in the cone-type domain D and let w = u - v. There then exists the finite or infinite limit $\lim_D w$.

Proof. If D is a hyperbolic domain, then $\lim_D w = \lim_D u - \lim_D v$, since both limits on the right are finite. The argument proceeds in the same manner if D is a parabolic domain and one of the flows $\operatorname{str}_D u$, $\operatorname{str}_D v$, equals zero. Now let $\operatorname{str}_D u > \operatorname{str}_D v > 0$.

It can be assumed that $u|_{D(0)} > v|_{D(0)}$ (the function u + const otherwise being considered in place of u). We construct the positive harmonic function w_0 such that

$$w_0|_{D(0)} = u - v, \quad \operatorname{str}_D w_0 = \operatorname{str}_D u - \operatorname{str}_D v > 0$$

(w_0 is constructed as the limit of solutions of appropriate Dirichlet problems). However, we then have $v + w_0 = u$ by Lemma 2, whence $w_0 = w$. The function w there-

fore has a limit by Proposition 1.

Proof of Theorem (Uniqueness). Let u and v be positive harmonic functions on R for which the collections of numbers $(\operatorname{str}_{A_i}, \lim_{B_j})$ coincide. Assume w = u - v; then $\lim_{B_j} w = 0$, $\operatorname{str}_{A_i} w = 0$. We will prove that $w \geq 0$. By Corollary 5, there exist the limits $\lim_{A_i} w$. If they are not all negative, then $w \geq 0$ by the maximum principle. Let some of them be less than 0. We denote by I the collection of indices i, $1 \leq i \leq s$, for which the limit $\lim_{A_i} w$ is minimal. We find a negative number $-\varepsilon$ that is greater than this minimum but less than the other limits $\lim_{A_i} w$, $i \notin I$ (and that is a regular value of the function u). Consider for sufficiently large t the domain bounded by all sections $A_i(t)$, $B_j(t)$. On the sections $A_i(t)$, $i \in I$, we have $w < -\varepsilon$, while on other sections $w > -\varepsilon$. By Lemma 1, $-\sum_{i \in I} \operatorname{str}_{A_i} w > 0$, which contradicts the condition $\operatorname{str}_{A_i} w = 0$. Thus, $w \geq 0$. Similarly $w \leq 0$, so that $u \equiv v$.

Proof of Corollary 1. The theorem establishes a linear one-to-one correspondence between positive harmonic functions on R and points of an octant in R^n. The assertions of Corollary 1 follow from the corresponding facts of linear algebra. For example, the uniqueness of the collection $u_1, \dots, u_n$ follows from the fact that an (n - 1)-dimensional simplex has n extremal points.

Corollary 2 is a reformulation of Corollary 1.

REFERENCES

1. S. Kobayasi and K. Nomidzu, Principles of Differential Geometry [Russian translation], Moscow, vol. 1, 1981.
2. L. Sario, M. Nakai, C. Wang, and L. O. Chung, "Classification theory of Riemannian manifolds," Lect. Notes Math., vol. 605, 1977.
3. M. Brelot, Topologies and Boundaries in Potential Theory [Russian translation], Moscow, 1974.
4. J. Milnor, "On deciding whether a surface is parabolic or hyperbolic," Amer. Math. Monthly, vol. 84, no. 1, pp. 43-46, 1977.
5. A. A. Grigor'yan, "Existence of Green's function on manifold," UMN, vol. 38, no. 1, pp. 161-162, 1983.
6. S. Y. Cheng and S. T. Yau, "Differential equations on Riemannian manifolds and their geometric applications," Comm. Pure and Appl. Math., vol. 28, no. 3, pp. 333-354, 1975.
7. L. Karp, "Subharmonic functions, harmonic mappings and isometric immersions," Ann. Math. Studies, no. 102, pp. 133-142, 1982.
8. J. Moser, "On Harnak's theorem for elliptic differential equations," Comm. Pure and Appl. Math., vol. 14, no. 3, pp. 577-591, 1961.
9. A. A. Grigor'yan, "One Liouville theorem on Riemannian manifold," Tr. Tbilissk. Un-ta, Ser. Matem., Mekhan., Astr., vol. 232-233, no. 13-14, pp. 49-76, 1982.